Probability and Stochastic Processes

A Friendly Introduction for Electrical and Computer Engineers

Third Edition

STUDENT'S SOLUTION MANUAL

(Solutions to the odd-numbered problems)

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Comments on this Student Solutions Manual

- Matlab functions written as solutions to homework problems in this Student's Solution Manual (SSM) can be found in the archive `matsoln3student.zip` Other Matlab functions used in the text or in these homework solutions can be found in the archive `matcode3e.zip`. The archives `matcode3e.zip` and `matsoln3student.zip` are available for download from the John Wiley companion website. Two other documents of interest are also available for download:
  - A manual `probmatlab.pdf` describing the `.m` functions in `matcode.zip`.
  - The quiz solutions manual `quizsol.pdf`.
- This manual uses a page size matched to the screen of an iPad tablet. If you do print on paper and you have good eyesight, you may wish to print two pages per sheet in landscape mode. On the other hand, a "Fit to Paper" printing option will create "Large Print" output.
- Send error reports, suggestions, or comments to

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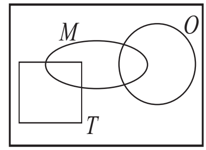
Problem Solutions – Chapter 1

Problem 1.1.1 Solution

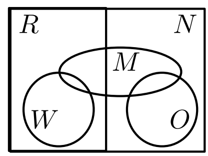
Based on the Venn diagram on the right, the complete Gerlandas pizza menu is



- Regular without toppings
- Regular with mushrooms
- Regular with onions
- Regular with mushrooms and onions
- Tuscan without toppings
- Tuscan with mushrooms

Problem 1.1.3 Solution

At Ricardo's, the pizza crust is either Roman (R) or Neapolitan (N). To draw the Venn diagram on the right, we make the following observations:



- The set $\{R, N\}$ is a partition so we can draw the Venn diagram with this partition.
- Only Roman pizzas can be white. Hence $W \subset R$.
- Only a Neapolitan pizza can have onions. Hence $O \subset N$.
- Both Neapolitan and Roman pizzas can have mushrooms so that event M straddles the $\{R, N\}$ partition.
- The Neapolitan pizza can have both mushrooms and onions so $M \cap O$ cannot be empty.
- The problem statement does not preclude putting mushrooms on a white Roman pizza. Hence the intersection $W \cap M$ should not be empty.

Problem 1.2.1 Solution

(a) An outcome specifies whether the connection speed is high (h), medium (m), or low (l) speed, and whether the signal is a mouse click (c) or a tweet (t). The sample space is

$$S = \{ht, hc, mt, mc, lt, lc\} . \tag{1}$$

(b) The event that the wi-fi connection is medium speed is $A_1 = \{mt, mc\}$.

(c) The event that a signal is a mouse click is $A_2 = \{hc, mc, lc\}$.

(d) The event that a connection is either high speed or low speed is $A_3 = \{ht, hc, lt, lc\}$.

(e) Since $A_1 \cap A_2 = \{mc\}$ and is not empty, A_1, A_2, and A_3 are not mutually exclusive.

(f) Since

$$A_1 \cup A_2 \cup A_3 = \{ht, hc, mt, mc, lt, lc\} = S, \tag{2}$$

the collection A_1, A_2, A_3 is collectively exhaustive.

Problem 1.2.3 Solution

The sample space is

$$S = \{A\clubsuit, \ldots, K\clubsuit, A\diamondsuit, \ldots, K\diamondsuit, A\heartsuit, \ldots, K\heartsuit, A\spadesuit, \ldots, K\spadesuit\} . \tag{1}$$

The event H that the first card is a heart is the set

$$H = \{A\heartsuit, \ldots, K\heartsuit\} . \tag{2}$$

The event H has 13 outcomes, corresponding to the 13 hearts in a deck.

Problem 1.2.5 Solution

Of course, there are many answers to this problem. Here are four partitions.

1. We can divide students into engineers or non-engineers. Let A_1 equal the set of engineering students and A_2 the non-engineers. The pair $\{A_1, A_2\}$ is a partition.

2. We can also separate students by GPA. Let B_i denote the subset of students with GPAs G satisfying $i-1 \leq G < i$. At Rutgers, $\{B_1, B_2, \ldots, B_5\}$ is a partition. Note that B_5 is the set of all students with perfect 4.0 GPAs. Of course, other schools use different scales for GPA.

3. We can also divide the students by age. Let C_i denote the subset of students of age i in years. At most universities, $\{C_{10}, C_{11}, \ldots, C_{100}\}$ would be an event space. Since a university may have prodigies either under 10 or over 100, we note that $\{C_0, C_1, \ldots\}$ is always a partition.

4. Lastly, we can categorize students by attendance. Let D_0 denote the number of students who have missed zero lectures and let D_1 denote all other students. Although it is likely that D_0 is an empty set, $\{D_0, D_1\}$ is a well defined partition.

Problem 1.3.1 Solution

(a) A and B mutually exclusive and collectively exhaustive imply $\mathrm{P}[A] + \mathrm{P}[B] = 1$. Since $\mathrm{P}[A] = 3\,\mathrm{P}[B]$, we have $\mathrm{P}[B] = 1/4$.

(b) Since $\mathrm{P}[A \cup B] = \mathrm{P}[A]$, we see that $B \subseteq A$. This implies $\mathrm{P}[A \cap B] = \mathrm{P}[B]$. Since $\mathrm{P}[A \cap B] = 0$, then $\mathrm{P}[B] = 0$.

(c) Since it's always true that $\mathrm{P}[A \cup B] = \mathrm{P}[A] + \mathrm{P}[B] - \mathrm{P}[AB]$, we have that

$$\mathrm{P}[A] + \mathrm{P}[B] - \mathrm{P}[AB] = \mathrm{P}[A] - \mathrm{P}[B]. \tag{1}$$

This implies $2\,\mathrm{P}[B] = \mathrm{P}[AB]$. However, since $AB \subset B$, we can conclude that $2\,\mathrm{P}[B] = \mathrm{P}[AB] \leq \mathrm{P}[B]$. This implies $\mathrm{P}[B] = 0$.

Problem 1.3.3 Solution

An outcome is a pair (i, j) where i is the value of the first die and j is the value of the second die. The sample space is the set

$$S = \{(1,1),(1,2),\ldots,(6,5),(6,6)\}. \tag{1}$$

with 36 outcomes, each with probability 1/36 Note that the event that the absolute value of the difference of the two rolls equals 3 is

$$D_3 = \{(1,4),(2,5),(3,6),(4,1),(5,2),(6,3)\}. \tag{2}$$

Since there are 6 outcomes in D_3, $P[D_3] = 6/36 = 1/6$.

Problem 1.3.5 Solution

The sample space of the experiment is

$$S = \{LF, BF, LW, BW\}. \tag{1}$$

From the problem statement, we know that $P[LF] = 0.5$, $P[BF] = 0.2$ and $P[BW] = 0.2$. This implies $P[LW] = 1-0.5-0.2-0.2 = 0.1$. The questions can be answered using Theorem 1.5.

(a) The probability that a program is slow is

$$P[W] = P[LW] + P[BW] = 0.1 + 0.2 = 0.3. \tag{2}$$

(b) The probability that a program is big is

$$P[B] = P[BF] + P[BW] = 0.2 + 0.2 = 0.4. \tag{3}$$

(c) The probability that a program is slow or big is

$$P[W \cup B] = P[W] + P[B] - P[BW] = 0.3 + 0.4 - 0.2 = 0.5. \tag{4}$$

Problem 1.3.7 Solution

A reasonable probability model that is consistent with the notion of a shuffled deck is that each card in the deck is equally likely to be the first card. Let H_i denote the event that the first card drawn is the ith heart where the first heart is the ace, the second heart is the deuce and so on. In that case, $P[H_i] = 1/52$ for $1 \le i \le 13$. The event H that the first card is a heart can be written as the mutually exclusive union

$$H = H_1 \cup H_2 \cup \cdots \cup H_{13}. \tag{1}$$

Using Theorem 1.1, we have

$$P[H] = \sum_{i=1}^{13} P[H_i] = 13/52. \tag{2}$$

This is the answer you would expect since 13 out of 52 cards are hearts. The point to keep in mind is that this is not just the common sense answer but is the result of a probability model for a shuffled deck and the axioms of probability.

Problem 1.3.9 Solution

Let s_i equal the outcome of the student's quiz. The sample space is then composed of all the possible grades that she can receive.

$$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}. \tag{1}$$

Since each of the 11 possible outcomes is equally likely, the probability of receiving a grade of i, for each $i = 0, 1, \ldots, 10$ is $P[s_i] = 1/11$. The probability that the student gets an A is the probability that she gets a score of 9 or higher. That is

$$P[\text{Grade of A}] = P[9] + P[10] = 1/11 + 1/11 = 2/11. \tag{2}$$

The probability of failing requires the student to get a grade less than 4.

$$\begin{aligned} P[\text{Failing}] &= P[3] + P[2] + P[1] + P[0] \\ &= 1/11 + 1/11 + 1/11 + 1/11 = 4/11. \end{aligned} \tag{3}$$

Problem 1.3.11 Solution

Specifically, we will use Theorem 1.4(c) which states that for any events A and B,

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]. \tag{1}$$

To prove the union bound by induction, we first prove the theorem for the case of $n = 2$ events. In this case, by Theorem 1.4(c),

$$P[A_1 \cup A_2] = P[A_1] + P[A_2] - P[A_1 \cap A_2]. \tag{2}$$

By the first axiom of probability, $P[A_1 \cap A_2] \geq 0$. Thus,

$$P[A_1 \cup A_2] \leq P[A_1] + P[A_2]. \tag{3}$$

which proves the union bound for the case $n = 2$.Now we make our induction hypothesis that the union-bound holds for any collection of $n - 1$ subsets. In this case, given subsets $A_1, \ldots, A_n$, we define

$$A = A_1 \cup A_2 \cup \cdots \cup A_{n-1}, \qquad B = A_n. \tag{4}$$

By our induction hypothesis,

$$P[A] = P[A_1 \cup A_2 \cup \cdots \cup A_{n-1}] \leq P[A_1] + \cdots + P[A_{n-1}]. \tag{5}$$

This permits us to write

$$\begin{aligned} P[A_1 \cup \cdots \cup A_n] &= P[A \cup B] \\ &\leq P[A] + P[B] \qquad \text{(by the union bound for } n = 2\text{)} \\ &= P[A_1 \cup \cdots \cup A_{n-1}] + P[A_n] \\ &\leq P[A_1] + \cdots P[A_{n-1}] + P[A_n] \end{aligned} \tag{6}$$

which completes the inductive proof.

Problem 1.3.13 Solution

Following the hint, we define the set of events $\{A_i | i = 1, 2, \ldots\}$ such that $i = 1, \ldots, m$, $A_i = B_i$ and for $i > m$, $A_i = \phi$. By construction, $\cup_{i=1}^{m} B_i = \cup_{i=1}^{\infty} A_i$. Axiom 3 then implies

$$P\left[\cup_{i=1}^{m} B_i\right] = P\left[\cup_{i=1}^{\infty} A_i\right] = \sum_{i=1}^{\infty} P\left[A_i\right]. \tag{1}$$

For $i > m$, $P[A_i] = P[\phi] = 0$, yielding the claim $P[\cup_{i=1}^{m} B_i] = \sum_{i=1}^{m} P[A_i] = \sum_{i=1}^{m} P[B_i]$.

Note that the fact that $P[\phi] = 0$ follows from Axioms 1 and 2. This problem is more challenging if you just use Axiom 3. We start by observing

$$P\left[\cup_{i=1}^{m} B_i\right] = \sum_{i=1}^{m-1} P\left[B_i\right] + \sum_{i=m}^{\infty} P\left[A_i\right]. \tag{2}$$

Now, we use Axiom 3 again on the countably infinite sequence $A_m, A_{m+1}, \ldots$ to write

$$\sum_{i=m}^{\infty} P\left[A_i\right] = P\left[A_m \cup A_{m+1} \cup \cdots\right] = P\left[B_m\right]. \tag{3}$$

Thus, we have used just Axiom 3 to prove Theorem 1.3:

$$P\left[\cup_{i=1}^{m} B_i\right] = \sum_{i=1}^{m} P\left[B_i\right]. \tag{4}$$

Problem 1.4.1 Solution

Each question requests a conditional probability.

(a) Note that the probability a call is brief is

$$P\left[B\right] = P\left[H_0 B\right] + P\left[H_1 B\right] + P\left[H_2 B\right] = 0.6. \tag{1}$$

The probability a brief call will have no handoffs is

$$P\left[H_0 | B\right] = \frac{P\left[H_0 B\right]}{P\left[B\right]} = \frac{0.4}{0.6} = \frac{2}{3}. \tag{2}$$

(b) The probability of one handoff is $P[H_1] = P[H_1B] + P[H_1L] = 0.2$. The probability that a call with one handoff will be long is

$$P[L|H_1] = \frac{P[H_1L]}{P[H_1]} = \frac{0.1}{0.2} = \frac{1}{2}. \qquad (3)$$

(c) The probability a call is long is $P[L] = 1 - P[B] = 0.4$. The probability that a long call will have one or more handoffs is

$$\begin{aligned} P[H_1 \cup H_2|L] &= \frac{P[H_1L \cup H_2L]}{P[L]} \\ &= \frac{P[H_1L] + P[H_2L]}{P[L]} = \frac{0.1 + 0.2}{0.4} = \frac{3}{4}. \end{aligned} \qquad (4)$$

Problem 1.4.3 Solution

Since the 2 of clubs is an even numbered card, $C_2 \subset E$ so that $P[C_2E] = P[C_2] = 1/3$. Since $P[E] = 2/3$,

$$P[C_2|E] = \frac{P[C_2E]}{P[E]} = \frac{1/3}{2/3} = 1/2. \qquad (1)$$

The probability that an even numbered card is picked given that the 2 is picked is

$$P[E|C_2] = \frac{P[C_2E]}{P[C_2]} = \frac{1/3}{1/3} = 1. \qquad (2)$$

Problem 1.4.5 Solution

The first generation consists of two plants each with genotype yg or gy. They are crossed to produce the following second generation genotypes, $S = \{yy, yg, gy, gg\}$. Each genotype is just as likely as any other so the probability of each genotype is consequently $1/4$. A pea plant has yellow seeds if it possesses at least one dominant y gene. The set of pea plants with yellow seeds is

$$Y = \{yy, yg, gy\}. \tag{1}$$

So the probability of a pea plant with yellow seeds is

$$P[Y] = P[yy] + P[yg] + P[gy] = 3/4. \tag{2}$$

Problem 1.4.7 Solution

The sample outcomes can be written ijk where the first card drawn is i, the second is j and the third is k. The sample space is

$$S = \{234, 243, 324, 342, 423, 432\}. \tag{1}$$

and each of the six outcomes has probability 1/6. The events E_1, E_2, E_3, O_1, O_2, O_3 are

$$E_1 = \{234, 243, 423, 432\}, \qquad O_1 = \{324, 342\}, \tag{2}$$
$$E_2 = \{243, 324, 342, 423\}, \qquad O_2 = \{234, 432\}, \tag{3}$$
$$E_3 = \{234, 324, 342, 432\}, \qquad O_3 = \{243, 423\}. \tag{4}$$

(a) The conditional probability the second card is even given that the first card is even is

$$P[E_2|E_1] = \frac{P[E_2E_1]}{P[E_1]} = \frac{P[243, 423]}{P[234, 243, 423, 432]} = \frac{2/6}{4/6} = 1/2. \tag{5}$$

(b) The conditional probability the first card is even given that the second card is even is

$$P[E_1|E_2] = \frac{P[E_1E_2]}{P[E_2]} = \frac{P[243, 423]}{P[243, 324, 342, 423]} = \frac{2/6}{4/6} = 1/2. \tag{6}$$

(c) The probability the first two cards are even given the third card is even is

$$P[E_1E_2|E_3] = \frac{P[E_1E_2E_3]}{P[E_3]} = 0. \tag{7}$$

(d) The conditional probabilities the second card is even given that the first card is odd is

$$P[E_2|O_1] = \frac{P[O_1E_2]}{P[O_1]} = \frac{P[O_1]}{P[O_1]} = 1. \tag{8}$$

(e) The conditional probability the second card is odd given that the first card is odd is

$$P[O_2|O_1] = \frac{P[O_1O_2]}{P[O_1]} = 0. \tag{9}$$

Problem 1.5.1 Solution

From the table we look to add all the mutually exclusive events to find each probability.

(a) The probability that a caller makes no hand-offs is

$$P[H_0] = P[LH_0] + P[BH_0] = 0.1 + 0.4 = 0.5. \tag{1}$$

(b) The probability that a call is brief is

$$P[B] = P[BH_0] + P[BH_1] + P[BH_2] = 0.4 + 0.1 + 0.1 = 0.6. \tag{2}$$

(c) The probability that a call is long or makes at least two hand-offs is

$$\begin{aligned} P[L \cup H_2] &= P[LH_0] + P[LH_1] + P[LH_2] + P[BH_2] \\ &= 0.1 + 0.1 + 0.2 + 0.1 = 0.5. \end{aligned} \tag{3}$$

Problem 1.5.3 Solution

(a) For convenience, let $p_i = P[FH_i]$ and $q_i = P[VH_i]$. Using this shorthand, the six unknowns $p_0, p_1, p_2, q_0, q_1, q_2$ fill the table as

$$\begin{array}{c|ccc} & H_0 & H_1 & H_2 \\ \hline F & p_0 & p_1 & p_2 \\ V & q_0 & q_1 & q_2 \end{array}. \tag{1}$$

However, we are given a number of facts:

$$p_0 + q_0 = 1/3, \qquad p_1 + q_1 = 1/3, \tag{2}$$
$$p_2 + q_2 = 1/3, \qquad p_0 + p_1 + p_2 = 5/12. \tag{3}$$

Other facts, such as $q_0 + q_1 + q_2 = 7/12$, can be derived from these facts. Thus, we have four equations and six unknowns, choosing p_0 and p_1 will specify the other unknowns. Unfortunately, arbitrary choices for either p_0 or p_1 will lead to negative values for the other probabilities. In terms of p_0 and p_1, the other unknowns are

$$q_0 = 1/3 - p_0, \qquad p_2 = 5/12 - (p_0 + p_1), \tag{4}$$
$$q_1 = 1/3 - p_1, \qquad q_2 = p_0 + p_1 - 1/12. \tag{5}$$

Because the probabilities must be nonnegative, we see that

$$0 \le p_0 \le 1/3, \tag{6}$$
$$0 \le p_1 \le 1/3, \tag{7}$$
$$1/12 \le p_0 + p_1 \le 5/12. \tag{8}$$

Although there are an infinite number of solutions, three possible solutions are:

$$p_0 = 1/3, \qquad p_1 = 1/12, \qquad p_2 = 0, \tag{9}$$
$$q_0 = 0, \qquad q_1 = 1/4, \qquad q_2 = 1/3. \tag{10}$$

and

$$p_0 = 1/4, \qquad p_1 = 1/12, \qquad p_2 = 1/12, \tag{11}$$
$$q_0 = 1/12, \qquad q_1 = 3/12, \qquad q_2 = 3/12. \tag{12}$$

and

$$p_0 = 0, \qquad p_1 = 1/12, \qquad p_2 = 1/3, \tag{13}$$
$$q_0 = 1/3, \qquad q_1 = 3/12, \qquad q_2 = 0. \tag{14}$$

(b) In terms of the p_i, q_i notation, the new facts are $p_0 = 1/4$ and $q_1 = 1/6$. These extra facts uniquely specify the probabilities. In this case,

$$p_0 = 1/4, \qquad p_1 = 1/6, \qquad p_2 = 0, \tag{15}$$
$$q_0 = 1/12, \qquad q_1 = 1/6, \qquad q_2 = 1/3. \tag{16}$$

Problem 1.6.1 Solution

This problems asks whether A and B can be independent events yet satisfy $A = B$? By definition, events A and B are independent if and only if $\mathrm{P}[AB] = \mathrm{P}[A]\,\mathrm{P}[B]$. We can see that if $A = B$, that is they are the same set, then

$$\mathrm{P}[AB] = \mathrm{P}[AA] = \mathrm{P}[A] = \mathrm{P}[B]. \tag{1}$$

Thus, for A and B to be the same set and also independent,

$$\mathrm{P}[A] = \mathrm{P}[AB] = \mathrm{P}[A]\,\mathrm{P}[B] = (\mathrm{P}[A])^2. \tag{2}$$

There are two ways that this requirement can be satisfied:

- $\mathrm{P}[A] = 1$ implying $A = B = S$.
- $\mathrm{P}[A] = 0$ implying $A = B = \phi$.

Problem 1.6.3 Solution

Let A_i and B_i denote the events that the ith phone sold is an Apricot or a Banana respectively. The works "each phone sold is twice as likely to be an Apricot than a Banana" tells us that

$$P[A_i] = 2P[B_i]. \tag{1}$$

However, since each phone sold is either an Apricot or a Banana, A_i and B_i are a partition and

$$P[A_i] + P[B_i] = 1. \tag{2}$$

Combining these equations, we have $P[A_i] = 2/3$ and $P[B_i] = 1/3$. The probability that two phones sold are the same is

$$P[A_1A_2 \cup B_1B_2] = P[A_1A_2] + P[B_1B_2]. \tag{3}$$

Since "each phone sale is independent,"

$$P[A_1A_2] = P[A_1]P[A_2] = \frac{4}{9}, \qquad P[B_1B_2] = P[B_1]P[B_2] = \frac{1}{9}. \tag{4}$$

Thus the probability that two phones sold are the same is

$$P[A_1A_2 \cup B_1B_2] = P[A_1A_2] + P[B_1B_2] = \frac{4}{9} + \frac{1}{9} = \frac{5}{9}. \tag{5}$$

Problem 1.6.5 Solution

(a) Since A and B are mutually exclusive, $P[A \cap B] = 0$. Since $P[A \cap B] = 0$,

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] = 3/8. \tag{1}$$

A Venn diagram should convince you that $A \subset B^c$ so that $A \cap B^c = A$. This implies

$$P[A \cap B^c] = P[A] = 1/4. \tag{2}$$

It also follows that $P[A \cup B^c] = P[B^c] = 1 - 1/8 = 7/8$.

(b) Events A and B are dependent since $P[AB] \neq P[A]P[B]$.

Problem 1.6.7 Solution

(a) Since $A \cap B = \emptyset$, $\mathrm{P}[A \cap B] = 0$. To find $\mathrm{P}[B]$, we can write

$$\mathrm{P}[A \cup B] = \mathrm{P}[A] + \mathrm{P}[B] - \mathrm{P}[A \cap B] \tag{1}$$

or

$$5/8 = 3/8 + \mathrm{P}[B] - 0. \tag{2}$$

Thus, $\mathrm{P}[B] = 1/4$. Since A is a subset of B^c, $\mathrm{P}[A \cap B^c] = \mathrm{P}[A] = 3/8$. Furthermore, since A is a subset of B^c, $\mathrm{P}[A \cup B^c] = \mathrm{P}[B^c] = 3/4$.

(b) The events A and B are dependent because

$$\mathrm{P}[AB] = 0 \neq 3/32 = \mathrm{P}[A]\,\mathrm{P}[B]. \tag{3}$$

Problem 1.6.9 Solution

For a sample space $S = \{1, 2, 3, 4\}$ with equiprobable outcomes, consider the events

$$A_1 = \{1, 2\} \qquad A_2 = \{2, 3\} \qquad A_3 = \{3, 1\}. \tag{1}$$

Each event A_i has probability $1/2$. Moreover, each pair of events is independent since

$$\mathrm{P}[A_1 A_2] = \mathrm{P}[A_2 A_3] = \mathrm{P}[A_3 A_1] = 1/4. \tag{2}$$

However, the three events A_1, A_2, A_3 are not independent since

$$\mathrm{P}[A_1 A_2 A_3] = 0 \neq \mathrm{P}[A_1]\,\mathrm{P}[A_2]\,\mathrm{P}[A_3]. \tag{3}$$

Problem 1.6.11 Solution

(a) For any events A and B, we can write the law of total probability in the form of

$$P[A] = P[AB] + P[AB^c]. \tag{1}$$

Since A and B are independent, $P[AB] = P[A]P[B]$. This implies

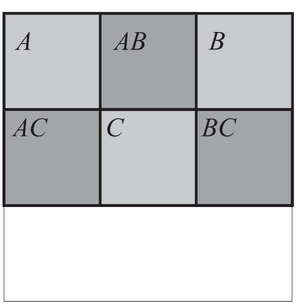
$$P[AB^c] = P[A] - P[A]P[B] = P[A](1 - P[B]) = P[A]P[B^c]. \tag{2}$$

Thus A and B^c are independent.

(b) Proving that A^c and B are independent is not really necessary. Since A and B are arbitrary labels, it is really the same claim as in part (a). That is, simply reversing the labels of A and B proves the claim. Alternatively, one can construct exactly the same proof as in part (a) with the labels A and B reversed.

(c) To prove that A^c and B^c are independent, we apply the result of part (a) to the sets A and B^c. Since we know from part (a) that A and B^c are independent, part (b) says that A^c and B^c are independent.

Problem 1.6.13 Solution



In the Venn diagram at right, assume the sample space has area 1 corresponding to probability 1. As drawn, A, B, and C each have area 1/3 and thus probability 1/3. The three way intersection ABC has zero probability, implying A, B, and C are not mutually independent since

$$P[ABC] = 0 \neq P[A]P[B]P[C]. \tag{1}$$

However, AB, BC, and AC each has area 1/9. As a result, each pair of events is independent since

$$P[AB] = P[A] P[B], \quad P[BC] = P[B] P[C], \quad P[AC] = P[A] P[C]. \quad (2)$$

Problem 1.7.1 Solution

We can generate the 200×1 vector $\mathbf{T}$, denoted `T` in MATLAB, via the command

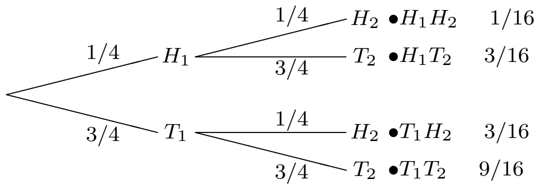
```
T=50+ceil(50*rand(200,1))
```

Keep in mind that `50*rand(200,1)` produces a 200×1 vector of random numbers, each in the interval $(0, 50)$. Applying the ceiling function converts these random numbers to rndom integers in the set $\{1, 2, \ldots, 50\}$. Finally, we add 50 to produce random numbers between 51 and 100.

Problem Solutions – Chapter 2

Problem 2.1.1 Solution

A sequential sample space for this experiment is



(a) From the tree, we observe

$$P[H_2] = P[H_1H_2] + P[T_1H_2] = 1/4. \tag{1}$$

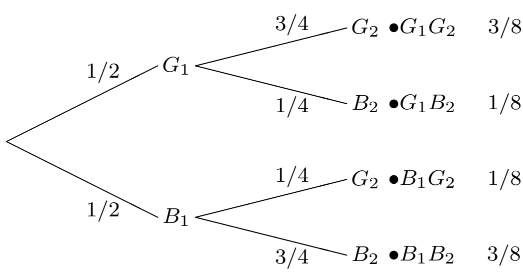
This implies

$$P[H_1|H_2] = \frac{P[H_1H_2]}{P[H_2]} = \frac{1/16}{1/4} = 1/4. \tag{2}$$

(b) The probability that the first flip is heads and the second flip is tails is $P[H_1T_2] = 3/16$.

Problem 2.1.3 Solution

Let G_i and B_i denote events indicating whether free throw i was good (G_i) or bad (B_i). The tree for the free throw experiment is



The game goes into overtime if exactly one free throw is made. This event has probability

$$P[O] = P[G_1B_2] + P[B_1G_2] = 1/8 + 1/8 = 1/4. \tag{1}$$

Problem 2.1.5 Solution

The $P[-|H]$ is the probability that a person who has HIV tests negative for the disease. This is referred to as a false-negative result. The case where a person who does not have HIV but tests positive for the disease, is called a false-positive result and has probability $P[+|H^c]$. Since the test is correct 99% of the time,

$$P[-|H] = P[+|H^c] = 0.01. \tag{1}$$

Now the probability that a person who has tested positive for HIV actually has the disease is

$$P[H|+] = \frac{P[+,H]}{P[+]} = \frac{P[+,H]}{P[+,H] + P[+,H^c]}. \tag{2}$$

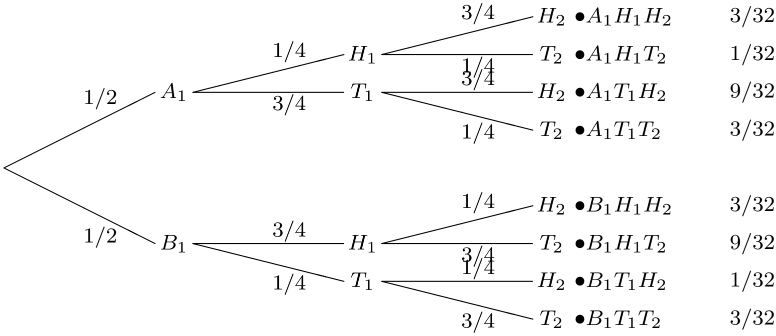
We can use Bayes' formula to evaluate these joint probabilities.

$$\begin{aligned} P[H|+] &= \frac{P[+|H]\,P[H]}{P[+|H]\,P[H] + P[+|H^c]\,P[H^c]} \\ &= \frac{(0.99)(0.0002)}{(0.99)(0.0002) + (0.01)(0.9998)} \\ &= 0.0194. \end{aligned} \tag{3}$$

Thus, even though the test is correct 99% of the time, the probability that a random person who tests positive actually has HIV is less than 0.02. The reason this probability is so low is that the a priori probability that a person has HIV is very small.

Problem 2.1.7 Solution

The tree for this experiment is



The event H_1H_2 that heads occurs on both flips has probability

$$P[H_1H_2] = P[A_1H_1H_2] + P[B_1H_1H_2] = 6/32. \tag{1}$$

The probability of H_1 is

$$\begin{aligned} P[H_1] &= P[A_1H_1H_2] + P[A_1H_1T_2] + P[B_1H_1H_2] + P[B_1H_1T_2] \\ &= 1/2. \end{aligned} \tag{2}$$

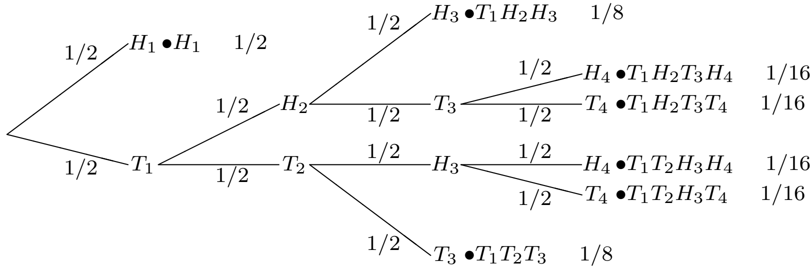
Similarly,

$$\begin{aligned} P[H_2] &= P[A_1H_1H_2] + P[A_1T_1H_2] + P[B_1H_1H_2] + P[B_1T_1H_2] \\ &= 1/2. \end{aligned} \tag{3}$$

Thus $P[H_1H_2] \neq P[H_1]\,P[H_2]$, implying H_1 and H_2 are not independent. This result should not be surprising since if the first flip is heads, it is likely that coin B was picked first. In this case, the second flip is less likely to be heads since it becomes more likely that the second coin flipped was coin A.

Problem 2.1.9 Solution

(a) The primary difficulty in this problem is translating the words into the correct tree diagram. The tree for this problem is shown below.



(b) From the tree,

$$\begin{aligned} P[H_3] &= P[T_1H_2H_3] + P[T_1T_2H_3H_4] + P[T_1T_2H_3H_4] \\ &= 1/8 + 1/16 + 1/16 = 1/4. \end{aligned} \tag{1}$$

Similarly,

$$\begin{aligned} P[T_3] &= P[T_1H_2T_3H_4] + P[T_1H_2T_3T_4] + P[T_1T_2T_3] \\ &= 1/8 + 1/16 + 1/16 = 1/4. \end{aligned} \tag{2}$$

(c) The event that Dagwood must diet is

$$D = (T_1H_2T_3T_4) \cup (T_1T_2H_3T_4) \cup (T_1T_2T_3). \tag{3}$$

The probability that Dagwood must diet is

$$\begin{aligned} P[D] &= P[T_1H_2T_3T_4] + P[T_1T_2H_3T_4] + P[T_1T_2T_3] \\ &= 1/16 + 1/16 + 1/8 = 1/4. \end{aligned} \tag{4}$$

The conditional probability of heads on flip 1 given that Dagwood must diet is

$$P[H_1|D] = \frac{P[H_1D]}{P[D]} = 0. \tag{5}$$

Remember, if there was heads on flip 1, then Dagwood always postpones his diet.

(d) From part (b), we found that $P[H_3] = 1/4$. To check independence, we calculate

$$\begin{aligned} P[H_2] &= P[T_1H_2H_3] + P[T_1H_2T_3] + P[T_1H_2T_4T_4] = 1/4 \\ P[H_2H_3] &= P[T_1H_2H_3] = 1/8. \end{aligned} \tag{6}$$

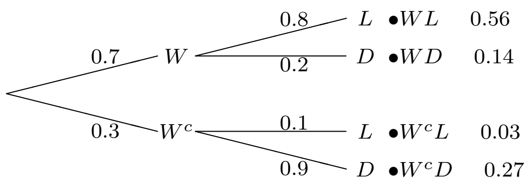
Now we find that

$$P[H_2H_3] = 1/8 \neq P[H_2]P[H_3]. \tag{7}$$

Hence, H_2 and H_3 are dependent events. In fact, $P[H_3|H_2] = 1/2$ while $P[H_3] = 1/4$. The reason for the dependence is that given H_2 occurred, then we know there will be a third flip which may result in H_3. That is, knowledge of H_2 tells us that the experiment didn't end after the first flip.

Problem 2.1.11 Solution

The starting point is to draw a tree of the experiment. We define the events W that the plant is watered, L that the plant lives, and D that the plant dies. The tree diagram is



It follows that

(a) $P[L] = P[WL] + P[W^cL] = 0.56 + 0.03 = 0.59$.

(b)

$$P[W^c|D] = \frac{P[W^cD]}{P[D]} = \frac{0.27}{0.14 + 0.27} = \frac{27}{41}. \tag{1}$$

(c) $P[D|W^c] = 0.9$.

In informal conversation, it can be confusing to distinguish between $P[D|W^c]$ and $P[W^c|D]$; however, they are simple once you draw the tree.

Problem 2.2.1 Solution

Technically, a gumball machine has a finite number of gumballs, but the problem description models the drawing of gumballs as sampling from the machine without replacement. This is a reasonable model when the machine has a very large gumball capacity and we have no knowledge beforehand of how many gumballs of each color are in the machine. Under this model, the requested probability is given by the multinomial probability

$$\begin{aligned} P[R_2 Y_2 G_2 B_2] &= \frac{8!}{2!\,2!\,2!\,2!} \left(\frac{1}{4}\right)^2 \left(\frac{1}{4}\right)^2 \left(\frac{1}{4}\right)^2 \left(\frac{1}{4}\right)^2 \\ &= \frac{8!}{4^{10}} \approx 0.0385. \end{aligned} \tag{1}$$

Problem 2.2.3 Solution

(a) Let B_i, L_i, O_i and C_i denote the events that the ith piece is Berry, Lemon, Orange, and Cherry respectively. Let F_1 denote the event that all three pieces draw are the same flavor. Thus,

$$F_1 = \{S_1S_2S_3, L_1L_2L_3, O_1O_2O_3, C_1C_2C_3\} \tag{1}$$

$$P[F_1] = P[S_1S_2S_3] + P[L_1L_2L_3] + P[O_1O_2O_3] + P[C_1C_2C_3] \tag{2}$$

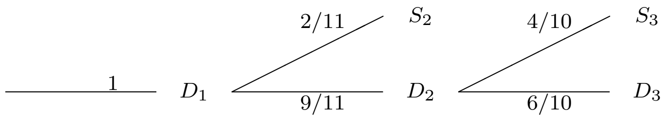
Note that

$$P[L_1L_2L_3] = \frac{3}{12} \cdot \frac{2}{11} \cdot \frac{1}{10} = \frac{1}{220} \tag{3}$$

and by symmetry,

$$P[F_1] = 4\,P[L_1L_2L_3] = \frac{1}{55}. \tag{4}$$

(b) Let D_i denote the event that the ith piece is a different flavor from all the prior pieces. Let S_i denote the event that piece i is the same flavor as a previous piece. A tree for this experiment is



Note that:

- $P[D_1] = 1$ because the first piece is "different" since there haven't been any prior pieces.
- The second piece is the same as the first piece with probability 2/11 because in the remaining 11 pieces there are 2 pieces that are the same as the first piece. Alternatively, out of 11 pieces left, there are 3 colors each with 3 pieces (that is, 9 pieces out of 11) that are different from the first piece.
- Given the first two pieces are different, there are 2 colors, each with 3 pieces (6 pieces) out of 10 remaining pieces that are a different flavor from the first two pieces. Thus $P[D_3|D_2D_1] = 6/10$.

It follows that the three pieces are different with probability

$$P[D_1D_2D_3] = 1\left(\frac{9}{11}\right)\left(\frac{6}{10}\right) = \frac{27}{55}. \quad (5)$$

Problem 2.2.5 Solution

Since there are $H = \binom{52}{7}$ equiprobable seven-card hands, each probability is just the number of hands of each type divided by H.

(a) Since there are 26 red cards, there are $\binom{26}{7}$ seven-card hands with all red cards. The probability of a seven-card hand of all red cards is

$$P[R_7] = \frac{\binom{26}{7}}{\binom{52}{7}} = \frac{26!\,45!}{52!\,19!} = 0.0049. \quad (1)$$

(b) There are 12 face cards in a 52 card deck and there are $\binom{12}{7}$ seven card hands with all face cards. The probability of drawing only face cards is

$$\mathrm{P}[F] = \frac{\binom{12}{7}}{\binom{52}{7}} = \frac{12!\,45!}{5!\,52!} = 5.92 \times 10^{-6}. \tag{2}$$

(c) There are 6 red face cards (J, Q, K of diamonds and hearts) in a 52 card deck. Thus it is impossible to get a seven-card hand of red face cards: $\mathrm{P}[R_7F] = 0$.

Problem 2.2.7 Solution

There are $2^5 = 32$ different binary codes with 5 bits. The number of codes with exactly 3 zeros equals the number of ways of choosing the bits in which those zeros occur. Therefore there are $\binom{5}{3} = 10$ codes with exactly 3 zeros.

Problem 2.2.9 Solution

We can break down the experiment of choosing a starting lineup into a sequence of subexperiments:

1. Choose 1 of the 10 pitchers. There are $N_1 = \binom{10}{1} = 10$ ways to do this.
2. Choose 1 of the 15 field players to be the designated hitter (DH). There are $N_2 = \binom{15}{1} = 15$ ways to do this.
3. Of the remaining 14 field players, choose 8 for the remaining field positions. There are $N_3 = \binom{14}{8}$ to do this.
4. For the 9 batters (consisting of the 8 field players and the designated hitter), choose a batting lineup. There are $N_4 = 9!$ ways to do this.

So the total number of different starting lineups when the DH is selected among the field players is

$$N = N_1 N_2 N_3 N_4 = (10)(15)\binom{14}{8} 9! = 163{,}459{,}296{,}000. \tag{1}$$

Note that this overestimates the number of combinations the manager must really consider because most field players can play only one or two positions. Although these constraints on the manager reduce the number of possible lineups, it typically makes the manager's job more difficult. As for the counting, we note that our count did not need to specify the positions played by the field players. Although this is an important consideration for the manager, it is not part of our counting of different lineups. In fact, the 8 nonpitching field players are allowed to switch positions at any time in the field. For example, the shortstop and second baseman could trade positions in the middle of an inning. Although the DH can go play the field, there are some coomplicated rules about this. Here is an excerpt from Major League Baseball Rule 6.10:

> The Designated Hitter may be used defensively, continuing to bat in the same position in the batting order, but the pitcher must then bat in the place of the substituted defensive player, unless more than one substitution is made, and the manager then must designate their spots in the batting order.

If you're curious, you can find the complete rule on the web.

Problem 2.2.11 Solution

(a) This is just the multinomial probability

$$P[A] = \binom{40}{19, 19, 2} \left(\frac{19}{40}\right)^{19} \left(\frac{19}{40}\right)^{19} \left(\frac{2}{40}\right)^{2} = \frac{40!}{19!19!2!} \left(\frac{19}{40}\right)^{19} \left(\frac{19}{40}\right)^{19} \left(\frac{2}{40}\right)^{2}. \tag{1}$$

(b) Each spin is either green (with probability 19/40) or not (with probability 21/40). If we call landing on greeen a success, then G_{19} is the probability of 19 successes in 40 trials. Thus

$$P[G_{19}] = \binom{40}{19} \left(\frac{19}{40}\right)^{19} \left(\frac{21}{40}\right)^{21}. \tag{2}$$

(c) If you bet on red, the probability you win is 19/40. If you bet green, the probability that you win is 19/40. If you first make a random choice to bet red or green, (say by flipping a coin), the probability you win is still $p = 19/40$.

Problem 2.2.13 Solution

What our design must specify is the number of boxes on the ticket, and the number of specially marked boxes. Suppose each ticket has n boxes and $5+k$ specially marked boxes. Note that when $k > 0$, a winning ticket will still have k unscratched boxes with the special mark. A ticket is a winner if each time a box is scratched off, the box has the special mark. Assuming the boxes are scratched off randomly, the first box scratched off has the mark with probability $(5+k)/n$ since there are $5+k$ marked boxes out of n boxes. Moreover, if the first scratched box has the mark, then there are $4+k$ marked boxes out of $n-1$ remaining boxes. Continuing this argument, the probability that a ticket is a winner is

$$p = \frac{5+k}{n}\frac{4+k}{n-1}\frac{3+k}{n-2}\frac{2+k}{n-3}\frac{1+k}{n-4} = \frac{(k+5)!(n-5)!}{k!n!}. \qquad (1)$$

By careful choice of n and k, we can choose p close to 0.01. For example,

n	9	11	14	17
k	0	1	2	3
p	0.0079	0.012	0.0105	0.0090

(2)

A gamecard with $N = 14$ boxes and $5 + k = 7$ shaded boxes would be quite reasonable.

Problem 2.3.1 Solution

(a) Since the probability of a zero is 0.8, we can express the probability of the code word 00111 as 2 occurrences of a 0 and three occurrences of a 1. Therefore

$$P[00111] = (0.8)^2(0.2)^3 = 0.00512. \qquad (1)$$

(b) The probability that a code word has exactly three 1's is

$$P[\text{three 1's}] = \binom{5}{3}(0.8)^2(0.2)^3 = 0.0512. \tag{2}$$

Problem 2.3.3 Solution

We know that the probability of a green and red light is 7/16, and that of a yellow light is 1/8. Since there are always 5 lights, G, Y, and R obey the multinomial probability law:

$$P[G = 2, Y = 1, R = 2] = \frac{5!}{2!1!2!}\left(\frac{7}{16}\right)^2\left(\frac{1}{8}\right)\left(\frac{7}{16}\right)^2. \tag{1}$$

The probability that the number of green lights equals the number of red lights

$$\begin{aligned} P[G = R] &= P[G = 1, R = 1, Y = 3] + P[G = 2, R = 2, Y = 1] \\ &\quad + P[G = 0, R = 0, Y = 5] \\ &= \frac{5!}{1!1!3!}\left(\frac{7}{16}\right)\left(\frac{7}{16}\right)\left(\frac{1}{8}\right)^3 + \frac{5!}{2!1!2!}\left(\frac{7}{16}\right)^2\left(\frac{7}{16}\right)^2\left(\frac{1}{8}\right) \\ &\quad + \frac{5!}{0!0!5!}\left(\frac{1}{8}\right)^5 \\ &\approx 0.1449. \end{aligned} \tag{2}$$

Problem 2.3.5 Solution

(a) There are 3 group 1 kickers and 6 group 2 kickers. Using G_i to denote that a group i kicker was chosen, we have

$$P[G_1] = 1/3, \qquad P[G_2] = 2/3. \tag{1}$$

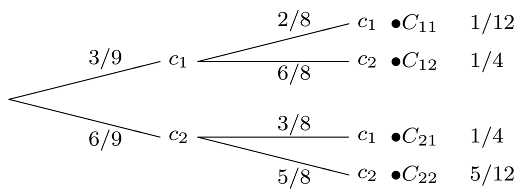
In addition, the problem statement tells us that

$$\mathrm{P}[K|G_1] = 1/2, \qquad \mathrm{P}[K|G_2] = 1/3. \tag{2}$$

Combining these facts using the Law of Total Probability yields

$$\begin{aligned}\mathrm{P}[K] &= \mathrm{P}[K|G_1]\,\mathrm{P}[G_1] + \mathrm{P}[K|G_2]\,\mathrm{P}[G_2] \\ &= (1/2)(1/3) + (1/3)(2/3) = 7/18.\end{aligned} \tag{3}$$

(b) To solve this part, we need to identify the groups from which the first and second kicker were chosen. Let c_i indicate whether a kicker was chosen from group i and let C_{ij} indicate that the first kicker was chosen from group i and the second kicker from group j. The experiment to choose the kickers is described by the sample tree:



Since a kicker from group 1 makes a kick with probability 1/2 while a kicker from group 2 makes a kick with probability 1/3,

$$\mathrm{P}[K_1K_2|C_{11}] = (1/2)^2, \qquad \mathrm{P}[K_1K_2|C_{12}] = (1/2)(1/3), \tag{4}$$

$$\mathrm{P}[K_1K_2|C_{21}] = (1/3)(1/2), \qquad \mathrm{P}[K_1K_2|C_{22}] = (1/3)^2. \tag{5}$$

By the law of total probability,

$$\begin{aligned}\mathrm{P}[K_1K_2] &= \mathrm{P}[K_1K_2|C_{11}]\,\mathrm{P}[C_{11}] + \mathrm{P}[K_1K_2|C_{12}]\,\mathrm{P}[C_{12}] \\ &\quad + \mathrm{P}[K_1K_2|C_{21}]\,\mathrm{P}[C_{21}] + \mathrm{P}[K_1K_2|C_{22}]\,\mathrm{P}[C_{22}] \\ &= \frac{1}{4}\frac{1}{12} + \frac{1}{6}\frac{1}{4} + \frac{1}{6}\frac{1}{4} + \frac{1}{9}\frac{5}{12} = 15/96.\end{aligned} \tag{6}$$

It should be apparent that $\mathrm{P}[K_1] = \mathrm{P}[K]$ from part (a). Symmetry should also make it clear that $\mathrm{P}[K_1] = \mathrm{P}[K_2]$ since for any ordering of two kickers, the reverse ordering is equally likely. If this is not clear, we derive this result by calculating $\mathrm{P}[K_2|C_{ij}]$ and using the law of total probability to calculate $\mathrm{P}[K_2]$.

$$\mathrm{P}[K_2|C_{11}] = 1/2, \qquad \mathrm{P}[K_2|C_{12}] = 1/3, \tag{7}$$
$$\mathrm{P}[K_2|C_{21}] = 1/2, \qquad \mathrm{P}[K_2|C_{22}] = 1/3. \tag{8}$$

By the law of total probability,

$$\begin{aligned}\mathrm{P}[K_2] &= \mathrm{P}[K_2|C_{11}]\,\mathrm{P}[C_{11}] + \mathrm{P}[K_2|C_{12}]\,\mathrm{P}[C_{12}] \\ &\quad + \mathrm{P}[K_2|C_{21}]\,\mathrm{P}[C_{21}] + \mathrm{P}[K_2|C_{22}]\,\mathrm{P}[C_{22}] \\ &= \frac{1}{2}\frac{1}{12} + \frac{1}{3}\frac{1}{4} + \frac{1}{2}\frac{1}{4} + \frac{1}{3}\frac{5}{12} = \frac{7}{18}. \end{aligned} \tag{9}$$

We observe that K_1 and K_2 are not independent since

$$\mathrm{P}[K_1 K_2] = \frac{15}{96} \neq \left(\frac{7}{18}\right)^2 = \mathrm{P}[K_1]\,\mathrm{P}[K_2]. \tag{10}$$

Note that $15/96$ and $(7/18)^2$ are close but not exactly the same. The reason K_1 and K_2 are dependent is that if the first kicker is successful, then it is more likely that kicker is from group 1. This makes it more likely that the second kicker is from group 2 and is thus more likely to miss.

(c) Once a kicker is chosen, each of the 10 field goals is an independent trial. If the kicker is from group 1, then the success probability is 1/2. If the kicker is from group 2, the success probability is 1/3. Out of 10 kicks, there are 5 misses iff there are 5 successful kicks. Given the type of kicker chosen, the probability of 5 misses is

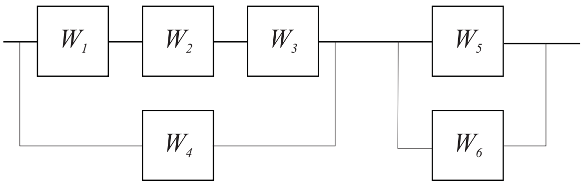
$$\mathrm{P}[M|G_1] = \binom{10}{5}(1/2)^5(1/2)^5, \quad \mathrm{P}[M|G_2] = \binom{10}{5}(1/3)^5(2/3)^5. \tag{11}$$

We use the Law of Total Probability to find

$$\begin{aligned} P[M] &= P[M|G_1]\,P[G_1] + P[M|G_2]\,P[G_2] \\ &= \binom{10}{5}\left((1/3)(1/2)^{10} + (2/3)(1/3)^5(2/3)^5\right). \qquad (12) \end{aligned}$$

Problem 2.4.1 Solution

From the problem statement, we can conclude that the device components are configured in the following way.

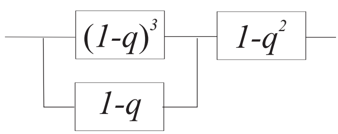


To find the probability that the device works, we replace series devices 1, 2, and 3, and parallel devices 5 and 6 each with a single device labeled with the probability that it works. In particular,

$$P[W_1W_2W_3] = (1-q)^3, \qquad (1)$$

$$P[W_5 \cup W_6] = 1 - P[W_5^cW_6^c] = 1 - q^2. \qquad (2)$$

This yields a composite device of the form



The probability $P[W']$ that the two devices in parallel work is 1 minus the probability that neither works:

$$P[W'] = 1 - q(1-(1-q)^3). \qquad (3)$$

Finally, for the device to work, both composite device in series must work. Thus, the probability the device works is

$$P[W] = [1 - q(1-(1-q)^3)][1-q^2]. \qquad (4)$$

Problem 2.4.3 Solution

Note that each digit 0 through 9 is mapped to the 4 bit binary representation of the digit. That is, 0 corresponds to 0000, 1 to 0001, up to 9 which corresponds to 1001. Of course, the 4 bit binary numbers corresponding to numbers 10 through 15 go unused, however this is unimportant to our problem. the 10 digit number results in the transmission of 40 bits. For each bit, an independent trial determines whether the bit was correct, a deletion, or an error. In Problem 2.4.2, we found the probabilities of these events to be

$$\mathrm{P}[C] = \gamma = 0.91854, \qquad \mathrm{P}[D] = \delta = 0.081, \qquad \mathrm{P}[E] = \epsilon = 0.00046. \quad (1)$$

Since each of the 40 bit transmissions is an independent trial, the joint probability of c correct bits, d deletions, and e erasures has the multinomial probability

$$\mathrm{P}[C = c, D = d, E = d] = \begin{cases} \frac{40!}{c!d!e!}\gamma^c\delta^d\epsilon^e & c + d + e = 40; c, d, e \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

Problem 2.5.1 Solution

Rather than just solve the problem for 50 trials, we can write a function that generates vectors **C** and **H** for an arbitrary number of trials n. The code for this task is

```
function [C,H]=twocoin(n);
C=ceil(2*rand(n,1));
P=1-(C/4);
H=(rand(n,1)< P);
```

The first line produces the $n \times 1$ vector `C` such that `C(i)` indicates whether coin 1 or coin 2 is chosen for trial i. Next, we generate the vector `P` such that `P(i)=0.75` if `C(i)=1`; otherwise, if `C(i)=2`, then `P(i)=0.5`. As a result, `H(i)` is the simulated result of a coin flip with heads, corresponding to `H(i)=1`, occurring with probability `P(i)`.

Problem 2.5.3 Solution

To test n 6-component devices, (such that each component works with probability q) we use the following function:

```
function N=reliable6(n,q);
% n is the number of 6 component devices
%N is the number of working devices
W=rand(n,6)>q;
D=(W(:,1)&W(:,2)&W(:,3))|W(:,4);
D=D&(W(:,5)|W(:,6));
N=sum(D);
```

The $n \times 6$ matrix `W` is a *logical* matrix such that `W(i,j)=1` if component `j` of device `i` works properly. Because `W` is a logical matrix, we can use the MATLAB logical operators `|` and `&` to implement the logic requirements for a working device. By applying these logical operators to the $n \times 1$ columns of `W`, we simulate the test of `n` circuits. Note that `D(i)=1` if device `i` works. Otherwise, `D(i)=0`. Lastly, we count the number `N` of working devices. The following code snippet produces ten sample runs, where each sample run tests `n=100` devices for $q = 0.2$.

```
>> for n=1:10, w(n)=reliable6(100,0.2); end
>> w
w =
    82   87   87   92   91   85   85   83   90   89
>>
```

As we see, the number of working devices is typically around 85 out of 100. Solving Problem 2.4.1, will show that the probability the device works is actually 0.8663.

Problem 2.5.5 Solution

For arbitrary number of trials n and failure probability q, the following functions evaluates replacing each of the six components by an ultrareliable device.

```
function N=ultrareliable6(n,q);
% n is the number of 6 component devices
%N is the number of working devices
for r=1:6,
    W=rand(n,6)>q;
    R=rand(n,1)>(q/2);
    W(:,r)=R;
    D=(W(:,1)&W(:,2)&W(:,3))|W(:,4);
    D=D&(W(:,5)|W(:,6));
    N(r)=sum(D);
end
```

This code is based on the code for the solution of Problem 2.5.3. The $n \times 6$ matrix `W` is a *logical* matrix such that `W(i,j)=1` if component `j` of device `i` works properly. Because `W` is a logical matrix, we can use the MATLAB logical operators `|` and `&` to implement the logic requirements for a working device. By applying these logical opeators to the $n \times 1$ columns of `W`, we simulate the test of `n` circuits. Note that `D(i)=1` if device `i` works. Otherwise, `D(i)=0`. Note that in the code, we first generate the matrix `W` such that each component has failure probability q. To simulate the replacement of the jth device by the ultrareliable version by replacing the jth column of `W` by the column vector `R` in which a device has failure probability $q/2$. Lastly, for each column replacement, we count the number `N` of working devices. A sample run for $n = 100$ trials and $q = 0.2$ yielded these results:

```
>> ultrareliable6(100,0.2)
ans =
    93    89    91    92    90    93
```

From the above, we see, for example, that replacing the third component with an ultrareliable component resulted in 91 working devices. The results are fairly inconclusive in that replacing devices 1, 2, or 3 should yield the same probability of device failure. If we experiment with $n = 10,000$ runs, the results are more definitive:

```
>> ultrareliable6(10000,0.2)
ans =
        8738      8762      8806      9135      8800      8796
>> >> ultrareliable6(10000,0.2)
ans =
        8771      8795      8806      9178      8886      8875
>>
```

In both cases, it is clear that replacing component 4 maximizes the device reliability. The somewhat complicated solution of Problem 2.4.4 will confirm this observation.

Problem Solutions – Chapter 3

Problem 3.2.1 Solution

(a) We wish to find the value of c that makes the PMF sum up to one.

$$P_N(n) = \begin{cases} c(1/2)^n & n = 0, 1, 2, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Therefore, $\sum_{n=0}^{2} P_N(n) = c + c/2 + c/4 = 1$, implying $c = 4/7$.

(b) The probability that $N \leq 1$ is

$$P[N \leq 1] = P[N = 0] + P[N = 1] = 4/7 + 2/7 = 6/7. \tag{2}$$

Problem 3.2.3 Solution

(a) We choose c so that the PMF sums to one.

$$\sum_x P_X(x) = \frac{c}{2} + \frac{c}{4} + \frac{c}{8} = \frac{7c}{8} = 1. \tag{1}$$

Thus $c = 8/7$.

(b)

$$P[X = 4] = P_X(4) = \frac{8}{7 \cdot 4} = \frac{2}{7}. \tag{2}$$

(c)

$$P[X < 4] = P_X(2) = \frac{8}{7 \cdot 2} = \frac{4}{7}. \tag{3}$$

(d)

$$P[3 \leq X \leq 9] = P_X(4) + P_X(8) = \frac{8}{7 \cdot 4} + \frac{8}{7 \cdot 8} = \frac{3}{7}. \tag{4}$$

Problem 3.2.5 Solution

(a) To find c, we apply the constraint $\sum_n P_N(n) = 1$, yielding

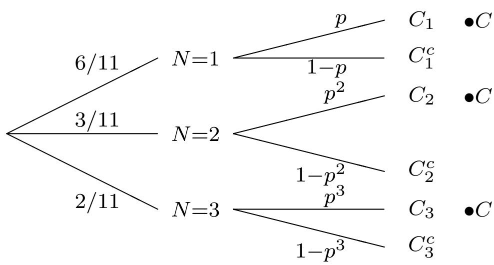
$$1 = \sum_{n=1}^{3} \frac{c}{n} = c\left(1 + \frac{1}{2} + \frac{1}{3}\right) = c\left(\frac{11}{6}\right). \tag{1}$$

Thus $c = 6/11$.

(b) The probability that N is odd is

$$P[N \text{ is odd}] = P_N(1) + P_N(3) = c\left(1 + \frac{1}{3}\right) = c\left(\frac{4}{3}\right) = \frac{24}{33}. \tag{2}$$

(c) We can view this as a sequential experiment: first we divide the file into N packets and then we check that all N packets are received correctly. In the second stage, we could specify how many packets are received correctly; however, it is sufficient to just specify whether the N packets are all received correctly or not. Using C_n to denote the event that n packets are transmitted and received correctly, we have

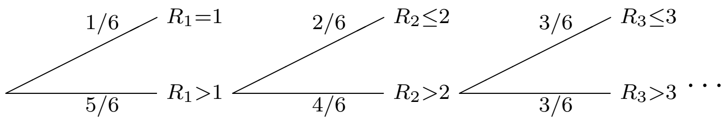


We see that

$$\begin{aligned} P[C] &= P[C_1] + P[C_2] + P[C_3] \\ &= \frac{6p}{11} + \frac{3p^2}{11} + \frac{2p^3}{11} = \frac{p(6 + 3p + 2p^2)}{11}. \end{aligned} \tag{3}$$

Problem 3.2.7 Solution

Note that $N > 3$ if we roll three rolls satisfying $R_1 > 1$, $R_2 > 2$ and $R_3 > 3$. The tree for this experiment is



We note that

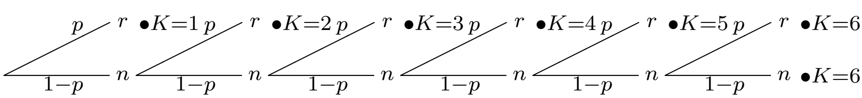
$$\mathrm{P}\left[N > 3\right] = \mathrm{P}\left[R_1 > 1, R_2 > 2, R_3 > 3\right] = \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} = \frac{5}{18}. \tag{1}$$

Problem 3.2.9 Solution

In Problem 3.2.8, each caller is willing to make 3 attempts to get through. An attempt is a failure if all n operators are busy, which occurs with probability $q = (0.8)^n$. Assuming call attempts are independent, a caller will suffer three failed attempts with probability $q^3 = (0.8)^{3n}$. The problem statement requires that $(0.8)^{3n} \le 0.05$. This implies $n \ge 4.48$ and so we need 5 operators.

Problem 3.2.11 Solution

(a) In the setup of a mobile call, the phone will send the "SETUP" message up to six times. Each time the setup message is sent, we have a Bernoulli trial with success probability p. Of course, the phone stops trying as soon as there is a success. Using r to denote a successful response, and n a non-response, the sample tree is



(b) We can write the PMF of K, the number of "SETUP" messages sent as

$$P_K(k) = \begin{cases} (1-p)^{k-1}p & k = 1, 2, \ldots, 5, \\ (1-p)^5 p + (1-p)^6 = (1-p)^5 & k = 6, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Note that the expression for $P_K(6)$ is different because $K = 6$ if either there was a success or a failure on the sixth attempt. In fact, $K = 6$ whenever there were failures on the first five attempts which is why $P_K(6)$ simplifies to $(1-p)^5$.

(c) Let B denote the event that a busy signal is given after six failed setup attempts. The probability of six consecutive failures is $\mathrm{P}[B] = (1-p)^6$.

(d) To be sure that $\mathrm{P}[B] \leq 0.02$, we need $p \geq 1 - (0.02)^{1/6} = 0.479$.

Problem 3.3.1 Solution

(a) If it is indeed true that Y, the number of yellow M&M's in a package, is uniformly distributed between 5 and 15, then the PMF of Y, is

$$P_Y(y) = \begin{cases} 1/11 & y = 5, 6, 7, \ldots, 15 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

(b)

$$\mathrm{P}[Y < 10] = P_Y(5) + P_Y(6) + \cdots + P_Y(9) = 5/11. \quad (2)$$

(c)

$$\mathrm{P}[Y > 12] = P_Y(13) + P_Y(14) + P_Y(15) = 3/11. \quad (3)$$

(d)

$$\mathrm{P}[8 \leq Y \leq 12] = P_Y(8) + P_Y(9) + \cdots + P_Y(12) = 5/11. \quad (4)$$

Problem 3.3.3 Solution

(a) Each paging attempt is an independent Bernoulli trial with success probability p. The number of times K that the pager receives a message is the number of successes in n Bernoulli trials and has the binomial PMF

$$P_K(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & k = 0, 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(b) Let R denote the event that the paging message was received at least once. The event R has probability

$$\mathrm{P}[R] = \mathrm{P}[B > 0] = 1 - \mathrm{P}[B = 0] = 1 - (1-p)^n. \tag{2}$$

To ensure that $\mathrm{P}[R] \geq 0.95$ requires that $n \geq \ln(0.05)/\ln(1-p)$. For $p = 0.8$, we must have $n \geq 1.86$. Thus, $n = 2$ pages would be necessary.

Problem 3.3.5 Solution

Whether a hook catches a fish is an independent trial with success probability h. The the number of fish hooked, K, has the binomial PMF

$$P_K(k) = \begin{cases} \binom{m}{k} h^k (1-h)^{m-k} & k = 0, 1, \ldots, m, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Problem 3.3.7 Solution

Each paging attempt is a Bernoulli trial with success probability p where a success occurs if the pager receives the paging message.

(a) The paging message is sent again and again until a success occurs. Hence the number of paging messages is $N = n$ if there are $n-1$ paging failures followed by a paging success. That is, N has the geometric PMF

$$P_N(n) = \begin{cases} (1-p)^{n-1} p & n = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(b) The probability that no more three paging attempts are required is

$$P[N \leq 3] = 1 - P[N > 3] = 1 - \sum_{n=4}^{\infty} P_N(n) = 1 - (1-p)^3. \quad (2)$$

This answer can be obtained without calculation since $N > 3$ if the first three paging attempts fail and that event occurs with probability $(1-p)^3$. Hence, we must choose p to satisfy $1 - (1-p)^3 \geq 0.95$ or $(1-p)^3 \leq 0.05$. This implies

$$p \geq 1 - (0.05)^{1/3} \approx 0.6316. \quad (3)$$

Problem 3.3.9 Solution

(a) K is a Pascal $(5, p = 0.1)$ random variable and has PMF

$$P_K(k) = \binom{k-1}{4} p^5 (1-p)^{k-5} = \binom{k-1}{4} (0.1)^5 (0.9)^{k-5}. \quad (1)$$

(b) L is a Pascal $(k = 33, p = 1/2)$ random variable and so its PMF is

$$P_L(l) = \binom{l-1}{32} p^{33} (1-p)^{l-33} = \binom{l-1}{32} \left(\frac{1}{2}\right)^l. \quad (2)$$

(c) M is a geometric $(p = 0.01)$ random variable, You should know that that M has PMF

$$P_M(m) = \begin{cases} (1-p)^{m-1} p & m = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

Problem 3.3.11 Solution

(a) If each message is transmitted 8 times and the probability of a successful transmission is p, then the PMF of N, the number of successful transmissions has the binomial PMF

$$P_N(n) = \begin{cases} \binom{8}{n} p^n (1-p)^{8-n} & n = 0, 1, \ldots, 8, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

(b) The indicator random variable I equals zero if and only if $N = 8$. Hence,

$$\mathrm{P}[I = 0] = \mathrm{P}[N = 0] = 1 - \mathrm{P}[I = 1] \quad (2)$$

Thus, the complete expression for the PMF of I is

$$P_I(i) = \begin{cases} (1-p)^8 & i = 0, \\ 1 - (1-p)^8 & i = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

Problem 3.3.13 Solution

(a) Each of the four m&m's is equally likely to be red or green. Hence the number of red m&m's is a binomial ($n = 4, p = 1/2$) random variable N with PMF

$$P_N(n) = \binom{4}{n} (1/2)^n (1/2)^{4-n} = \binom{4}{n} \left(\frac{1}{16}\right). \quad (1)$$

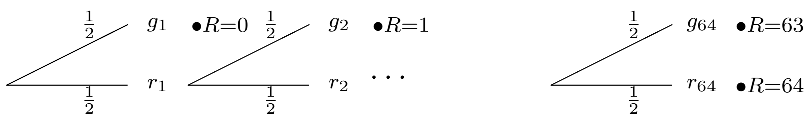
The probability of an equal number of red and green is

$$\mathrm{P}[E] = \mathrm{P}[N = 2] = P_N(2) = \binom{4}{2} \left(\frac{1}{16}\right) = \frac{3}{8}. \quad (2)$$

(b) In the bag of 64 m&m's, each m&m is green with probability 1/2 so that G is a binomial ($n = 64, p = 1/2$) random variable with PMF

$$P_G(g) = \binom{64}{g} (1/2)^g (1/2)^{64-g} = \binom{64}{g} 2^{-64}. \tag{3}$$

(c) This is similar to the number of geometric number number of trials needed for the first success, except things are a little trickier because the bag may have all red m&m's. To be more clear, we will use r_i and g_i to denote the color of the ith m&m eaten. The tree is



From the tree, we see that

$$P[R = 0] = 2^{-1}, \quad P[R = 1] = 2^{-2}, \cdots \quad P[R = 63] = 2^{-64}, \tag{4}$$

and $P[R = 64] = 2^{-64}$. The complete PMF of R is

$$P_R(r) = \begin{cases} (1/2)^{r+1} & r = 0, 1, 2, \ldots, 63, \\ (1/2)^{64} & r = 64, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Problem 3.3.15 Solution

The packets are delay sensitive and can only be retransmitted d times. For $t < d$, a packet is transmitted t times if the first $t - 1$ attempts fail followed by a successful transmission on attempt t. Further, the packet is transmitted d times if there are failures on the first $d - 1$ transmissions, no matter what the outcome of attempt d. So the random variable T, the number of times that a packet is transmitted, can be represented by the following PMF.

$$P_T(t) = \begin{cases} p(1-p)^{t-1} & t = 1, 2, \ldots, d-1, \\ (1-p)^{d-1} & t = d, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Problem 3.3.17 Solution

(a) Since each day is independent of any other day, $\mathrm{P}[W_{33}]$ is just the probability that a winning lottery ticket was bought. Similarly for $\mathrm{P}[L_{87}]$ and $\mathrm{P}[N_{99}]$ become just the probability that a losing ticket was bought and that no ticket was bought on a single day, respectively. Therefore

$$\mathrm{P}[W_{33}] = p/2, \qquad \mathrm{P}[L_{87}] = (1-p)/2, \qquad \mathrm{P}[N_{99}] = 1/2. \tag{1}$$

(b) Suppose we say a success occurs on the kth trial if on day k we buy a ticket. Otherwise, a failure occurs. The probability of success is simply $1/2$. The random variable K is just the number of trials until the first success and has the geometric PMF

$$P_K(k) = \begin{cases} (1/2)(1/2)^{k-1} = (1/2)^k & k = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(c) The probability that you decide to buy a ticket and it is a losing ticket is $(1-p)/2$, independent of any other day. If we view buying a losing ticket as a Bernoulli success, R, the number of losing lottery tickets bought in m days, has the binomial PMF

$$P_R(r) = \begin{cases} \binom{m}{r}[(1-p)/2]^r[(1+p)/2]^{m-r} & r = 0, 1, \ldots, m, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

(d) Letting D be the day on which the j-th losing ticket is bought, we can find the probability that $D = d$ by noting that $j-1$ losing tickets must have been purchased in the $d-1$ previous days. Therefore D has the Pascal PMF

$$P_D(d) = \begin{cases} \binom{d-1}{j-1}[(1-p)/2]^j[(1+p)/2]^{d-j} & d = j, j+1, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Problem 3.3.19 Solution

Since a and b are positive, let K be a binomial random variable for n trials and success probability $p = a/(a+b)$. First, we observe that the sum of over all possible values of the PMF of K is

$$\begin{aligned} \sum_{k=0}^{n} P_K(k) &= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{a}{a+b}\right)^k \left(\frac{b}{a+b}\right)^{n-k} \\ &= \frac{\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}}{(a+b)^n}. \end{aligned} \tag{1}$$

Since $\sum_{k=0}^{n} P_K(k) = 1$, we see that

$$(a+b)^n = (a+b)^n \sum_{k=0}^{n} P_K(k) = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}. \tag{2}$$

Problem 3.4.1 Solution

Using the CDF given in the problem statement we find that

(a) $P[Y < 1] = 0$ and $P[Y \leq 1] = 1/4$.

(b)

$$P[Y > 2] = 1 - P[Y \leq 2] = 1 - 1/2 = 1/2. \tag{1}$$

$$P[Y \geq 2] = 1 - P[Y < 2] = 1 - 1/4 = 3/4. \tag{2}$$

(c)

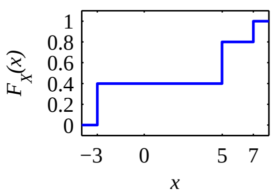
$$P[Y = 3] = F_Y(3^+) - F_Y(3^-) = 1/2. \tag{3}$$
$$P[Y > 3] = 1 - F_Y(3) = 0. \tag{4}$$

(d) From the staircase CDF of Problem 3.4.1, we see that Y is a discrete random variable. The jumps in the CDF occur at at the values that Y can take on. The height of each jump equals the probability of that value. The PMF of Y is

$$P_Y(y) = \begin{cases} 1/4 & y = 1, \\ 1/4 & y = 2, \\ 1/2 & y = 3, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Problem 3.4.3 Solution

(a) Similar to the previous problem, the graph of the CDF is shown below.



$$F_X(x) = \begin{cases} 0 & x < -3, \\ 0.4 & -3 \le x < 5, \\ 0.8 & 5 \le x < 7, \\ 1 & x \ge 7. \end{cases} \tag{1}$$

(b) The corresponding PMF of X is

$$P_X(x) = \begin{cases} 0.4 & x = -3 \\ 0.4 & x = 5 \\ 0.2 & x = 7 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Problem 3.4.5 Solution

Since mushrooms occur with probability 2/3, the number of pizzas sold before the first mushroom pizza is $N = n < 100$ if the first n pizzas do not have mushrooms followed by mushrooms on pizza $n+1$. Also, it is possible that $N = 100$ if all 100 pizzas are sold without mushrooms. the resulting PMF is

$$P_N(n) = \begin{cases} (1/3)^n(2/3) & n = 0, 1, \ldots, 99, \\ (1/3)^{100} & n = 100, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

For integers $n < 100$, the CDF of N obeys

$$F_N(n) = \sum_{i=0}^{n} P_N(i) = \sum_{i=0}^{n} (1/3)^i(2/3) = 1 - (1/3)^{n+1}. \tag{2}$$

A complete expression for $F_N(n)$ must give a valid answer for every value of n, including non-integer values. We can write the CDF using the floor function $\lfloor x \rfloor$ which denote the largest integer less than or equal to X. The complete expression for the CDF is

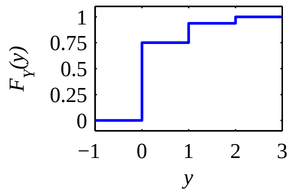
$$F_N(x) = \begin{cases} 0 & x < 0, \\ 1 - (1/3)^{\lfloor x \rfloor + 1} & 0 \le x < 100, \\ 1 & x \ge 100. \end{cases} \tag{3}$$

Problem 3.4.7 Solution

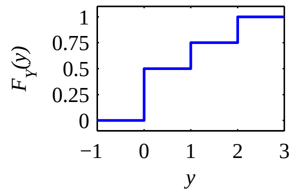
In Problem 3.2.6, we found the PMF of Y. This PMF, and its corresponding CDF are

$$P_Y(y) = \begin{cases} 1-p & y = 0, \\ p(1-p) & y = 1, \\ p^2 & y = 2, \\ 0 & \text{otherwise,} \end{cases} \qquad F_Y(y) = \begin{cases} 0 & y < 0, \\ 1-p & 0 \le y < 1, \\ 1-p^2 & 1 \le y < 2, \\ 1 & y \ge 2. \end{cases} \tag{1}$$

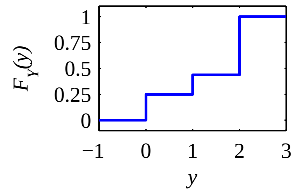
For the three values of p, the CDF resembles



$p = 1/4$



$p = 1/2$



$p = 3/4$

Problem 3.5.1 Solution

For this problem, we just need to pay careful attention to the definitions of mode and median.

(a) The mode must satisfy $P_X(x_{\text{mod}}) \geq P_X(x)$ for all x. In the case of the uniform PMF, any integer x' between 1 and 100 is a mode of the random variable X. Hence, the set of all modes is

$$X_{\text{mod}} = \{1, 2, \ldots, 100\}. \tag{1}$$

(b) The median must satisfy $\text{P}[X < x_{\text{med}}] = \text{P}[X > x_{\text{med}}]$. Since

$$\text{P}[X \leq 50] = \text{P}[X \geq 51] = 1/2. \tag{2}$$

we observe that $x_{\text{med}} = 50.5$ is a median since it satisfies

$$\text{P}[X < x_{\text{med}}] = \text{P}[X > x_{\text{med}}] = 1/2. \tag{3}$$

In fact, for any x' satisfying $50 < x' < 51$, $\text{P}[X < x'] = \text{P}[X > x'] = 1/2$. Thus,

$$X_{\text{med}} = \{x | 50 < x < 51\}. \tag{4}$$

Problem 3.5.3 Solution

(a) J has the Poisson PMF

$$P_J(j) = \begin{cases} t^j e^{-t}/j! & j = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

It follows that

$$0.9 = P[J > 0] = 1 - P_J(0) = 1 - e^{-t} \implies t = \ln(10) = 2.3. \tag{2}$$

(b) For $k = 0, 1, 2, \ldots$, $P_K(k) = 10^k e^{-10}/k!$. Thus

$$P[K = 10] = P_K(10) = 10^{10} e^{-10} = 0.1251. \tag{3}$$

(c) L is a Poisson ($\alpha = E[L] = 2$) random variable. Thus its PMF is

$$P_L(l) = \begin{cases} 2^l e^{-2}/l! & l = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

It follows that

$$P[L \leq 1] = P_L(0) + P_L(1) = 3e^{-2} = 0.406. \tag{5}$$

Problem 3.5.5 Solution

(a) Each packet transmission is a Bernoulli trial with success probability 0.95 and X is the number of packet failures (received in error) in 10 trials. Since the failure probability is $p = 0.05$, X has the binomial $(n = 10, p = 0.05)$ PMF

$$P_X(x) = \binom{10}{x} (0.05)^x (0.95)^{10-x}. \tag{1}$$

(b) When you send 12 thousand packets, the number of packets received in error, Y, is a binomial $(n = 12000, p = 0.05)$ random variable. The expected number received in error is $\mathrm{E}[Y] = np = 600$ per hour, or about 10 packets per minute. Keep in mind this is a reasonable figure if you are an active data user.

Problem 3.5.7 Solution

From the solution to Problem 3.4.2, the PMF of X is

$$P_X(x) = \begin{cases} 0.2 & x = -1, \\ 0.5 & x = 0, \\ 0.3 & x = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The expected value of X is

$$\mathrm{E}[X] = \sum_x x P_X(x) = -1(0.2) + 0(0.5) + 1(0.3) = 0.1. \tag{2}$$

Problem 3.5.9 Solution

From Definition 3.6, random variable X has PMF

$$P_X(x) = \begin{cases} \binom{4}{x}(1/2)^4 & x = 0, 1, 2, 3, 4, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The expected value of X is

$$\begin{aligned} \mathrm{E}[X] &= \sum_{x=0}^{4} x P_X(x) \\ &= 0\binom{4}{0}\frac{1}{2^4} + 1\binom{4}{1}\frac{1}{2^4} + 2\binom{4}{2}\frac{1}{2^4} + 3\binom{4}{3}\frac{1}{2^4} + 4\binom{4}{4}\frac{1}{2^4} \\ &= [4 + 12 + 12 + 4]/2^4 = 2. \end{aligned} \tag{2}$$

Problem 3.5.11 Solution

K has expected value $E[K] = 1/p = 11$ and PMF

$$P_K(k) = \begin{cases} (1-p)^{k-1}p & k = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) From these facts,

$$\begin{aligned} P[K = E[K]] = P_K(11) &= (1-p)^{10}p \\ &= (10/11)^{10}(1/11) = 10^{10}/11^{11} = 0.035. \end{aligned}$$

(b)

$$\begin{aligned} P[K > E[K]] &= P[K > 11] \\ &= \sum_{x=12}^{\infty} P_K(x) \\ &= \sum_{x=12}^{\infty} (1-p)^{x-1}p \\ &= p[(1-p)^{11} + (1-p)^{12} + \cdots] \\ &= p(1-p)^{11}[1 + (1-p) + (1-p)^2 + \cdots] \\ &= (1-p)^{11} = (10/11)^{11} = 0.3505. \end{aligned} \tag{2}$$

The answer $(1-p)^{11}$ can also be found by recalling that $K > 11$ if and only if there are 11 failures before the first success, an event which has probability $(1-p)^{11}$.

(c)

$$\begin{aligned} P[K < E[K]] &= 1 - P[K \geq E[K]] \\ &= 1 - (P[K = E[K]] + P[K > E[K]]) \\ &= 1 - \left((10/11)^{10}(1/11) + (10/11)^{11}\right) \\ &= 1 - (10/11)^{10}. \end{aligned} \tag{3}$$

Note that $(10/11)^{10}$ is the probability of ten straight failures. As long as this does NOT occur, then $K < 11$.

Problem 3.5.13 Solution

The following experiments are based on a common model of packet transmissions in data networks. In these networks, each data packet contains a cylic redundancy check (CRC) code that permits the receiver to determine whether the packet was decoded correctly. In the following, we assume that a packet is corrupted with probability $\epsilon = 0.001$, independent of whether any other packet is corrupted.

(a) Let $X = 1$ if a data packet is decoded correctly; otherwise $X = 0$. Random variable X is a Bernoulli random variable with PMF

$$P_X(x) = \begin{cases} 0.001 & x = 0, \\ 0.999 & x = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The parameter $\epsilon = 0.001$ is the probability a packet is corrupted. The expected value of X is

$$E[X] = 1 - \epsilon = 0.999. \tag{2}$$

(b) Let Y denote the number of packets received in error out of 100 packets transmitted. Y has the binomial PMF

$$P_Y(y) = \begin{cases} \binom{100}{y}(0.001)^y(0.999)^{100-y} & y = 0, 1, \ldots, 100, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The expected value of Y is

$$E[Y] = 100\epsilon = 0.1. \tag{4}$$

(c) Let L equal the number of packets that must be received to decode 5 packets in error. L has the Pascal PMF

$$P_L(l) = \begin{cases} \binom{l-1}{4}(0.001)^5(0.999)^{l-5} & l = 5, 6, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

The expected value of L is

$$\mathrm{E}[L] = \frac{5}{\epsilon} = \frac{5}{0.001} = 5000. \tag{6}$$

(d) If packet arrivals obey a Poisson model with an average arrival rate of 1000 packets per second, then the number N of packets that arrive in 5 seconds has the Poisson PMF

$$P_N(n) = \begin{cases} 5000^n e^{-5000}/n! & n = 0, 1, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

The expected value of N is $\mathrm{E}[N] = 5000$.

Problem 3.5.15 Solution

In this "double-or-nothing" type game, there are only two possible payoffs. The first is zero dollars, which happens when we lose 6 straight bets, and the second payoff is 64 dollars which happens unless we lose 6 straight bets. So the PMF of Y is

$$P_Y(y) = \begin{cases} (1/2)^6 = 1/64 & y = 0, \\ 1 - (1/2)^6 = 63/64 & y = 64, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

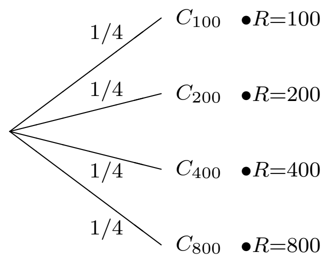
The expected amount you take home is

$$\mathrm{E}[Y] = 0(1/64) + 64(63/64) = 63. \tag{2}$$

So, on the average, we can expect to break even, which is not a very exciting proposition.

Problem 3.5.17 Solution

(a) Since you ignore the host, you pick your suitcase and open it. The tree is



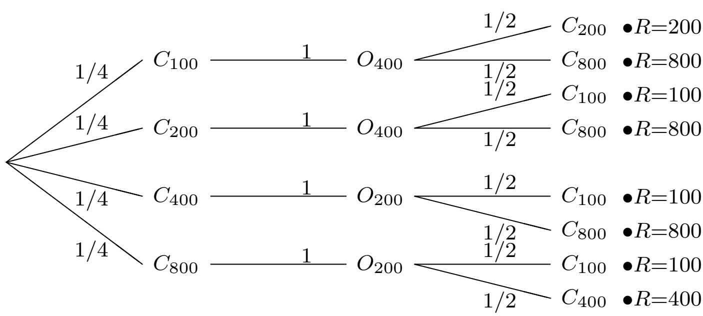
The PMF of R is just

$$P_R(r) = \begin{cases} 1/4 & r = 100, 200, 400, 800, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The expected value of R is

$$\mathrm{E}[R] = \sum_r r P_R(r) = \frac{1}{4}(100 + 200 + 400 + 800) = 375. \tag{2}$$

(b) In this case, the tree diagram is



All eight outcomes are equally likely. It follows that

r	100	200	400	800
$P_R(r)$	3/8	1/8	1/8	3/8

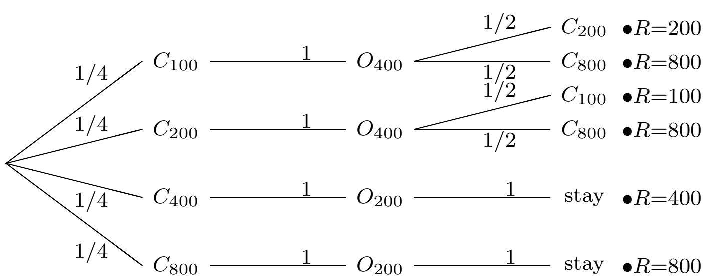
The expected value of R is

$$\mathrm{E}[R] = \frac{3}{8}(100) + \frac{1}{8}(200 + 400) + \frac{3}{8}(800) = 412.5. \tag{3}$$

(c) You can do better by making your decision whether to switch to one of the unopened suitcases depend on what suitcase the host opened. In particular, studying the tree from part (b), we see that if the host opens the \$200 suitcase, then your originally chosen suitcase is either the \$400 suitcase or the \$800 suitcase. That is, you learn you have already picked one of the two best suitcases and it seems likely that you would be better to not switch. On the other hand, if the host opens the \$400 suitcase, then you have learned that your original choice was either the \$100 or \$200 suitcase. In this case, switching gives you a chance to win the \$800 suitcase. In this case switching seems like a good idea. Thus, our intuition suggests that

- switch if the host opens the \$400 suitcase;
- stay with your original suitcase if the host opens the \$200 suitcase.

To verify that our intuition is correct, we construct the tree to evaluate this new switching policy:



It follows that

r	100	200	400	800
$P_R(r)$	1/8	1/8	1/4	1/2

The expected value of R is

$$\mathrm{E}[R] = \frac{1}{8}(100 + 200) + \frac{1}{4}(400) + \frac{1}{2}(800) = 537.5. \tag{4}$$

Problem 3.5.19 Solution

By the definition of the expected value,

$$\begin{aligned} \mathrm{E}[X_n] &= \sum_{x=1}^{n} x \binom{n}{x} p^x (1-p)^{n-x} && (1) \\ &= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-1-(x-1))!} p^{x-1}(1-p)^{n-1-(x-1)}. && (2) \end{aligned}$$

With the substitution $x' = x - 1$, we have

$$\mathrm{E}[X_n] = np \underbrace{\sum_{x'=0}^{n-1} \binom{n-1}{x'} p^{x'} (1-p)^{n-x'}}_{1} = np \sum_{x'=0}^{n-1} P_{X_{n-1}}(x) = np. \tag{3}$$

The above sum is 1 because it is the sum of a binomial random variable for $n-1$ trials over all possible values.

Problem 3.5.21 Solution

(a) A geometric (p) random variable has expected value $1/p$. Since R is a geometric random variable with $\mathrm{E}[R] = 100/m$, we can conclude that R is a geometric ($p = m/100$) random variable. Thus the PMF of R is

$$P_R(r) = \begin{cases} (1 - m/100)^{r-1}(m/100) & r = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(b) From the problem statement, $E[R] = 100/m$. Thus

$$E[W] = E[5mR] = 5m\,E[R] = 5m\frac{100}{m} = 500. \tag{2}$$

(c) She wins the money if she does work $W \geq 1000$, which has probability

$$P[W \geq 1000] = P[5mR \geq 1000] = P\left[R \geq \frac{200}{m}\right]. \tag{3}$$

Note that for a geometric (p) random variable X and an integer x_0,

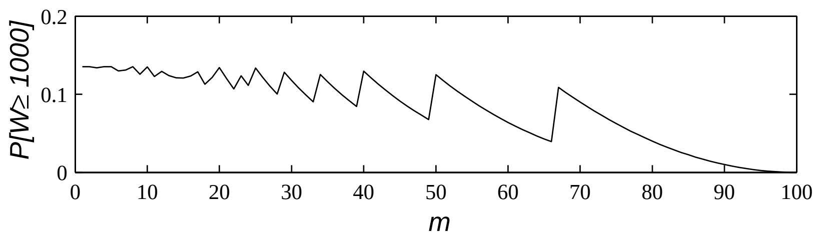
$$\begin{aligned} P[X \geq x_0] &= \sum_{x=x_0}^{\infty} P_X(x) \\ &= (1-p)^{x_0-1} p \left(1 + (1-p) + (1-p)^2 + \cdots\right) \\ &= (1-p)^{x_0-1}. \end{aligned} \tag{4}$$

Thus for the geometric $(p = m/100)$ random variable R,

$$P\left[R \geq \frac{200}{m}\right] = P\left[R \geq \left\lceil \frac{200}{m} \right\rceil\right] = \left(1 - \frac{m}{100}\right)^{\left\lceil \frac{200}{m} \right\rceil - 1}, \tag{5}$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

As a function of m, the probability of winning is an odd sawtooth function that has a peak each time $200/m$ is close to an integer. Here is a plot if you're curious:



It does happen to be true in this case that $P[W \geq 1000]$ is maximized at $m = 1$. For $m = 1$,

$$P[W \geq 1000] = P[R \geq 200] = (0.99)^{199} = 0.1353. \tag{6}$$

Problem 3.6.1 Solution

From the solution to Problem 3.4.1, the PMF of Y is

$$P_Y(y) = \begin{cases} 1/4 & y = 1, \\ 1/4 & y = 2, \\ 1/2 & y = 3, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) Since Y has range $S_Y = \{1, 2, 3\}$, the range of $U = Y^2$ is $S_U = \{1, 4, 9\}$. The PMF of U can be found by observing that

$$\mathrm{P}[U = u] = \mathrm{P}\left[Y^2 = u\right] = \mathrm{P}\left[Y = \sqrt{u}\right] + \mathrm{P}\left[Y = -\sqrt{u}\right]. \tag{2}$$

Since Y is never negative, $P_U(u) = P_Y(\sqrt{u})$. Hence,

$$P_U(1) = P_Y(1) = 1/4, \tag{3}$$
$$P_U(4) = P_Y(2) = 1/4, \tag{4}$$
$$P_U(9) = P_Y(3) = 1/2. \tag{5}$$

For all other values of u, $P_U(u) = 0$. The complete expression for the PMF of U is

$$P_U(u) = \begin{cases} 1/4 & u = 1, \\ 1/4 & u = 4, \\ 1/2 & u = 9, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

(b) From the PMF, it is straightforward to write down the CDF

$$F_U(u) = \begin{cases} 0 & u < 1, \\ 1/4 & 1 \le u < 4, \\ 1/2 & 4 \le u < 9, \\ 1 & u \ge 9. \end{cases} \tag{7}$$

(c) From Definition 3.13, the expected value of U is

$$E[U] = \sum_u u P_U(u) = 1(1/4) + 4(1/4) + 9(1/2) = 5.75. \tag{8}$$

From Theorem 3.10, we can calculate the expected value of U as

$$\begin{aligned} E[U] = E\left[Y^2\right] &= \sum_y y^2 P_Y(y) \\ &= 1^2(1/4) + 2^2(1/4) + 3^2(1/2) = 5.75. \end{aligned} \tag{9}$$

As we expect, both methods yield the same answer.

Problem 3.6.3 Solution

From the solution to Problem 3.4.3, the PMF of X is

$$P_X(x) = \begin{cases} 0.4 & x = -3, \\ 0.4 & x = 5, \\ 0.2 & x = 7, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) The PMF of $W = -X$ satisfies

$$P_W(w) = P[-X = w] = P_X(-w). \tag{2}$$

This implies

$$P_W(-7) = P_X(7) = 0.2 \tag{3}$$
$$P_W(-5) = P_X(5) = 0.4 \tag{4}$$
$$P_W(3) = P_X(-3) = 0.4. \tag{5}$$

The complete PMF for W is

$$P_W(w) = \begin{cases} 0.2 & w = -7, \\ 0.4 & w = -5, \\ 0.4 & w = 3, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

(b) From the PMF, the CDF of W is

$$F_W(w) = \begin{cases} 0 & w < -7, \\ 0.2 & -7 \leq w < -5, \\ 0.6 & -5 \leq w < 3, \\ 1 & w \geq 3. \end{cases} \tag{7}$$

(c) From the PMF, W has expected value

$$E[W] = \sum_w w P_W(w) = -7(0.2) + -5(0.4) + 3(0.4) = -2.2. \tag{8}$$

Problem 3.6.5 Solution

(a) The source continues to transmit packets until one is received correctly. Hence, the total number of times that a packet is transmitted is $X = x$ if the first $x - 1$ transmissions were in error. Therefore the PMF of X is

$$P_X(x) = \begin{cases} q^{x-1}(1-q) & x = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(b) The time required to send a packet is a millisecond and the time required to send an acknowledgment back to the source takes another millisecond. Thus, if X transmissions of a packet are needed to send the packet correctly, then the packet is correctly received after $T = 2X - 1$ milliseconds. Therefore, for an odd integer $t > 0$, $T = t$ iff $X = (t + 1)/2$. Thus,

$$P_T(t) = P_X((t+1)/2) = \begin{cases} q^{(t-1)/2}(1-q) & t = 1, 3, 5, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Problem 3.6.7 Solution

(a) A student is properly counted with probability p, independent of any other student being counted. Hence, we have 70 Bernoulli trials and N is a binomial $(70, p)$ random variable with PMF

$$P_N(n) = \binom{70}{n} p^n (1-p)^{70-n}. \tag{1}$$

(b) There are two ways to find this. The first way is to observe that

$$\begin{aligned} \mathrm{P}[U = u] = \mathrm{P}[N = 70 - u] &= P_N(70 - u) \\ &= \binom{70}{70-u} p^{70-u} (1-p)^{70-(70-u)} \\ &= \binom{70}{u} (1-p)^u p^{70-u}. \end{aligned} \tag{2}$$

We see that U is a binomial $(70, 1-p)$. The second way is to argue this directly since U is counting overlooked students. If we call an overlooked student a "success" with probability $1-p$, then U, the number of successes in n trials, is binomial $(70, 1-p)$.

(c)

$$\begin{aligned} \mathrm{P}[U \geq 2] &= 1 - \mathrm{P}[U < 2] \\ &= 1 - (P_U(0) + P_U(1)) \\ &= 1 - (p^{70} + 70(1-p)p^{69}). \end{aligned} \tag{3}$$

(d) The binomial $(n = 70, 1-p)$ random variable U has $\mathrm{E}[U] = 70(1-p)$. Solving $70(1-p) = 2$ yields $p = 34/35$.

Problem 3.7.1 Solution

Let W_n equal the number of winning tickets you purchase in n days. Since each day is an independent trial, W_n is a binomial $(n, p = 0.1)$ random variable. Since each ticket costs 1 dollar, you spend n dollars on tickets in n days. Since each winning ticket is cashed for 5 dollars, your profit after n days is

$$X_n = 5W_n - n. \tag{1}$$

It follows that

$$E[X_n] = 5 E[W_n] - n = 5np - n = (5p - 1)n = -n/2. \tag{2}$$

On average, you lose about 50 cents per day.

Problem 3.7.3 Solution

Whether a lottery ticket is a winner is a Bernoulli trial with a success probability of 0.001. If we buy one every day for 50 years for a total of $50 \cdot 365 = 18250$ tickets, then the number of winning tickets T is a binomial random variable with mean

$$E[T] = 18250(0.001) = 18.25. \tag{1}$$

Since each winning ticket grosses \$1000, the revenue we collect over 50 years is $R = 1000T$ dollars. The expected revenue is

$$E[R] = 1000 E[T] = 18250. \tag{2}$$

But buying a lottery ticket everyday for 50 years, at \$2.00 a pop isn't cheap and will cost us a total of $18250 \cdot 2 = \$36500$. Our net profit is then $Q = R - 36500$ and the result of our loyal 50 year patronage of the lottery system, is disappointing expected loss of

$$E[Q] = E[R] - 36500 = -18250. \tag{3}$$

Problem 3.7.5 Solution

Given the distributions of D, the waiting time in days and the resulting cost, C, we can answer the following questions.

(a) The expected waiting time is simply the expected value of D.

$$\mathrm{E}[D] = \sum_{d=1}^{4} d \cdot P_D(d) = 1(0.2) + 2(0.4) + 3(0.3) + 4(0.1) = 2.3. \quad (1)$$

(b) The expected deviation from the waiting time is

$$\mathrm{E}[D - \mu_D] = \mathrm{E}[D] - \mathrm{E}[\mu_d] = \mu_D - \mu_D = 0. \quad (2)$$

(c) C can be expressed as a function of D in the following manner.

$$C(D) = \begin{cases} 90 & D = 1, \\ 70 & D = 2, \\ 40 & D = 3, \\ 40 & D = 4. \end{cases} \quad (3)$$

(d) The expected service charge is

$$\mathrm{E}[C] = 90(0.2) + 70(0.4) + 40(0.3) + 40(0.1) = 62 \text{ dollars}. \quad (4)$$

Problem 3.7.7 Solution

As a function of the number of minutes used, M, the monthly cost is

$$C(M) = \begin{cases} 20 & M \leq 30 \\ 20 + (M - 30)/2 & M \geq 30 \end{cases} \quad (1)$$

The expected cost per month is

$$
\begin{aligned}
\mathrm{E}[C] &= \sum_{m=1}^{\infty} C(m) P_M(m) \\
&= \sum_{m=1}^{30} 20 P_M(m) + \sum_{m=31}^{\infty} (20 + (m-30)/2) P_M(m) \\
&= 20 \sum_{m=1}^{\infty} P_M(m) + \frac{1}{2} \sum_{m=31}^{\infty} (m-30) P_M(m). \qquad (2)
\end{aligned}
$$

Since $\sum_{m=1}^{\infty} P_M(m) = 1$ and since $P_M(m) = (1-p)^{m-1} p$ for $m \geq 1$, we have

$$
\mathrm{E}[C] = 20 + \frac{(1-p)^{30}}{2} \sum_{m=31}^{\infty} (m-30)(1-p)^{m-31} p. \qquad (3)
$$

Making the substitution $j = m - 30$ yields

$$
\mathrm{E}[C] = 20 + \frac{(1-p)^{30}}{2} \sum_{j=1}^{\infty} j (1-p)^{j-1} p = 20 + \frac{(1-p)^{30}}{2p}. \qquad (4)
$$

Problem 3.7.9 Solution

We consider the cases of using standard devices or ultra reliable devices separately. In both cases, the methodology is the same. We define random variable W such that $W = 1$ if the circuit works or $W = 0$ if the circuit is defective. (In the probability literature, W is called an indicator random variable.) The PMF of W depends on whether the circuit uses standard devices or ultra reliable devices. We then define the revenue as a function $R(W)$ and we evaluate $\mathrm{E}[R(W)]$.

The circuit with standard devices works with probability $(1-q)^{10}$ and generates revenue of k dollars if all of its 10 constituent devices work. In this case, $W = W_s$ has PMF

$$
P_{W_s}(w) = \begin{cases} 1 - (1-q)^{10} & w = 0, \\ (1-q)^{10} & w = 1, \\ 0 & \text{otherwise.} \end{cases} \qquad (1)
$$

In addition, let R_s denote the profit on a circuit with standard devices. We observe that we can we can express R_s as a function $r_s(W_s)$:

$$R_s = r_s(W_s) = \begin{cases} -10 & W_s = 0, \\ k - 10 & W_s = 1. \end{cases} \tag{2}$$

Thus we can express the expected profit as

$$\begin{aligned} \mathrm{E}[R_s] &= \mathrm{E}[r_s(W)] \\ &= \sum_{w=0}^{1} P_{W_s}(w)\, r_s(w) \\ &= P_{W_s}(0)\,(-10) + P_{W_s}(1)\,(k-10) \\ &= (1-(1-q)^{10})(-10) + (1-q)^{10}(k-10) = (0.9)^{10}k - 10. \end{aligned} \tag{3}$$

To examine the circuit with ultra reliable devices, let $W = W_u$ indicate whether the circuit works and let $R_u = r_u(W_u)$ denote the profit on a circuit with ultrareliable devices. W_u has PMF

$$P_{W_u}(w) = \begin{cases} 1-(1-q/2)^{10} & w = 0, \\ (1-q/2)^{10} & w = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The revenue function is

$$R_u = r_u(W_u) = \begin{cases} -30 & W_u = 0, \\ k - 30 & W_u = 1. \end{cases} \tag{5}$$

Thus we can express the expected profit as

$$\begin{aligned} \mathrm{E}[R_u] &= \mathrm{E}[r_u(W_u)] \\ &= \sum_{w=0}^{1} P_{W_u}(w)\, r_u(w) \\ &= P_{W_u}(0)\,(-30) + P_{W_u}(1)\,(k-30) \\ &= (1-(1-q/2)^{10})(-30) + (1-q/2)^{10}(k-30) \\ &= (0.95)^{10}k - 30. \end{aligned} \tag{6}$$

Now we can compare $\mathrm{E}[R_s]$ and $\mathrm{E}[R_u]$ to decide which circuit implementation offers the highest expected profit. The inequality $\mathrm{E}[R_u] \geq \mathrm{E}[R_s]$, holds if and only if

$$k \geq 20/[(0.95)^{10} - (0.9)^{10}] = 80.21. \tag{7}$$

So for $k < \$80.21$ using all standard devices results in greater revenue, while for $k > \$80.21$ more revenue will be generated by implementing the circuit with all ultra-reliable devices. That is, when the price commanded for a working circuit is sufficiently high, we should build more-expensive higher-reliability circuits.

If you have read ahead to Section 7.1 and learned about conditional expected values, you might prefer the following solution. If not, you might want to come back and review this alternate approach after reading Section 7.1.

Let W denote the event that a circuit works. The circuit works and generates revenue of k dollars if all of its 10 constituent devices work. For each implementation, standard or ultra-reliable, let R denote the profit on a device. We can express the expected profit as

$$\mathrm{E}[R] = \mathrm{P}[W]\,\mathrm{E}[R|W] + \mathrm{P}[W^c]\,\mathrm{E}[R|W^c]. \tag{8}$$

Let's first consider the case when only standard devices are used. In this case, a circuit works with probability $\mathrm{P}[W] = (1-q)^{10}$. The profit made on a working device is $k - 10$ dollars while a nonworking circuit has a profit of -10 dollars. That is, $\mathrm{E}[R|W] = k - 10$ and $\mathrm{E}[R|W^c] = -10$. Of course, a negative profit is actually a loss. Using R_s to denote the profit using standard circuits, the expected profit is

$$\begin{aligned}\mathrm{E}[R_s] &= (1-q)^{10}(k-10) + (1-(1-q)^{10})(-10) \\ &= (0.9)^{10}k - 10.\end{aligned} \tag{9}$$

And for the ultra-reliable case, the circuit works with probability $\mathrm{P}[W] = (1-q/2)^{10}$. The profit per working circuit is $\mathrm{E}[R|W] = k - 30$ dollars while the profit for a nonworking circuit is $\mathrm{E}[R|W^c] = -30$ dollars. The expected profit is

$$\begin{aligned} \mathrm{E}[R_u] &= (1-q/2)^{10}(k-30) + (1-(1-q/2)^{10})(-30) \\ &= (0.95)^{10}k - 30. \end{aligned} \tag{10}$$

Not surprisingly, we get the same answers for $\mathrm{E}[R_u]$ *and* $\mathrm{E}[R_s]$ *as in the first solution by performing essentially the same calculations. it should be apparent that indicator random variable* W *in the first solution indicates the occurrence of the conditioning event* W *in the second solution. That is, indicators are a way to track conditioning events.*

Problem 3.7.11 Solution

(a) There are $\binom{46}{6}$ equally likely winning combinations so that

$$q = \frac{1}{\binom{46}{6}} = \frac{1}{9{,}366{,}819} \approx 1.07 \times 10^{-7}. \tag{1}$$

(b) Assuming each ticket is chosen randomly, each of the $2n-1$ other tickets is independently a winner with probability q. The number of other winning tickets K_n has the binomial PMF

$$P_{K_n}(k) = \begin{cases} \binom{2n-1}{k} q^k (1-q)^{2n-1-k} & k = 0, 1, \ldots, 2n-1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Since the pot has $n+r$ dollars, the expected amount that you win on your ticket is

$$\mathrm{E}[V] = 0(1-q) + q\,\mathrm{E}\left[\frac{n+r}{K_n+1}\right] = q(n+r)\,\mathrm{E}\left[\frac{1}{K_n+1}\right]. \tag{3}$$

Note that $E[1/K_n+1]$ was also evaluated in Problem 3.7.10. For completeness, we repeat those steps here.

$$
\begin{aligned}
E\left[\frac{1}{K_n+1}\right] &= \sum_{k=0}^{2n-1} \frac{1}{k+1} \frac{(2n-1)!}{k!(2n-1-k)!} q^k (1-q)^{2n-1-k} \\
&= \frac{1}{2n} \sum_{k=0}^{2n-1} \frac{(2n)!}{(k+1)!(2n-(k+1))!} q^k (1-q)^{2n-(k+1)}. \qquad (4)
\end{aligned}
$$

By factoring out $1/q$, we obtain

$$
\begin{aligned}
E\left[\frac{1}{K_n+1}\right] &= \frac{1}{2nq} \sum_{k=0}^{2n-1} \binom{2n}{k+1} q^{k+1} (1-q)^{2n-(k+1)} \\
&= \frac{1}{2nq} \underbrace{\sum_{j=1}^{2n} \binom{2n}{j} q^j (1-q)^{2n-j}}_{A}. \qquad (5)
\end{aligned}
$$

We observe that the above sum labeled A is the sum of a binomial PMF for $2n$ trials and success probability q over all possible values except $j=0$. Thus $A = 1 - \binom{2n}{0} q^0 (1-q)^{2n-0}$, which implies

$$
E\left[\frac{1}{K_n+1}\right] = \frac{A}{2nq} = \frac{1-(1-q)^{2n}}{2nq}. \qquad (6)
$$

The expected value of your ticket is

$$
\begin{aligned}
E[V] &= \frac{q(n+r)[1-(1-q)^{2n}]}{2nq} \\
&= \frac{1}{2}\left(1+\frac{r}{n}\right)[1-(1-q)^{2n}]. \qquad (7)
\end{aligned}
$$

Each ticket tends to be more valuable when the carryover pot r is large and the number of new tickets sold, $2n$, is small. For any fixed number n, corresponding to $2n$ tickets sold, a sufficiently large pot r will guarantee that $\mathrm{E}[V] > 1$. For example if $n = 10^7$, (20 million tickets sold) then

$$\mathrm{E}[V] = 0.44\left(1 + \frac{r}{10^7}\right). \tag{8}$$

If the carryover pot r is 30 million dollars, then $\mathrm{E}[V] = 1.76$. This suggests that buying a one dollar ticket is a good idea. This is an unusual situation because normally a carryover pot of 30 million dollars will result in far more than 20 million tickets being sold.

(c) So that we can use the results of the previous part, suppose there were $2n-1$ tickets sold before you must make your decision. If you buy one of each possible ticket, you are guaranteed to have one winning ticket. From the other $2n-1$ tickets, there will be K_n winners. The total number of winning tickets will be K_n+1. In the previous part we found that

$$\mathrm{E}\left[\frac{1}{K_n+1}\right] = \frac{1-(1-q)^{2n}}{2nq}. \tag{9}$$

Let R denote the expected return from buying one of each possible ticket. The pot had r dollars beforehand. The $2n-1$ other tickets are sold add $n-1/2$ dollars to the pot. Furthermore, you must buy $1/q$ tickets, adding $1/(2q)$ dollars to the pot. Since the cost of the tickets is $1/q$ dollars, your expected profit

$$\begin{aligned}\mathrm{E}[R] &= \mathrm{E}\left[\frac{r+n-1/2+1/(2q)}{K_n+1}\right] - \frac{1}{q}\\ &= \frac{q(2r+2n-1)+1}{2q}\,\mathrm{E}\left[\frac{1}{K_n+1}\right] - \frac{1}{q}\\ &= \frac{[q(2r+2n-1)+1](1-(1-q)^{2n})}{4nq^2} - \frac{1}{q}. \end{aligned} \tag{10}$$

For fixed n, sufficiently large r will make $\mathrm{E}[R] > 0$. On the other hand, for fixed r, $\lim_{n\to\infty}\mathrm{E}[R] = -1/(2q)$. That is, as n approaches infinity, your expected loss will be quite large.

Problem 3.8.1 Solution

Given the following PMF

$$P_N(n) = \begin{cases} 0.2 & n = 0, \\ 0.7 & n = 1, \\ 0.1 & n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

the calculations are straightforward:

(a) $E[N] = (0.2)0 + (0.7)1 + (0.1)2 = 0.9$.

(b) $E[N^2] = (0.2)0^2 + (0.7)1^2 + (0.1)2^2 = 1.1$.

(c) $\text{Var}[N] = E[N^2] - E[N]^2 = 1.1 - (0.9)^2 = 0.29$.

(d) $\sigma_N = \sqrt{\text{Var}[N]} = \sqrt{0.29}$.

Problem 3.8.3 Solution

From the solution to Problem 3.4.2, the PMF of X is

$$P_X(x) = \begin{cases} 0.2 & x = -1, \\ 0.5 & x = 0, \\ 0.3 & x = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The expected value of X is

$$E[X] = \sum_x x P_X(x) = (-1)(0.2) + 0(0.5) + 1(0.3) = 0.1. \tag{2}$$

The expected value of X^2 is

$$E\left[X^2\right] = \sum_x x^2 P_X(x) = (-1)^2(0.2) + 0^2(0.5) + 1^2(0.3) = 0.5. \tag{3}$$

The variance of X is

$$\text{Var}[X] = E\left[X^2\right] - (E[X])^2 = 0.5 - (0.1)^2 = 0.49. \tag{4}$$

Problem 3.8.5 Solution

(a) The expected value of X is

$$\begin{aligned}
\mathrm{E}[X] &= \sum_{x=0}^{4} x P_X(x) \\
&= 0\binom{4}{0}\frac{1}{2^4} + 1\binom{4}{1}\frac{1}{2^4} + 2\binom{4}{2}\frac{1}{2^4} + 3\binom{4}{3}\frac{1}{2^4} + 4\binom{4}{4}\frac{1}{2^4} \\
&= [4 + 12 + 12 + 4]/2^4 = 2. \qquad (1)
\end{aligned}$$

The expected value of X^2 is

$$\begin{aligned}
\mathrm{E}\left[X^2\right] &= \sum_{x=0}^{4} x^2 P_X(x) \\
&= 0^2\binom{4}{0}\frac{1}{2^4} + 1^2\binom{4}{1}\frac{1}{2^4} + 2^2\binom{4}{2}\frac{1}{2^4} + 3^2\binom{4}{3}\frac{1}{2^4} + 4^2\binom{4}{4}\frac{1}{2^4} \\
&= [4 + 24 + 36 + 16]/2^4 = 5. \qquad (2)
\end{aligned}$$

The variance of X is

$$\mathrm{Var}[X] = \mathrm{E}\left[X^2\right] - (\mathrm{E}[X])^2 = 5 - 2^2 = 1. \qquad (3)$$

Thus, X has standard deviation $\sigma_X = \sqrt{\mathrm{Var}[X]} = 1$.

(b) The probability that X is within one standard deviation of its expected value is

$$\begin{aligned}
\mathrm{P}[\mu_X - \sigma_X \le X \le \mu_X + \sigma_X] &= \mathrm{P}[2 - 1 \le X \le 2 + 1] \\
&= \mathrm{P}[1 \le X \le 3]. \qquad (4)
\end{aligned}$$

This calculation is easy using the PMF of X:

$$\mathrm{P}[1 \le X \le 3] = P_X(1) + P_X(2) + P_X(3) = 7/8. \qquad (5)$$

Problem 3.8.7 Solution

For $Y = aX + b$, we wish to show that $\text{Var}[Y] = a^2 \text{Var}[X]$. We begin by noting that Theorem 3.12 says that $\text{E}[aX + b] = a\,\text{E}[X] + b$. Hence, by the definition of variance.

$$\begin{aligned} \text{Var}[Y] &= \text{E}\left[(aX + b - (a\,\text{E}[X] + b))^2\right] \\ &= \text{E}\left[a^2(X - \text{E}[X])^2\right] \\ &= a^2\,\text{E}\left[(X - \text{E}[X])^2\right]. \end{aligned} \tag{1}$$

Since $\text{E}[(X - \text{E}[X])^2] = \text{Var}[X]$, the assertion is proved.

Problem 3.8.9 Solution

With our measure of jitter being σ_T, and the fact that $T = 2X - 1$, we can express the jitter as a function of q by realizing that

$$\text{Var}[T] = 4\,\text{Var}[X] = \frac{4q}{(1-q)^2}. \tag{1}$$

Therefore, our maximum permitted jitter is

$$\sigma_T = \frac{2\sqrt{q}}{(1-q)} = 2 \text{ ms}. \tag{2}$$

Solving for q yields $q^2 - 3q + 1 = 0$. By solving this quadratic equation, we obtain

$$q = \frac{3 \pm \sqrt{5}}{2} = 3/2 \pm \sqrt{5}/2. \tag{3}$$

Since q must be a value between 0 and 1, we know that a value of $q = 3/2 - \sqrt{5}/2 \approx 0.382$ will ensure a jitter of at most 2 milliseconds.

Problem 3.8.11 Solution

The standard deviation can be expressed as

$$\sigma_D = \sqrt{\text{Var}[D]} = \sqrt{\text{E}\left[D^2\right] - \text{E}\left[D\right]^2}, \tag{1}$$

where

$$\mathrm{E}\left[D^2\right] = \sum_{d=1}^{4} d^2 P_D(d) = 0.2 + 1.6 + 2.7 + 1.6 = 6.1. \tag{2}$$

So finally we have

$$\sigma_D = \sqrt{6.1 - 2.3^2} = \sqrt{0.81} = 0.9. \tag{3}$$

Problem 3.9.1 Solution

For a binomial (n, p) random variable X, the solution in terms of math is

$$\mathrm{P}\left[E_2\right] = \sum_{x=0}^{\lfloor \sqrt{n} \rfloor} P_X\left(x^2\right). \tag{1}$$

In terms of MATLAB, the efficient solution is to generate the vector of perfect squares $x =$ `[0 1 4 9 16 ...]` and then to pass that vector to the `binomialpmf.m`. In this case, the values of the binomial PMF are calculated only once. Here is the code:

```
function q=perfectbinomial(n,p);
i=0:floor(sqrt(n));
x=i.^2;
q=sum(binomialpmf(n,p,x));
```

For a binomial $(100, 0.5)$ random variable X, the probability X is a perfect square is

```
>> perfectbinomial(100,0.5)
ans =
    0.0811
```

Problem 3.9.3 Solution

Recall that in Example 3.27 that the weight in pounds X of a package and the cost $Y = g(X)$ of shipping a package were described by

$$P_X(x) = \begin{cases} 0.15 & x = 1,2,3,4, \\ 0.1 & x = 5,6,7,8, \\ 0 & \text{otherwise,} \end{cases} \qquad Y = \begin{cases} 105X - 5X^2 & 1 \le X \le 5, \\ 500 & 6 \le X \le 10. \end{cases} \tag{1}$$

```
%shipcostpmf.m
sx=(1:8)';
px=[0.15*ones(4,1); ...
        0.1*ones(4,1)];
gx=(sx<=5).* ...
    (105*sx-5*(sx.^2))...
   + ((sx>5).*500);
sy=unique(gx)'
py=finitepmf(gx,px,sy)'
```

The `shipcostpmf` script on the left calculates the PMF of Y. The vector `gx` is the mapping $g(x)$ for each $x \in S_X$. In `gx`, the element 500 appears three times, corresponding to $x = 6$, $x = 7$, and $x = 8$. The function `sy=unique(gx)` extracts the unique elements of `gx` while `finitepmf(gx,px,sy)` calculates the probability of each element of `sy`.

Here is the output:

```
>> shipcostpmf
sy =
   100   190   270   340   400   500
py =
   0.15    0.15    0.15    0.15    0.10    0.30
```

Problem 3.9.5 Solution

Suppose X_n is a Zipf $(n, \alpha = 1)$ random variable and thus has PMF

$$P_X(x) = \begin{cases} c(n)/x & x = 1,2,\ldots,n, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The problem asks us to find the smallest value of k such that $P[X_n \le k] \ge 0.75$. That is, if the server caches the k most popular files, then with

$P[X_n \le k]$ the request is for one of the k cached files. First, we might as well solve this problem for any probability p rather than just $p = 0.75$. Thus, in math terms, we are looking for

$$k = \min \{k' | \, P[X_n \le k'] \ge p\}. \tag{2}$$

What makes the Zipf distribution hard to analyze is that there is no closed form expression for

$$c(n) = \left(\sum_{x=1}^{n} \frac{1}{x} \right)^{-1}. \tag{3}$$

Thus, we use MATLAB to grind through the calculations. The following simple program generates the Zipf distributions and returns the correct value of k.

```
function k=zipfcache(n,p);
%Usage: k=zipfcache(n,p);
%for the Zipf (n,alpha=1) distribution, returns the smallest k
%such that the first k items have total probability p
pmf=1./(1:n);
pmf=pmf/sum(pmf); %normalize to sum to 1
cdf=cumsum(pmf);
k=1+sum(cdf<=p);
```

The program `zipfcache` generalizes 0.75 to be the probability p. Although this program is sufficient, the problem asks us to find k for all values of n from 1 to 10^3!. One way to do this is to call `zipfcache` a thousand times to find k for each value of n. A better way is to use the properties of the Zipf PDF. In particular,

$$P[X_n \le k'] = c(n) \sum_{x=1}^{k'} \frac{1}{x} = \frac{c(n)}{c(k')}. \tag{4}$$

Thus we wish to find

$$k = \min \left\{ k' | \frac{c(n)}{c(k')} \ge p \right\} = \min \left\{ k' | \frac{1}{c(k')} \ge \frac{p}{c(n)} \right\}. \tag{5}$$

Note that the definition of k implies that

$$\frac{1}{c(k')} < \frac{p}{c(n)}, \qquad k' = 1, \ldots, k-1. \tag{6}$$

Using the notation $|A|$ to denote the number of elements in the set A, we can write

$$k = 1 + \left| \left\{ k' | \frac{1}{c(k')} < \frac{p}{c(n)} \right\} \right|. \tag{7}$$

This is the basis for a very short MATLAB program:

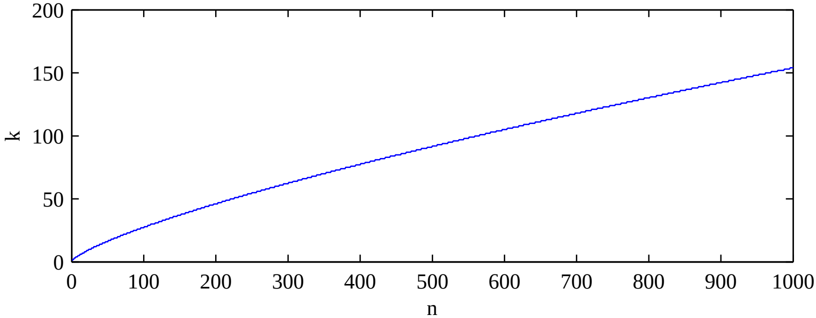
```
function k=zipfcacheall(n,p);
%Usage: k=zipfcacheall(n,p);
%returns vector k such that the first
%k(m) items have total probability >= p
%for the Zipf(m,1) distribution.
c=1./cumsum(1./(1:n));
k=1+countless(1./c,p./c);
```

Note that `zipfcacheall` uses a short MATLAB program `countless.m` that is almost the same as `count.m` introduced in Example 3.40.If `n=countless(x,y)`, then `n(i)` is the number of elements of `x` that are strictly less than `y(i)` while `count` returns the number of elements less than or equal to `y(i)`.

In any case, the commands

```
k=zipfcacheall(1000,0.75);
plot(1:1000,k);
```

is sufficient to produce this figure of k as a function of m:



We see in the figure that the number of files that must be cached grows slowly with the total number of files n.

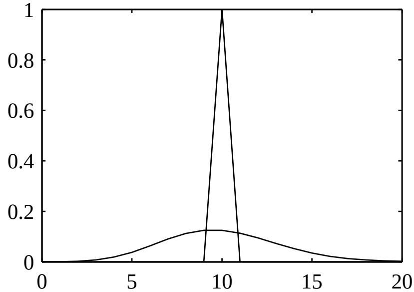
Finally, we make one last observation. It is generally desirable for MATLAB to execute operations in parallel. The program `zipfcacheall` generally will run faster than n calls to `zipfcache`. However, to do its counting all at once, `countless` generates and $n \times n$ array. When n is not too large, say $n \leq 1000$, the resulting array with $n^2 = 1{,}000{,}000$ elements fits in memory. For much larger values of n, say $n = 10^6$ (as was proposed in the original printing of this edition of the text, `countless` will cause an "out of memory" error.

Problem 3.9.7 Solution

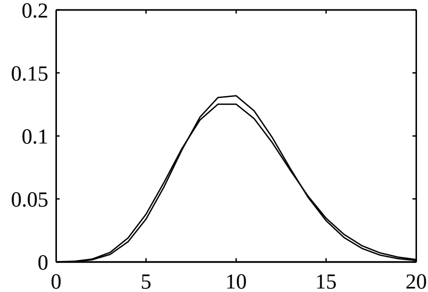
We can compare the binomial and Poisson PMFs for $(n, p) = (100, 0.1)$ using the following MATLAB code:

```
x=0:20;
p=poissonpmf(100,x);
b=binomialpmf(100,0.1,x);
plot(x,p,x,b);
```

For $(n, p) = (10, 1)$, the binomial PMF has no randomness. For $(n, p) = (100, 0.1)$, the approximation is reasonable:

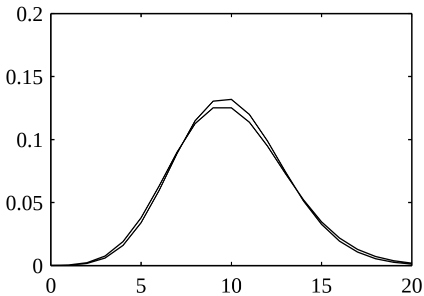


(a) $n = 10, p = 1$

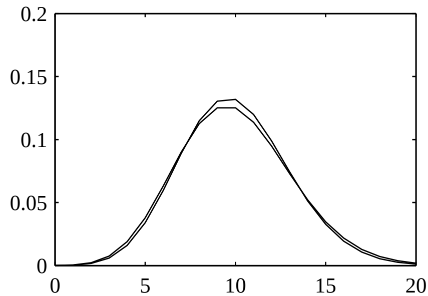


(b) $n = 100, p = 0.1$

Finally, for $(n, p) = (1000, 0.01)$, and $(n, p) = (10000, 0.001)$, the approximation is very good:



(a) $n = 1000, p = 0.01$



(b) $n = 10000, p = 0.001$

Problem 3.9.9 Solution

For the PC version of MATLAB employed for this test, `poissonpmf(n,n)` reported `Inf` for $n = n^* = 714$. The problem with the `poissonpmf` function in Example 3.37 is that the cumulative product that calculated $n^k/k!$ can have an overflow. Following the hint, we can write an alternate `poissonpmf` function as follows:

```
function pmf=poissonpmf(alpha,x)
%Poisson (alpha) rv X,
%out=vector pmf: pmf(i)=P[X=x(i)]
x=x(:);
if (alpha==0)
    pmf=1.0*(x==0);
else
    k=(1:ceil(max(x)))';
    logfacts =cumsum(log(k));
    pb=exp([-alpha; ...
        -alpha+ (k*log(alpha))-logfacts]);
    okx=(x>=0).*(x==floor(x));
    x=okx.*x;
    pmf=okx.*pb(x+1);
end
%pmf(i)=0 for zero-prob x(i)
```

By summing logarithms, the intermediate terms are much less likely to overflow.

Problem Solutions – Chapter 4

Problem 4.2.1 Solution

The CDF of X is

$$F_X(x) = \begin{cases} 0 & x < -1, \\ (x+1)/2 & -1 \leq x < 1, \\ 1 & x \geq 1. \end{cases} \quad (1)$$

Each question can be answered by expressing the requested probability in terms of $F_X(x)$.

(a)

$$\begin{aligned} P[X > 1/2] &= 1 - P[X \leq 1/2] \\ &= 1 - F_X(1/2) = 1 - 3/4 = 1/4. \end{aligned} \quad (2)$$

(b) This is a little trickier than it should be. Being careful, we can write

$$\begin{aligned} P[-1/2 \leq X < 3/4] &= P[-1/2 < X \leq 3/4] \\ &\quad + P[X = -1/2] - P[X = 3/4]. \end{aligned} \quad (3)$$

Since the CDF of X is a continuous function, the probability that X takes on any specific value is zero. This implies $P[X = 3/4] = 0$ and $P[X = -1/2] = 0$. (If this is not clear at this point, it will become clear in Section 4.7.) Thus,

$$\begin{aligned} P[-1/2 \leq X < 3/4] &= P[-1/2 < X \leq 3/4] \\ &= F_X(3/4) - F_X(-1/2) = 5/8. \end{aligned} \quad (4)$$

(c)

$$\begin{aligned} P[|X| \leq 1/2] &= P[-1/2 \leq X \leq 1/2] \\ &= P[X \leq 1/2] - P[X < -1/2]. \end{aligned} \quad (5)$$

Note that $P[X \leq 1/2] = F_X(1/2) = 3/4$. Since the probability that $P[X = -1/2] = 0$, $P[X < -1/2] = P[X \leq 1/2]$. Hence $P[X < -1/2] = F_X(-1/2) = 1/4$. This implies

$$\begin{aligned} P[|X| \leq 1/2] &= P[X \leq 1/2] - P[X < -1/2] \\ &= 3/4 - 1/4 = 1/2. \end{aligned} \tag{6}$$

(d) Since $F_X(1) = 1$, we must have $a \leq 1$. For $a \leq 1$, we need to satisfy

$$P[X \leq a] = F_X(a) = \frac{a+1}{2} = 0.8. \tag{7}$$

Thus $a = 0.6$.

Problem 4.2.3 Solution

(a) By definition, $\lceil nx \rceil$ is the smallest integer that is greater than or equal to nx. This implies $nx \leq \lceil nx \rceil \leq nx + 1$.

(b) By part (a),

$$\frac{nx}{n} \leq \frac{\lceil nx \rceil}{n} \leq \frac{nx+1}{n}. \tag{1}$$

That is,

$$x \leq \frac{\lceil nx \rceil}{n} \leq x + \frac{1}{n}. \tag{2}$$

This implies

$$x \leq \lim_{n\to\infty} \frac{\lceil nx \rceil}{n} \leq \lim_{n\to\infty} x + \frac{1}{n} = x. \tag{3}$$

(c) In the same way, $\lfloor nx \rfloor$ is the largest integer that is less than or equal to nx. This implies $nx - 1 \leq \lfloor nx \rfloor \leq nx$. It follows that

$$\frac{nx-1}{n} \leq \frac{\lfloor nx \rfloor}{n} \leq \frac{nx}{n}. \tag{4}$$

That is,

$$x - \frac{1}{n} \leq \frac{\lfloor nx \rfloor}{n} \leq x. \tag{5}$$

This implies

$$\lim_{n\to\infty} x - \frac{1}{n} = x \leq \lim_{n\to\infty} \frac{\lceil nx \rceil}{n} \leq x. \tag{6}$$

Problem 4.3.1 Solution

$$f_X(x) = \begin{cases} cx & 0 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) From the above PDF we can determine the value of c by integrating the PDF and setting it equal to 1, yielding

$$\int_0^2 cx\,dx = 2c = 1. \tag{2}$$

Therefore $c = 1/2$.

(b) $P[0 \leq X \leq 1] = \int_0^1 \frac{x}{2}\,dx = 1/4$.

(c) $P[-1/2 \leq X \leq 1/2] = \int_0^{1/2} \frac{x}{2}\,dx = 1/16$.

(d) The CDF of X is found by integrating the PDF from 0 to x.

$$F_X(x) = \int_0^x f_X(x')\,dx' = \begin{cases} 0 & x < 0, \\ x^2/4 & 0 \leq x \leq 2, \\ 1 & x > 2. \end{cases} \tag{3}$$

Problem 4.3.3 Solution

We find the PDF by taking the derivative of $F_U(u)$ on each piece that $F_U(u)$ is defined. The CDF and corresponding PDF of U are

$$F_U(u) = \begin{cases} 0 & u < -5, \\ (u+5)/8 & -5 \le u < -3, \\ 1/4 & -3 \le u < 3, \\ 1/4 + 3(u-3)/8 & 3 \le u < 5, \\ 1 & u \ge 5, \end{cases} \tag{1}$$

$$f_U(u) = \begin{cases} 0 & u < -5, \\ 1/8 & -5 \le u < -3, \\ 0 & -3 \le u < 3, \\ 3/8 & 3 \le u < 5, \\ 0 & u \ge 5. \end{cases} \tag{2}$$

Problem 4.3.5 Solution

For $x > 2$,

$$f_X(x) = (1/2)f_2(x) = (c_2/2)e^{-x}. \tag{1}$$

The non-negativity requirement $f_X(x) \ge 0$ for all x implies $c_2 \ge 0$. For $0 \le x \le 2$, non-negativity implies

$$\frac{c_1}{2} + \frac{c_2}{2}e^{-x} \ge 0, \qquad 0 \le x \le 2. \tag{2}$$

Since $c_2 \ge 0$, we see that this condition is satisfied if and only if

$$\frac{c_1}{2} + \frac{c_2}{2}e^{-2} \ge 0, \tag{3}$$

which simplifies to $c_1 \ge -c_2 e^{-2}$. Finally the requirement that the PDF integrates to unity yields

$$\begin{aligned}1 &= \frac{1}{2}\int_{-\infty}^{\infty} f_1(x)\,dx + \frac{1}{2}\int_{-\infty}^{\infty} f_2(x)\,dx \\ &= \frac{1}{2}\int_{0}^{2} c_1\,dx + \frac{1}{2}\int_{0}^{\infty} c_2 e^{-x}\,dx \\ &= c_1 + c_2/2. \end{aligned} \tag{4}$$

Thus $c_1 = 1 - c_2/2$ and we can represent our three constraints in terms of c_2 as

$$c_2 \geq 0, \qquad 1 - c_2/2 \geq -c_2 e^{-2}. \tag{5}$$

This can be simplified to

$$c_1 = 1 - c_2/2, \qquad 0 \leq c_2 \leq \frac{1}{1/2 - e^{-2}} = 2.742. \tag{6}$$

We note that this problem is tricky because $f_X(x)$ can be a valid PDF even if $c_1 < 0$.

Problem 4.4.1 Solution

$$f_X(x) = \begin{cases} 1/4 & -1 \leq x \leq 3, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We recognize that X is a uniform random variable from [-1,3].

(a) $\mathrm{E}[X] = 1$ and $\mathrm{Var}[X] = \frac{(3+1)^2}{12} = 4/3$.

(b) The new random variable Y is defined as $Y = h(X) = X^2$. Therefore

$$h(\mathrm{E}[X]) = h(1) = 1 \tag{2}$$

and

$$\mathrm{E}[h(X)] = \mathrm{E}\left[X^2\right] = \mathrm{Var}[X] + \mathrm{E}[X]^2 = 4/3 + 1 = 7/3. \tag{3}$$

(c) Finally

$$\mathrm{E}[Y] = \mathrm{E}[h(X)] = \mathrm{E}\left[X^2\right] = 7/3, \tag{4}$$

$$\mathrm{Var}[Y] = \mathrm{E}\left[X^4\right] - \mathrm{E}\left[X^2\right]^2 = \int_{-1}^{3} \frac{x^4}{4}\,dx - \frac{49}{9} = \frac{61}{5} - \frac{49}{9}. \tag{5}$$

Problem 4.4.3 Solution

The CDF of X is

$$F_X(x) = \begin{cases} 0 & x < 0, \\ x/2 & 0 \le x < 2, \\ 1 & x \ge 2. \end{cases} \tag{1}$$

(a) To find E[X], we first find the PDF by differentiating the above CDF.

$$f_X(x) = \begin{cases} 1/2 & 0 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The expected value is then

$$\mathrm{E}[X] = \int_0^2 \frac{x}{2}\,dx = 1. \tag{3}$$

(b)

$$\mathrm{E}\left[X^2\right] = \int_0^2 \frac{x^2}{2}\,dx = 8/3, \tag{4}$$

$$\mathrm{Var}[X] = \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2 = 8/3 - 1 = 5/3. \tag{5}$$

Problem 4.4.5 Solution

The CDF of Y is

$$F_Y(y) = \begin{cases} 0 & y < -1, \\ (y+1)/2 & -1 \le y < 1, \\ 1 & y \ge 1. \end{cases} \tag{1}$$

(a) We can find the expected value of Y by first differentiating the above CDF to find the PDF

$$f_Y(y) = \begin{cases} 1/2 & -1 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

It follows that

$$\mathrm{E}[Y] = \int_{-1}^{1} y/2\, dy = 0. \tag{3}$$

(b)

$$\mathrm{E}\left[Y^2\right] = \int_{-1}^{1} \frac{y^2}{2}\, dy = 1/3, \tag{4}$$

$$\mathrm{Var}[Y] = \mathrm{E}\left[Y^2\right] - \mathrm{E}[Y]^2 = 1/3 - 0 = 1/3. \tag{5}$$

Problem 4.4.7 Solution

To find the moments, we first find the PDF of U by taking the derivative of $F_U(u)$. The CDF and corresponding PDF are

$$F_U(u) = \begin{cases} 0 & u < -5, \\ (u+5)/8 & -5 \le u < -3, \\ 1/4 & -3 \le u < 3, \\ 1/4 + 3(u-3)/8 & 3 \le u < 5, \\ 1 & u \ge 5. \end{cases} \tag{1}$$

$$f_U(u) = \begin{cases} 0 & u < -5, \\ 1/8 & -5 \le u < -3, \\ 0 & -3 \le u < 3, \\ 3/8 & 3 \le u < 5, \\ 0 & u \ge 5. \end{cases} \tag{2}$$

(a) The expected value of U is

$$\mathrm{E}[U] = \int_{-\infty}^{\infty} u f_U(u)\, du = \int_{-5}^{-3} \frac{u}{8}\, du + \int_{3}^{5} \frac{3u}{8}\, du$$
$$= \left.\frac{u^2}{16}\right|_{-5}^{-3} + \left.\frac{3u^2}{16}\right|_{3}^{5} = 2. \tag{3}$$

(b) The second moment of U is

$$\mathrm{E}\left[U^2\right] = \int_{-\infty}^{\infty} u^2 f_U(u)\, du = \int_{-5}^{-3} \frac{u^2}{8}\, du + \int_{3}^{5} \frac{3u^2}{8}\, du$$
$$= \left.\frac{u^3}{24}\right|_{-5}^{-3} + \left.\frac{u^3}{8}\right|_{3}^{5} = 49/3. \tag{4}$$

The variance of U is $\mathrm{Var}[U] = \mathrm{E}[U^2] - (\mathrm{E}[U])^2 = 37/3$.

(c) Note that $2^U = e^{(\ln 2)U}$. This implies that

$$\int 2^u\, du = \int e^{(\ln 2)u}\, du = \frac{1}{\ln 2} e^{(\ln 2)u} = \frac{2^u}{\ln 2}. \tag{5}$$

The expected value of 2^U is then

$$\mathrm{E}\left[2^U\right] = \int_{-\infty}^{\infty} 2^u f_U(u)\, du$$
$$= \int_{-5}^{-3} \frac{2^u}{8}\, du + \int_{3}^{5} \frac{3 \cdot 2^u}{8}\, du$$
$$= \left.\frac{2^u}{8 \ln 2}\right|_{-5}^{-3} + \left.\frac{3 \cdot 2^u}{8 \ln 2}\right|_{3}^{5} = \frac{2307}{256 \ln 2} = 13.001. \tag{6}$$

Problem 4.5.1 Solution

Since Y is a continuous uniform $(a = 1, b = 5)$ random variable, we know that

$$f_Y(y) = \begin{cases} 1/4 & 1 \le y \le 5, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

and that

$$E[Y] = \frac{a+b}{2} = 3, \qquad \text{Var}[Y] = \frac{(b-a)^2}{12} = \frac{4}{3}. \tag{2}$$

With these facts, the remaining calculations are straightforward:

(a) $P[Y > E[Y]] = P[Y > 3] = \int_3^5 \frac{1}{4}\, dy = \frac{1}{2}$.

(b) $P[Y \le \text{Var}[Y]] = P[X \le 4/3] = \int_1^{4/3} \frac{1}{4}\, dy = \frac{1}{12}$.

Problem 4.5.3 Solution

The reflected power Y has an exponential $(\lambda = 1/P_0)$ PDF. From Theorem 4.8, $E[Y] = P_0$. The probability that an aircraft is correctly identified is

$$P[Y > P_0] = \int_{P_0}^{\infty} \frac{1}{P_0} e^{-y/P_0}\, dy = e^{-1}. \tag{1}$$

Fortunately, real radar systems offer better performance.

Problem 4.5.5 Solution

An exponential (λ) random variable has PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

and has expected value $E[Y] = 1/\lambda$. Although λ was not specified in the problem, we can still solve for the probabilities:

(a) $P[Y \geq E[Y]] = \int_{1/\lambda}^{\infty} \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\Big|_{1/\lambda}^{\infty} = e^{-1}.$

(b) $P[Y \geq 2\,E[Y]] = \int_{2/\lambda}^{\infty} \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\Big|_{2/\lambda}^{\infty} = e^{-2}.$

Problem 4.5.7 Solution

Since Y is an Erlang random variable with parameters $\lambda = 2$ and $n = 2$, we find in Appendix A that

$$f_Y(y) = \begin{cases} 4ye^{-2y} & y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) Appendix A tells us that $E[Y] = n/\lambda = 1$.

(b) Appendix A also tells us that $\text{Var}[Y] = n/\lambda^2 = 1/2$.

(c) The probability that $1/2 \leq Y < 3/2$ is

$$P[1/2 \leq Y < 3/2] = \int_{1/2}^{3/2} f_Y(y)\; dy = \int_{1/2}^{3/2} 4ye^{-2y}\, dy. \tag{2}$$

This integral is easily completed using the integration by parts formula $\int u\, dv = uv - \int v\, du$ with

$$u = 2y, \qquad dv = 2e^{-2y},$$
$$du = 2dy, \qquad v = -e^{-2y}.$$

Making these substitutions, we obtain

$$\begin{aligned} P[1/2 \leq Y < 3/2] &= -2ye^{-2y}\Big|_{1/2}^{3/2} + \int_{1/2}^{3/2} 2e^{-2y}\, dy \\ &= 2e^{-1} - 4e^{-3} = 0.537. \end{aligned} \tag{3}$$

Problem 4.5.9 Solution

Since U is a continuous uniform random variable with $E[U] = 10$, we know that $u = 10$ is the midpoint of a uniform (a, b) PDF. That is, for some constant $c > 0$, U is a continuous uniform $(10 - c, 10 + c)$ random variable with PDF

$$f_U(u) = \begin{cases} 1/(2c) & 10 - c \le u \le 10 + c, \\ 0 & \text{otherwise.} \end{cases}$$

This implies

$$\begin{aligned} \frac{1}{4} = P[U > 12] &= \int_{12}^{\infty} f_U(u)\, du \\ &= \int_{12}^{10+c} \frac{1}{2c}\, du = \frac{10 + c - 12}{2c} = \frac{1}{2} - \frac{1}{c}. \end{aligned} \tag{1}$$

This implies $1/c = 1/4$ or $c = 4$. Thus U is a uniform $(6, 14)$ random variable with PDF

$$f_U(u) = \begin{cases} 1/8 & 6 \le u \le 14, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$P[U < 9] = \int_{-\infty}^{9} f_U(u)\, du = \int_{6}^{9} \frac{1}{8}\, du = \frac{3}{8}.$$

Problem 4.5.11 Solution

For a uniform $(-a, a)$ random variable X,

$$\operatorname{Var}[X] = (a - (-a))^2/12 = a^2/3. \tag{1}$$

Hence $P[|X| \le \operatorname{Var}[X]] = P[|X| \le a^2/3]$. Keep in mind that

$$f_X(x) = \begin{cases} 1/(2a) & -a \le x \le a, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

If $a^2/3 > a$, that is $a > 3$, then we have $\mathrm{P}[|X| \leq \mathrm{Var}[X]] = 1$. Otherwise, if $a \leq 3$,

$$\mathrm{P}\left[|X| \leq \mathrm{Var}[X]\right] = \mathrm{P}\left[|X| \leq a^2/3\right] = \int_{-a^2/3}^{a^2/3} \frac{1}{2a}\,dx = a/3. \tag{3}$$

Problem 4.5.13 Solution

Given that

$$f_X(x) = \begin{cases} (1/2)e^{-x/2} & x \geq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

(a)

$$\mathrm{P}\left[1 \leq X \leq 2\right] = \int_1^2 (1/2)e^{-x/2}\,dx = e^{-1/2} - e^{-1} = 0.2387. \tag{2}$$

(b) The CDF of X may be be expressed as

$$F_X(x) = \begin{cases} 0 & x < 0, \\ \int_0^x (1/2)e^{-x/2}\,d\tau & x \geq 0, \end{cases} = \begin{cases} 0 & x < 0, \\ 1 - e^{-x/2} & x \geq 0. \end{cases} \tag{3}$$

(c) X is an exponential random variable with parameter $a = 1/2$. By Theorem 4.8, the expected value of X is $\mathrm{E}[X] = 1/a = 2$.

(d) By Theorem 4.8, the variance of X is $\mathrm{Var}[X] = 1/a^2 = 4$.

Problem 4.5.15 Solution

Let X denote the holding time of a call. The PDF of X is

$$f_X(x) = \begin{cases} (1/\tau)e^{-x/\tau} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We will use $C_A(X)$ and $C_B(X)$ to denote the cost of a call under the two plans. From the problem statement, we note that $C_A(X) = 10X$ so that $\mathrm{E}[C_A(X)] = 10\,\mathrm{E}[X] = 10\tau$. On the other hand

$$C_B(X) = 99 + 10(X - 20)^+, \tag{2}$$

where $y^+ = y$ if $y \geq 0$; otherwise $y^+ = 0$ for $y < 0$. Thus,

$$\begin{aligned}\mathrm{E}\left[C_B(X)\right] &= \mathrm{E}\left[99 + 10(X-20)^+\right] \\ &= 99 + 10\,\mathrm{E}\left[(X-20)^+\right] \\ &= 99 + 10\,\mathrm{E}\left[(X-20)^+ | X \leq 20\right] \mathrm{P}\left[X \leq 20\right] \\ &\qquad + 10\,\mathrm{E}\left[(X-20)^+ | X > 20\right] \mathrm{P}\left[X > 20\right].\end{aligned} \tag{3}$$

Given $X \leq 20$, $(X-20)^+ = 0$. Thus $\mathrm{E}[(X-20)^+|X \leq 20] = 0$ and

$$\mathrm{E}\left[C_B(X)\right] = 99 + 10\,\mathrm{E}\left[(X-20)|X > 20\right] \mathrm{P}\left[X > 20\right]. \tag{4}$$

Finally, we observe that $\mathrm{P}[X > 20] = e^{-20/\tau}$ and that

$$\mathrm{E}\left[(X-20)|X > 20\right] = \tau \tag{5}$$

since given $X \geq 20$, $X - 20$ has a PDF identical to X by the memoryless property of the exponential random variable. Thus,

$$\mathrm{E}\left[C_B(X)\right] = 99 + 10\tau e^{-20/\tau} \tag{6}$$

Some numeric comparisons show that $\mathrm{E}[C_B(X)] \leq \mathrm{E}[C_A(X)]$ if $\tau > 12.34$ minutes. That is, the flat price for the first 20 minutes is a good deal only if your average phone call is sufficiently long.

Problem 4.5.17 Solution

For an Erlang (n, λ) random variable X, the kth moment is

$$\begin{aligned}\mathrm{E}\left[X^k\right] &= \int_0^\infty x^k f_X(x)\; dt \\ &= \int_0^\infty \frac{\lambda^n x^{n+k-1}}{(n-1)!} e^{-\lambda x}\, dt = \frac{(n+k-1)!}{\lambda^k (n-1)!} \underbrace{\int_0^\infty \frac{\lambda^{n+k} x^{n+k-1}}{(n+k-1)!} e^{-\lambda t}\, dt}_{1}.\end{aligned} \tag{1}$$

The above integral equals 1 since it is the integral of an Erlang $(n+k, \lambda)$ PDF over all possible values. Hence,

$$\mathrm{E}\left[X^k\right] = \frac{(n+k-1)!}{\lambda^k (n-1)!}. \tag{2}$$

This implies that the first and second moments are

$$\mathrm{E}[X] = \frac{n!}{(n-1)!\lambda} = \frac{n}{\lambda}, \qquad \mathrm{E}\left[X^2\right] = \frac{(n+1)!}{\lambda^2 (n-1)!} = \frac{(n+1)n}{\lambda^2}. \tag{3}$$

It follows that the variance of X is n/λ^2.

Problem 4.5.19 Solution

For $n = 1$, we have the fact $\mathrm{E}[X] = 1/\lambda$ that is given in the problem statement. Now we assume that $\mathrm{E}[X^{n-1}] = (n-1)!/\lambda^{n-1}$. To complete the proof, we show that this implies that $\mathrm{E}[X^n] = n!/\lambda^n$. Specifically, we write

$$\mathrm{E}[X^n] = \int_0 x^n \lambda e^{-\lambda x}\, dx. \tag{1}$$

Now we use the integration by parts formula $\int u\, dv = uv - \int v\, du$ with $u = x^n$ and $dv = \lambda e^{-\lambda x}\, dx$. This implies $du = nx^{n-1}\, dx$ and $v = -e^{-\lambda x}$ so that

$$\begin{aligned} \mathrm{E}[X^n] &= -x^n e^{-\lambda x}\Big|_0^\infty + \int_0^\infty n x^{n-1} e^{-\lambda x}\, dx \\ &= 0 + \frac{n}{\lambda}\int_0^\infty x^{n-1}\lambda e^{-\lambda x}\, dx \\ &= \frac{n}{\lambda}\mathrm{E}\left[X^{n-1}\right]. \end{aligned} \tag{2}$$

By our induction hyothesis, $\mathrm{E}[X^{n-1}] = (n-1)!/\lambda^{n-1}$ which implies

$$\mathrm{E}[X^n] = n!/\lambda^n. \tag{3}$$

Problem 4.6.1 Solution

Given that the peak temperature, T, is a Gaussian random variable with mean 85 and standard deviation 10 we can use the fact that $F_T(t) = \Phi((t-\mu_T)/\sigma_T)$ and Table 4.2 on page 143 to evaluate:

$$\begin{aligned} P[T > 100] &= 1 - P[T \leq 100] \\ &= 1 - F_T(100) \\ &= 1 - \Phi\left(\frac{100-85}{10}\right) \\ &= 1 - \Phi(1.5) = 1 - 0.933 = 0.066, && (1) \\ P[T < 60] &= \Phi\left(\frac{60-85}{10}\right) \\ &= \Phi(-2.5) = 1 - \Phi(2.5) = 1 - .993 = 0.007, && (2) \\ P[70 \leq T \leq 100] &= F_T(100) - F_T(70) \\ &= \Phi(1.5) - \Phi(-1.5) = 2\Phi(1.5) - 1 = .866. && (3) \end{aligned}$$

Problem 4.6.3 Solution

(a)

$$\begin{aligned} P[V > 4] = 1 - P[V \leq 4] &= 1 - P\left[\frac{V-0}{\sigma} \leq \frac{4-0}{\sigma}\right] \\ &= 1 - \Phi(4/\sigma) \\ &= 1 - \Phi(2) = 0.023. && (1) \end{aligned}$$

(b)

$$P[W \leq 2] = P\left[\frac{W-2}{5} \leq \frac{2-2}{5}\right] = \Phi(0) = \frac{1}{2}. \quad (2)$$

(c)

$$
\begin{aligned}
\mathrm{P}[X \le \mu + 1] &= \mathrm{P}[X - \mu \le 1] \\
&= \mathrm{P}\left[\frac{X-\mu}{\sigma} \le \frac{1}{\sigma}\right] \\
&= \Phi(1/\sigma) = \Phi(0.5) = 0.692. \qquad (3)
\end{aligned}
$$

(d)

$$
\begin{aligned}
\mathrm{P}[Y > 65] &= 1 - \mathrm{P}[Y \le 65] \\
&= 1 - \mathrm{P}\left[\frac{Y-50}{10} \le \frac{65-50}{10}\right] \\
&= 1 - \Phi(1.5) = 1 - 0.933 = 0.067. \qquad (4)
\end{aligned}
$$

Problem 4.6.5 Solution

Your body temperature T (in degrees Fahrenheit) satisfies

$$
\mathrm{P}[T > 100] = \mathrm{P}\left[\frac{T-98.6}{0.4} > \frac{100-98.6}{0.4}\right] = Q(3.5) = 2.32 \times 10^{-4}. \qquad (1)
$$

According to this model, if you were to record your body temperature every day for $10,000$ days (over 27 years), you would expect to measure a temperature over 100 perhaps 2 or 3 times. This seems very low since a 100 degree body temperature is a mild and not uncommon fever. What this suggests is that this is a good model for when you are healthy but is not a good model for when you are sick. When you are healthy, 2×10^{-4} might be a reasonable value for the probability of an elevated temperature. However, when you are sick, you need a new model for body temperatures such that $\mathrm{P}[T > 100]$ is much higher.

Problem 4.6.7 Solution

X is a Gaussian random variable with zero mean but unknown variance. We do know, however, that

$$
\mathrm{P}[|X| \le 10] = 0.1. \qquad (1)
$$

We can find the variance $\mathrm{Var}[X]$ by expanding the above probability in terms of the $\Phi(\cdot)$ function.

$$P[-10 \leq X \leq 10] = F_X(10) - F_X(-10) = 2\Phi\left(\frac{10}{\sigma_X}\right) - 1. \tag{2}$$

This implies $\Phi(10/\sigma_X) = 0.55$. Using Table 4.2 for the Gaussian CDF, we find that $10/\sigma_X = 0.15$ or $\sigma_X = 66.6$.

Problem 4.6.9 Solution

Moving to Antarctica, we find that the temperature, T is still Gaussian but with variance 225. We also know that with probability 1/2, T exceeds -75 degrees. First we would like to find the mean temperature, and we do so by looking at the second fact.

$$P[T > -75] = 1 - P[T \leq -75] = 1 - \Phi\left(\frac{-75 - \mu_T}{15}\right) = 1/2 \tag{1}$$

By looking at the table we find that if $\Phi(x) = 1/2$, then $x = 0$. Therefore,

$$\Phi\left(\frac{-75 - \mu_T}{15}\right) = 1/2 \tag{2}$$

implies that $(-75 - \mu_T)/15 = 0$ or $\mu_T = -75$. Now we have a Gaussian T with expected value -75 and standard deviation 15. So we are prepared to answer the following problems:

$$P[T > 0] = Q\left(\frac{0 - (-75)}{15}\right) = Q(5) = 2.87 \times 10^{-7}, \tag{3}$$

$$\begin{aligned} P[T < -100] &= F_T(-100) = \Phi\left(\frac{-100 - (-75)}{15}\right) \\ &= \Phi(-5/3) = 1 - \Phi(5/3) = 0.0478. \end{aligned} \tag{4}$$

Problem 4.6.11 Solution

We are given that there are 100,000,000 men in the United States and 23,000 of them are at least 7 feet tall, and the heights of U.S men are independent Gaussian random variables with mean $5'10''$.

(a) Let H denote the height in inches of a U.S male. To find σ_X, we look at the fact that the probability that $P[H \geq 84]$ is the number of men who are at least 7 feet tall divided by the total number of men (the frequency interpretation of probability). Since we measure H in inches, we have

$$P[H \geq 84] = \frac{23{,}000}{100{,}000{,}000} = \Phi\left(\frac{70-84}{\sigma_X}\right) = 0.00023. \tag{1}$$

Since $\Phi(-x) = 1 - \Phi(x) = Q(x)$,

$$Q(14/\sigma_X) = 2.3 \cdot 10^{-4}. \tag{2}$$

From Table 4.3, this implies $14/\sigma_X = 3.5$ or $\sigma_X = 4$.

(b) The probability that a randomly chosen man is at least 8 feet tall is

$$P[H \geq 96] = Q\left(\frac{96-70}{4}\right) = Q(6.5). \tag{3}$$

Unfortunately, Table 4.3 doesn't include $Q(6.5)$, although it should be apparent that the probability is very small. In fact, MATLAB will calculate $Q(6.5) = 4.0 \times 10^{-11}$.

(c) First we need to find the probability that a man is at least 7'6".

$$P[H \geq 90] = Q\left(\frac{90-70}{4}\right) = Q(5) \approx 3 \cdot 10^{-7} = \beta. \tag{4}$$

Although Table 4.3 stops at $Q(4.99)$, if you're curious, the exact value is $Q(5) = 2.87 \cdot 10^{-7}$.

Now we can begin to find the probability that no man is at least 7'6". This can be modeled as 100,000,000 repetitions of a Bernoulli trial with parameter $1-\beta$. The probability that no man is at least 7'6" is

$$(1-\beta)^{100{,}000{,}000} = 9.4 \times 10^{-14}. \tag{5}$$

(d) The expected value of N is just the number of trials multiplied by the probability that a man is at least 7'6".

$$\mathrm{E}[N] = 100{,}000{,}000 \cdot \beta = 30. \tag{6}$$

Problem 4.6.13 Solution

First we note that since W has an $N[\mu, \sigma^2]$ distribution, the integral we wish to evaluate is

$$I = \int_{-\infty}^{\infty} f_W(w)\, dw = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-(w-\mu)^2/2\sigma^2}\, dw. \tag{1}$$

(a) Using the substitution $x = (w-\mu)/\sigma$, we have $dx = dw/\sigma$ and

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx. \tag{2}$$

(b) When we write I^2 as the product of integrals, we use y to denote the other variable of integration so that

$$\begin{aligned} I^2 &= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx\right)\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\, dy\right) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dx\, dy. \end{aligned} \tag{3}$$

(c) By changing to polar coordinates, $x^2+y^2 = r^2$ and $dx\,dy = r\,dr\,d\theta$ so that

$$\begin{aligned} I^2 &= \frac{1}{2\pi} \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2} r\, dr\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} -e^{-r^2/2}\Big|_0^{\infty}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1. \end{aligned} \tag{4}$$

Problem 4.6.15 Solution

This problem is mostly calculus and only a little probability. The result is a famous formula in the analysis of radio systems. From the problem statement, the SNR Y is an exponential $(1/\gamma)$ random variable with PDF

$$f_Y(y) = \begin{cases} (1/\gamma)e^{-y/\gamma} & y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Thus, from the problem statement, the BER is

$$\begin{aligned} \overline{P}_e = \mathrm{E}[P_e(Y)] &= \int_{-\infty}^{\infty} Q(\sqrt{2y}) f_Y(y)\, dy \\ &= \int_0^{\infty} Q(\sqrt{2y}) \frac{y}{\gamma} e^{-y/\gamma}\, dy. \end{aligned} \tag{2}$$

Like most integrals with exponential factors, its a good idea to try integration by parts. Before doing so, we recall that if X is a Gaussian $(0,1)$ random variable with CDF $F_X(x)$, then

$$Q(x) = 1 - F_X(x). \tag{3}$$

It follows that $Q(x)$ has derivative

$$Q'(x) = \frac{dQ(x)}{dx} = -\frac{dF_X(x)}{dx} = -f_X(x) = -\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{4}$$

To solve the integral, we use the integration by parts formula

$$\int_a^b u\, dv = uv|_a^b - \int_a^b v\, du, \tag{5}$$

where

$$u = Q(\sqrt{2y}), \qquad dv = \frac{1}{\gamma} e^{-y/\gamma}\, dy, \tag{6}$$

$$du = Q'(\sqrt{2y}) \frac{1}{\sqrt{2y}} = -\frac{e^{-y}}{2\sqrt{\pi y}}, \qquad v = -e^{-y/\gamma}. \tag{7}$$

From integration by parts, it follows that

$$\begin{aligned}\overline{P}_e &= uv\big|_0^\infty - \int_0^\infty v\,du \\ &= -Q(\sqrt{2y})e^{-y/\gamma}\Big|_0^\infty - \int_0^\infty \frac{1}{\sqrt{y}} e^{-y[1+(1/\gamma)]}\,dy \\ &= 0 + Q(0)e^{-0} - \frac{1}{2\sqrt{\pi}}\int_0^\infty y^{-1/2}e^{-y/\bar{\gamma}}\,dy, \end{aligned} \tag{8}$$

where $\bar{\gamma} = \gamma/(1+\gamma)$. Next, recalling that $Q(0) = 1/2$ and making the substitution $t = y/\bar{\gamma}$, we obtain

$$\overline{P}_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\bar{\gamma}}{\pi}}\int_0^\infty t^{-1/2}e^{-t}\,dt. \tag{9}$$

From Math Fact B.11, we see that the remaining integral is the $\Gamma(z)$ function evaluated $z = 1/2$. Since $\Gamma(1/2) = \sqrt{\pi}$,

$$\overline{P}_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\bar{\gamma}}{\pi}}\Gamma(1/2) = \frac{1}{2}\left[1 - \sqrt{\bar{\gamma}}\right] = \frac{1}{2}\left[1 - \sqrt{\frac{\gamma}{1+\gamma}}\right]. \tag{10}$$

Problem 4.6.17 Solution

First we recall that the stock price at time t is X, a uniform $(k-t, k+t)$ random variable. The profit from the straddle is $R' = 2d - (V + W)$ where

$$V = (k - X)^+, \qquad W = (X - k)^+. \tag{1}$$

To find the CDF, we write

$$\begin{aligned} F_{R'}(r) = \mathrm{P}\left[R' \leq r\right] &= \mathrm{P}\left[2d - (V + W) \leq r\right] \\ &= \mathrm{P}\left[V + W \geq 2d - r\right]. \end{aligned} \tag{2}$$

Since $V + W$ is non-negative,

$$F_{R'}(r) = \mathrm{P}\left[V + W \geq 2d - r\right] = 1, \qquad r \geq 2d. \tag{3}$$

Now we focus on the case $r \leq 2d$. Here we observe that $V > 0$ and $W > 0$ are mutually exclusive events. Thus, for $2d - r \geq 0$,

$$F_{R'}(r) = \mathrm{P}[V \geq 2d - r] + \mathrm{P}[W \geq 2d - r] = 2\,\mathrm{P}[W \geq 2d - r]. \tag{4}$$

since W and V are identically distributed. Since $W = (X-k)^+$ and $2d-r \geq 0$,

$$\begin{aligned} \mathrm{P}[W \geq 2d - r] &= \mathrm{P}\left[(X - k)^+ \geq 2d - r\right] \\ &= \mathrm{P}[X - k \geq 2d - r] \\ &= \begin{cases} 0 & (2d - r) > t, \\ \frac{t-(2d-r)}{2t} & (2d - r) \leq t. \end{cases} \end{aligned} \tag{5}$$

We can combine the above results in the following statement:

$$F_{R'}(r) = 2\,\mathrm{P}[W \geq 2d - r] = \begin{cases} 0 & r < 2d - t, \\ \frac{t-2d+r}{t} & 2d - t \leq r \leq 2d, \\ 1 & r \geq 2d. \end{cases} \tag{6}$$

The PDF of R' is

$$f_{R'}(r) = \begin{cases} \frac{1}{t} & 2d - t \leq r \leq 2d, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

It might appear that this is a good strategy since you may expect to receive a return of $\mathrm{E}[R'] > 0$ dollars; however this is not free because you assume the risk of a significant loss. In a real investment, the PDF of the price X is not bounded and the loss can be very very large. However, in the case of this problem, the bounded PDF for X implies the loss is not so terrible. From part (a), or by examination of the PDF $f_{R'}(r)$, we see that

$$\mathrm{E}[R'] = \frac{4d - t}{2}.$$

Thus $\mathrm{E}[R'] > 0$ if and only if $d > t/4$. In the worst case of $d = t/4$, we observe that R' has a uniform PDF over $(-t/2, t/2)$ and the worst possible loss is $t/2$ dollars. Whether the risk of such a loss is worth taking for an expected return $\mathrm{E}[R']$ would depend mostly on your financial capital and your investment objectives, which were not included in the problem formulation.

Problem 4.7.1 Solution

(a) Using the given CDF

$$P[X < -1] = F_X(-1^-) = 0, \quad (1)$$
$$P[X \leq -1] = F_X(-1) = -1/3 + 1/3 = 0. \quad (2)$$

Where $F_X(-1^-)$ denotes the limiting value of the CDF found by approaching -1 from the left. Likewise, $F_X(-1^+)$ is interpreted to be the value of the CDF found by approaching -1 from the right. We notice that these two probabilities are the same and therefore the probability that X is exactly -1 is zero.

(b)

$$P[X < 0] = F_X(0^-) = 1/3, \quad (3)$$
$$P[X \leq 0] = F_X(0) = 2/3. \quad (4)$$

Here we see that there is a discrete jump at $X = 0$. Approached from the left the CDF yields a value of 1/3 but approached from the right the value is 2/3. This means that there is a non-zero probability that $X = 0$, in fact that probability is the difference of the two values.

$$P[X = 0] = P[X \leq 0] - P[X < 0] = 2/3 - 1/3 = 1/3. \quad (5)$$

(c)

$$P[0 < X \leq 1] = F_X(1) - F_X(0^+) = 1 - 2/3 = 1/3, \quad (6)$$
$$P[0 \leq X \leq 1] = F_X(1) - F_X(0^-) = 1 - 1/3 = 2/3. \quad (7)$$

The difference in the last two probabilities above is that the first was concerned with the probability that X was strictly greater then 0, and the second with the probability that X was greater than or equal to zero. Since the the second probability is a larger set (it includes the probability that $X = 0$) it should always be greater than or equal to the first probability. The two differ by the probability that $X = 0$, and this difference is non-zero only when the random variable exhibits a discrete jump in the CDF.

Problem 4.7.3 Solution

(a) By taking the derivative of the CDF $F_X(x)$ given in Problem 4.7.2, we obtain the PDF

$$f_X(x) = \begin{cases} \frac{\delta(x+1)}{4} + 1/4 + \frac{\delta(x-1)}{4} & -1 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(b) The first moment of X is

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f_X(x)\, dx \\ &= x/4|_{x=-1} + x^2/8|_{-1}^{1} + x/4|_{x=1} \\ &= -1/4 + 0 + 1/4 = 0. \end{aligned} \tag{2}$$

(c) The second moment of X is

$$\begin{aligned} E\left[X^2\right] &= \int_{-\infty}^{\infty} x^2 f_X(x)\, dx \\ &= x^2/4\big|_{x=-1} + x^3/12\big|_{-1}^{1} + x^2/4\big|_{x=1} \\ &= 1/4 + 1/6 + 1/4 = 2/3. \end{aligned} \tag{3}$$

Since $E[X] = 0$, $\text{Var}[X] = E[X^2] = 2/3$.

Problem 4.7.5 Solution

The PMF of a geometric random variable with mean $1/p$ is

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The corresponding PDF is

$$\begin{aligned} f_X(x) &= p\delta(x-1) + p(1-p)\delta(x-2) + \cdots \\ &= \sum_{j=1}^{\infty} p(1-p)^{j-1}\delta(x-j). \end{aligned} \tag{2}$$

Problem 4.7.7 Solution

The professor is on time 80 percent of the time and when he is late his arrival time is uniformly distributed between 0 and 300 seconds. The PDF of T, is

$$f_T(t) = \begin{cases} 0.8\delta(t-0) + \frac{0.2}{300} & 0 \le t \le 300, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The CDF can be found be integrating

$$F_T(t) = \begin{cases} 0 & t < -1, \\ 0.8 + \frac{0.2t}{300} & 0 \le t < 300, \\ 1 & t \ge 300. \end{cases} \tag{2}$$

Problem 4.7.9 Solution

The professor is on time and lectures the full 80 minutes with probability 0.7. In terms of math,

$$\mathrm{P}[T = 80] = 0.7. \tag{1}$$

Likewise when the professor is more than 5 minutes late, the students leave and a 0 minute lecture is observed. Since he is late 30% of the time and given that he is late, his arrival is uniformly distributed between 0 and 10 minutes, the probability that there is no lecture is

$$\mathrm{P}[T = 0] = (0.3)(0.5) = 0.15 \tag{2}$$

The only other possible lecture durations are uniformly distributed between 75 and 80 minutes, because the students will not wait longer then 5 minutes, and that probability must add to a total of $1 - 0.7 - 0.15 = 0.15$. So the PDF of T can be written as

$$f_T(t) = \begin{cases} 0.15\delta(t) & t = 0, \\ 0.03 & 75 \leq 7 < 80, \\ 0.7\delta(t - 80) & t = 80, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Problem 4.8.1 Solution

Taking the derivative of the CDF $F_Y(y)$ in Quiz 4.2, we obtain

$$f_Y(y) = \begin{cases} 1/4 & 0 \leq y \leq 4, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We see that Y is a uniform $(0, 4)$ random variable. By Theorem 6.3, if X is a uniform $(0, 1)$ random variable, then $Y = 4X$ is a uniform $(0, 4)$ random variable. Using `rand` as MATLAB's uniform $(0, 1)$ random variable, the program `quiz31rv` is essentially a one line program:

```
function y=quiz31rv(m)
%Usage y=quiz31rv(m)
%Returns the vector y holding m
%samples of the uniform (0,4) random
%variable Y of Quiz 3.1
y=4*rand(m,1);
```

Problem 4.8.3 Solution

By Theorem 4.9, if X is an exponential (λ) random variable, then $K = \lceil X \rceil$ is a geometric (p) random variable with $p = 1 - e^{-\lambda}$. Thus, given p, we can write $\lambda = -\ln(1-p)$ and $\lceil X \rceil$ is a geometric (p) random variable. Here is the MATLAB function that implements this technique:

```
function k=georv(p,m);
lambda= -log(1-p);
k=ceil(exponentialrv(lambda,m));
```

To compare this technique with that use in `geometricrv.m`, we first examine the code for `exponentialrv.m`:

```
function x=exponentialrv(lambda,m)
x=-(1/lambda)*log(1-rand(m,1));
```

To analyze how $m = 1$ random sample is generated, let $R =$ `rand(1,1)`. In terms of mathematics, `exponentialrv(lambda,1)` generates the random variable

$$X = -\frac{\ln(1-R)}{\lambda} \tag{1}$$

For $\lambda = -\ln(1-p)$, we have that

$$K = \lceil X \rceil = \left\lceil \frac{\ln(1-R)}{\ln(1-p)} \right\rceil \tag{2}$$

This is precisely the same function implemented by `geometricrv.m`. In short, the two methods for generating geometric (p) random samples are one in the same.

Problem Solutions – Chapter 5

Problem 5.1.1 Solution

(a) The probability $P[X \leq 2, Y \leq 3]$ can be found be evaluating the joint CDF $F_{X,Y}(x, y)$ at $x = 2$ and $y = 3$. This yields

$$P[X \leq 2, Y \leq 3] = F_{X,Y}(2, 3) = (1 - e^{-2})(1 - e^{-3}) \tag{1}$$

(b) To find the marginal CDF of X, $F_X(x)$, we simply evaluate the joint CDF at $y = \infty$.

$$F_X(x) = F_{X,Y}(x, \infty) = \begin{cases} 1 - e^{-x} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(c) Likewise for the marginal CDF of Y, we evaluate the joint CDF at $X = \infty$.

$$F_Y(y) = F_{X,Y}(\infty, y) = \begin{cases} 1 - e^{-y} & y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Problem 5.1.3 Solution

We wish to find $P[x_1 \leq X \leq x_2 \cup y_1 \leq Y \leq y_2]$. We define events

$$A = \{x_1 \leq X \leq x_2\}, \qquad B = \{y_1 \leq Y \leq y_2\} \tag{1}$$

so that $P[A \cup B]$ is the probability of observing an X, Y pair in the "cross" region. By Theorem 1.4(c),

$$P[A \cup B] = P[A] + P[B] - P[AB] \tag{2}$$

Keep in mind that the intersection of events A and B are all the outcomes such that both A and B occur, specifically, $AB = \{x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2\}$. It follows that

$$\begin{aligned} \mathrm{P}[A \cup B] = \mathrm{P}[x_1 \leq X \leq x_2] + \mathrm{P}[y_1 \leq Y \leq y_2] \\ - \mathrm{P}[x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2]. \end{aligned} \tag{3}$$

By Theorem 5.2,

$$\begin{aligned} &\mathrm{P}[x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2] \\ &\quad = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1). \end{aligned} \tag{4}$$

Expressed in terms of the marginal and joint CDFs,

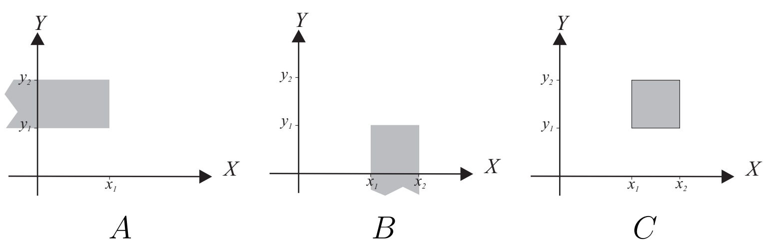
$$\begin{aligned} \mathrm{P}[A \cup B] = F_X(x_2) - F_X(x_1) + F_Y(y_2) - F_Y(y_1) \\ - F_{X,Y}(x_2, y_2) + F_{X,Y}(x_2, y_1) \\ + F_{X,Y}(x_1, y_2) - F_{X,Y}(x_1, y_1). \end{aligned} \tag{5}$$

Problem 5.1.5 Solution

In this problem, we prove Theorem 5.2 which states

$$\begin{aligned} \mathrm{P}[x_1 < X \leq x_2, y_1 < Y \leq y_2] = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) \\ - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1). \end{aligned} \tag{1}$$

(a) The events A, B, and C are



(2)

(b) In terms of the joint CDF $F_{X,Y}(x, y)$, we can write

$$P[A] = F_{X,Y}(x_1, y_2) - F_{X,Y}(x_1, y_1), \quad (3)$$
$$P[B] = F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_1), \quad (4)$$
$$P[A \cup B \cup C] = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_1, y_1). \quad (5)$$

(c) Since A, B, and C are mutually exclusive,

$$P[A \cup B \cup C] = P[A] + P[B] + P[C]. \quad (6)$$

However, since we want to express

$$P[C] = P[x_1 < X \leq x_2, y_1 < Y \leq y_2] \quad (7)$$

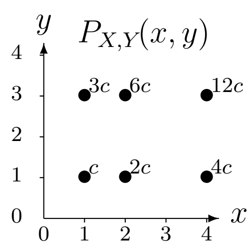
in terms of the joint CDF $F_{X,Y}(x, y)$, we write

$$\begin{aligned} P[C] &= P[A \cup B \cup C] - P[A] - P[B] \\ &= F_{X,Y}(x_2, y_2) - F_{X,Y}(x_1, y_2) - F_{X,Y}(x_2, y_1) + F_{X,Y}(x_1, y_1), \quad (8) \end{aligned}$$

which completes the proof of the theorem.

Problem 5.2.1 Solution

In this problem, it is helpful to label points with nonzero probability on the X, Y plane:



(a) We must choose c so the PMF sums to one:

$$\sum_{x=1,2,4} \sum_{y=1,3} P_{X,Y}(x,y) = c \sum_{x=1,2,4} x \sum_{y=1,3} y$$
$$= c\left[1(1+3) + 2(1+3) + 4(1+3)\right] = 28c. \quad (1)$$

Thus $c = 1/28$.

(b) The event $\{Y < X\}$ has probability

$$P[Y < X] = \sum_{x=1,2,4} \sum_{y<x} P_{X,Y}(x,y)$$
$$= \frac{1(0) + 2(1) + 4(1+3)}{28} = \frac{18}{28}. \quad (2)$$

(c) The event $\{Y > X\}$ has probability

$$P[Y > X] = \sum_{x=1,2,4} \sum_{y>x} P_{X,Y}(x,y)$$
$$= \frac{1(3) + 2(3) + 4(0)}{28} = \frac{9}{28}. \quad (3)$$

(d) There are two ways to solve this part. The direct way is to calculate

$$P[Y = X] = \sum_{x=1,2,4} \sum_{y=x} P_{X,Y}(x,y) = \frac{1(1) + 2(0)}{28} = \frac{1}{28}. \quad (4)$$

The indirect way is to use the previous results and the observation that

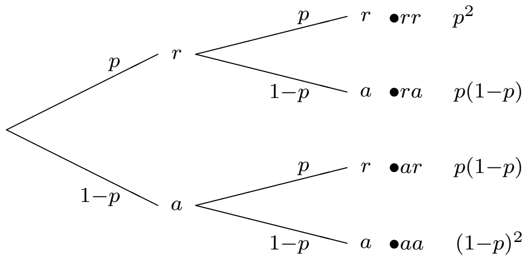
$$P[Y = X] = 1 - P[Y < X] - P[Y > X]$$
$$= 1 - 18/28 - 9/28 = 1/28. \quad (5)$$

(e)

$$P[Y = 3] = \sum_{x=1,2,4} P_{X,Y}(x,3)$$
$$= \frac{(1)(3) + (2)(3) + (4)(3)}{28} = \frac{21}{28} = \frac{3}{4}. \quad (6)$$

Problem 5.2.3 Solution

Let r (reject) and a (accept) denote the result of each test. There are four possible outcomes: rr, ra, ar, aa. The sample tree is



Now we construct a table that maps the sample outcomes to values of X and Y.

outcome	$P[\cdot]$	X	Y
rr	p^2	1	1
ra	$p(1-p)$	1	0
ar	$p(1-p)$	0	1
aa	$(1-p)^2$	0	0

(1)

This table is esentially the joint PMF $P_{X,Y}(x,y)$.

$$P_{X,Y}(x,y) = \begin{cases} p^2 & x=1, y=1, \\ p(1-p) & x=0, y=1, \\ p(1-p) & x=1, y=0, \\ (1-p)^2 & x=0, y=0, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Problem 5.2.5 Solution

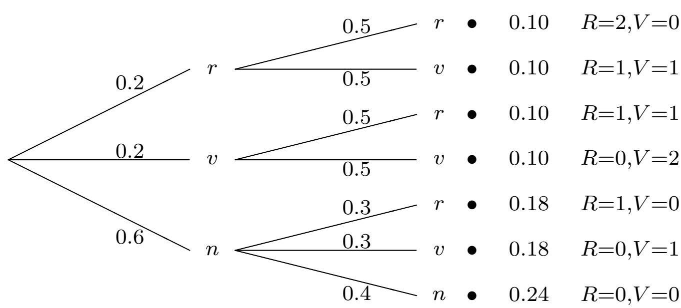
As the problem statement says, reasonable arguments can be made for the labels being X and Y or x and y. As we see in the arguments below, the lowercase choice of the text is somewhat arbitrary.

- *Lowercase axis labels:* For the lowercase labels, we observe that we are depicting the masses associated with the joint PMF $P_{X,Y}(x,y)$ whose arguments are x and y. Since the PMF function is defined in terms of x and y, the axis labels should be x and y.

- *Uppercase axis labels:* On the other hand, we are depicting the possible outcomes (labeled with their respective probabilities) of the pair of random variables X and Y. The corresponding axis labels should be X and Y just as in Figure 5.2. The fact that we have labeled the possible outcomes by their probabilities is irrelevant. Further, since the expression for the PMF $P_{X,Y}(x,y)$ given in the figure could just as well have been written $P_{X,Y}(\cdot,\cdot)$, it is clear that the lowercase x and y are not what matter.

Problem 5.2.7 Solution

(a) Using r, v, and n to denote the events that (r) Rutgers scores, (v) Villanova scores, and (n) neither scores in a particular minute, the tree is:



From the leaf probabilities, we can write down the joint PMF of R and V, taking care to note that the pair $R = 1, V = 1$ occurs for more than one outcome.

$P_{R,V}(r,v)$	$v=0$	$v=1$	$v=2$
$r=0$	0.24	0.18	0.1
$r=1$	0.18	0.2	0
$r=2$	0.1	0	0

(b)

$$\begin{aligned} \mathrm{P}[T] = \mathrm{P}[R=V] &= \sum_i P_{R,V}(i,i) \\ &= P_{R,V}(0,0) + P_{R,V}(1,1) \\ &= 0.24 + 0.2 = 0.44. \end{aligned} \tag{1}$$

(c) By summing across the rows of the table for $P_{R,V}(r,v)$, we obtain $P_R(0) = 0.52$, $P_R(1) = 0.38$, and $P_R(r)2 = 0.1$. The complete expresson for the marginal PMF is

r	0	1	2
$P_R(r)$	0.52	0.38	0.10

(d) For each pair (R,V), we have $G = R + V$. From first principles.

$$P_G(0) = P_{R,V}(0,0) = 0.24, \tag{2}$$

$$P_G(1) = P_{R,V}(1,0) + P_{R,V}(0,1) = 0.36, \tag{3}$$

$$P_G(2) = P_{R,V}(2,0) + P_{R,V}(1,1) + P_{R,V}(0,2) = 0.4. \tag{4}$$

The complete expression is

$$P_G(g) = \begin{cases} 0.24 & g = 0, \\ 0.36 & g = 1, \\ 0.40 & g = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Problem 5.2.9 Solution

Each circuit test produces an acceptable circuit with probability p. Let K denote the number of rejected circuits that occur in n tests and X is the number of acceptable circuits before the first reject. The joint PMF, $P_{K,X}(k,x) = \mathrm{P}[K = k, X = x]$ can be found by realizing that $\{K = k, X = x\}$ occurs if and only if the following events occur:

A The first x tests must be acceptable.

B Test $x+1$ must be a rejection since otherwise we would have $x+1$ acceptable at the beginnning.

C The remaining $n-x-1$ tests must contain $k-1$ rejections.

Since the events A, B and C are independent, the joint PMF for $x+k \le r$, $x \ge 0$ and $k \ge 0$ is

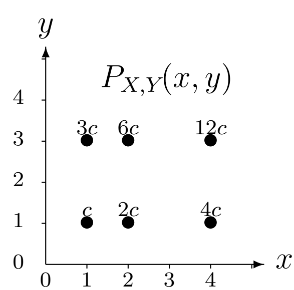
$$P_{K,X}(k,x) = \underbrace{p^x}_{\mathrm{P}[A]} \underbrace{(1-p)}_{\mathrm{P}[B]} \underbrace{\binom{n-x-1}{k-1}(1-p)^{k-1}p^{n-x-1-(k-1)}}_{\mathrm{P}[C]} \qquad (1)$$

After simplifying, a complete expression for the joint PMF is

$$P_{K,X}(k,x) = \begin{cases} \binom{n-x-1}{k-1} p^{n-k}(1-p)^k & x+k \le n, x \ge 0, k \ge 0, \\ 0 & \text{otherwise.} \end{cases} \qquad (2)$$

Problem 5.3.1 Solution

On the X, Y plane, the joint PMF $P_{X,Y}(x,y)$ is



By choosing $c = 1/28$, the PMF sums to one.

(a) The marginal PMFs of X and Y are

$$P_X(x) = \sum_{y=1,3} P_{X,Y}(x,y) = \begin{cases} 4/28 & x = 1, \\ 8/28 & x = 2, \\ 16/28 & x = 4, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

$$P_Y(y) = \sum_{x=1,2,4} P_{X,Y}(x,y) = \begin{cases} 7/28 & y = 1, \\ 21/28 & y = 3, \\ 0 & \text{otherwise}. \end{cases} \tag{2}$$

(b) The expected values of X and Y are

$$\mathrm{E}[X] = \sum_{x=1,2,4} x P_X(x) = (4/28) + 2(8/28) + 4(16/28) = 3, \tag{3}$$

$$\mathrm{E}[Y] = \sum_{y=1,3} y P_Y(y) = 7/28 + 3(21/28) = 5/2. \tag{4}$$

(c) The second moments are

$$\begin{aligned} \mathrm{E}\left[X^2\right] &= \sum_{x=1,2,4} x P_X(x) \\ &= 1^2(4/28) + 2^2(8/28) + 4^2(16/28) = 73/7, \end{aligned} \tag{5}$$

$$\mathrm{E}\left[Y^2\right] = \sum_{y=1,3} y P_Y(y) = 1^2(7/28) + 3^2(21/28) = 7. \tag{6}$$

The variances are

$$\mathrm{Var}[X] = \mathrm{E}\left[X^2\right] - (\mathrm{E}[X])^2 = 10/7, \tag{7}$$

$$\mathrm{Var}[Y] = \mathrm{E}\left[Y^2\right] - (\mathrm{E}[Y])^2 = 3/4. \tag{8}$$

The standard deviations are $\sigma_X = \sqrt{10/7}$ and $\sigma_Y = \sqrt{3/4}$.

Problem 5.3.3 Solution

We recognize that the given joint PMF is written as the product of two marginal PMFs $P_N(n)$ and $P_K(k)$ where

$$P_N(n) = \sum_{k=0}^{100} P_{N,K}(n,k) = \begin{cases} \frac{100^n e^{-100}}{n!} & n = 0, 1, \ldots, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

$$P_K(k) = \sum_{n=0}^{\infty} P_{N,K}(n,k) = \begin{cases} \binom{100}{k} p^k (1-p)^{100-k} & k = 0, 1, \ldots, 100, \\ 0 & \text{otherwise}. \end{cases} \tag{2}$$

Problem 5.3.5 Solution

The joint PMF of N, K is

$$P_{N,K}(n,k) = \begin{cases} (1-p)^{n-1} p/n & k = 1, 2, \ldots, n, \\ & n = 1, 2 \ldots, \\ o & \text{otherwise}. \end{cases} \tag{1}$$

For $n \geq 1$, the marginal PMF of N is

$$P_N(n) = \sum_{k=1}^{n} P_{N,K}(n,k) = \sum_{k=1}^{n} (1-p)^{n-1} p/n = (1-p)^{n-1} p. \tag{2}$$

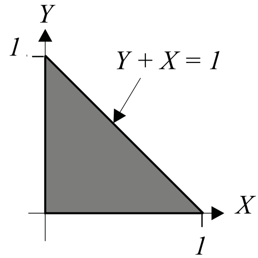
The marginal PMF of K is found by summing $P_{N,K}(n,k)$ over all possible N. Note that if $K = k$, then $N \geq k$. Thus,

$$P_K(k) = \sum_{n=k}^{\infty} \frac{1}{n} (1-p)^{n-1} p. \tag{3}$$

Unfortunately, this sum cannot be simplified.

Problem 5.4.1 Solution

(a) The joint PDF of X and Y is



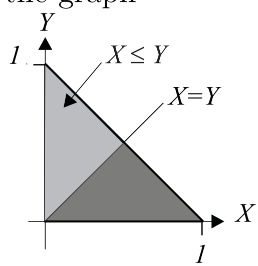
$$f_{X,Y}(x,y) = \begin{cases} c & x+y \leq 1, x, y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

To find the constant c we integrate over the region shown. This gives

$$\int_0^1 \int_0^{1-x} c\, dy\, dx = cx - \frac{cx}{2}\Big|_0^1 = \frac{c}{2} = 1. \tag{1}$$

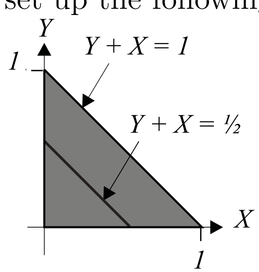
Therefore $c = 2$.

(b) To find the $P[X \leq Y]$ we look to integrate over the area indicated by the graph



$$\begin{aligned} P[X \leq Y] &= \int_0^{1/2} \int_x^{1-x} dy\, dx \\ &= \int_0^{1/2} (2-4x)\, dx \\ &= 1/2. \end{aligned} \tag{2}$$

(c) The probability $P[X + Y \leq 1/2]$ can be seen in the figure. Here we can set up the following integrals



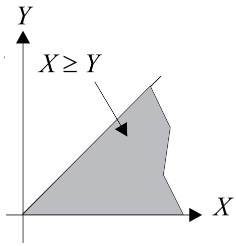
$$\begin{aligned} P[X + Y \leq 1/2] &= \int_0^{1/2} \int_0^{1/2-x} 2\, dy\, dx \\ &= \int_0^{1/2} (1-2x)\, dx \\ &= 1/2 - 1/4 = 1/4. \end{aligned} \tag{3}$$

Problem 5.4.3 Solution

The joint PDF of X and Y is

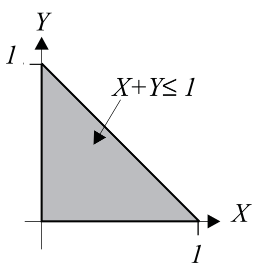
$$f_{X,Y}(x,y) = \begin{cases} 6e^{-(2x+3y)} & x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) The probability that $X \geq Y$ is:



$$\begin{aligned} \mathrm{P}[X \geq Y] &= \int_0^\infty \int_0^x 6e^{-(2x+3y)}\,dy\,dx \\ &= \int_0^\infty 2e^{-2x}\left(-e^{-3y}\Big|_{y=0}^{y=x}\right)dx \\ &= \int_0^\infty [2e^{-2x} - 2e^{-5x}]\,dx = 3/5. \end{aligned} \tag{2}$$

The probability $\mathrm{P}[X+Y \leq 1]$ is found by integrating over the region where $X+Y \leq 1$:



$$\begin{aligned} \mathrm{P}[X+Y \leq 1] &= \int_0^1 \int_0^{1-x} 6e^{-(2x+3y)}\,dy\,dx \\ &= \int_0^1 2e^{-2x}\left[-e^{-3y}\Big|_{y=0}^{y=1-x}\right]dx \\ &= \int_0^1 2e^{-2x}\left[1 - e^{-3(1-x)}\right]dx \\ &= -e^{-2x} - 2e^{x-3}\Big|_0^1 \\ &= 1 + 2e^{-3} - 3e^{-2}. \end{aligned} \tag{3}$$

(b) The event $\{\min(X,Y) \geq 1\}$ is the same as the event $\{X \geq 1, Y \geq 1\}$. Thus,

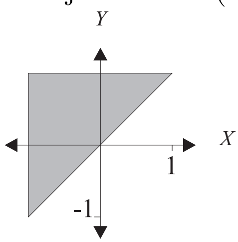
$$\mathrm{P}[\min(X,Y) \geq 1] = \int_1^\infty \int_1^\infty 6e^{-(2x+3y)}\,dy\,dx = e^{-(2+3)}. \tag{4}$$

(c) The event $\{\max(X,Y) \leq 1\}$ is the same as the event $\{X \leq 1, Y \leq 1\}$ so that

$$P[\max(X,Y) \leq 1] = \int_0^1 \int_0^1 6e^{-(2x+3y)}\, dy\, dx = (1-e^{-2})(1-e^{-3}). \tag{5}$$

Problem 5.5.1 Solution

The joint PDF (and the corresponding region of nonzero probability) are



$$f_{X,Y}(x,y) = \begin{cases} 1/2 & -1 \leq x \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a)

$$P[X > 0] = \int_0^1 \int_x^1 \frac{1}{2}\, dy\, dx = \int_0^1 \frac{1-x}{2}\, dx = 1/4 \tag{2}$$

This result can be deduced by geometry. The shaded triangle of the X, Y plane corresponding to the event $X > 0$ is $1/4$ of the total shaded area.

(b) For $x > 1$ or $x < -1$, $f_X(x) = 0$. For $-1 \leq x \leq 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_x^1 \frac{1}{2}\, dy = (1-x)/2. \tag{3}$$

The complete expression for the marginal PDF is

$$f_X(x) = \begin{cases} (1-x)/2 & -1 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

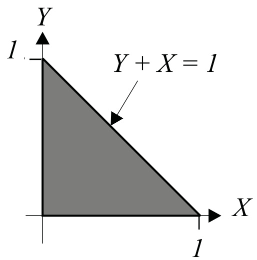
(c) From the marginal PDF $f_X(x)$, the expected value of X is

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \frac{1}{2}\int_{-1}^{1} x(1-x)\, dx$$
$$= \left.\frac{x^2}{4} - \frac{x^3}{6}\right|_{-1}^{1} = -\frac{1}{3}. \quad (5)$$

Problem 5.5.3 Solution

X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & x+y \leq 1, x, y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$



Using the figure to the left we can find the marginal PDFs by integrating over the appropriate regions.

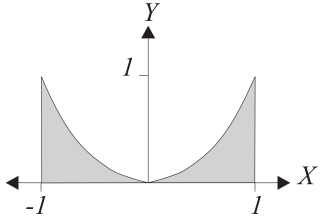
$$f_X(x) = \int_0^{1-x} 2\, dy = \begin{cases} 2(1-x) & 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

Likewise for $f_Y(y)$:

$$f_Y(y) = \int_0^{1-y} 2\, dx = \begin{cases} 2(1-y) & 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

Problem 5.5.5 Solution

The joint PDF of X and Y and the region of nonzero probability are



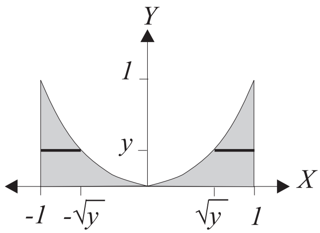
$$f_{X,Y}(x,y) = \begin{cases} 5x^2/2 & -1 \leq x \leq 1, 0 \leq y \leq x^2, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

We can find the appropriate marginal PDFs by integrating the joint PDF.

(a) The marginal PDF of X is

$$f_X(x) = \int_0^{x^2} \frac{5x^2}{2}\, dy = \begin{cases} 5x^4/2 & -1 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(b) Note that $f_Y(y) = 0$ for $y > 1$ or $y < 0$. For $0 \le y \le 1$,



$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx \\ &= \int_{-1}^{-\sqrt{y}} \frac{5x^2}{2}\, dx + \int_{\sqrt{y}}^{1} \frac{5x^2}{2}\, dx \\ &= 5(1 - y^{3/2})/3. \end{aligned} \tag{3}$$

The complete expression for the marginal CDF of Y is

$$f_Y(y) = \begin{cases} 5(1 - y^{3/2})/3 & 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Problem 5.5.7 Solution

First, we observe that Y has mean $\mu_Y = a\mu_X + b$ and variance $\text{Var}[Y] = a^2 \text{Var}[X]$. The covariance of X and Y is

$$\begin{aligned} \text{Cov}[X,Y] &= \text{E}[(X - \mu_X)(aX + b - a\mu_X - b)] \\ &= a\,\text{E}\left[(X - \mu_X)^2\right] \\ &= a\,\text{Var}[X]. \end{aligned} \tag{1}$$

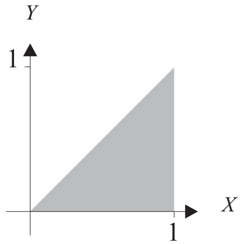
The correlation coefficient is

$$\rho_{X,Y} = \frac{\text{Cov}[X,Y]}{\sqrt{\text{Var}[X]}\sqrt{\text{Var}[Y]}} = \frac{a\,\text{Var}[X]}{\sqrt{\text{Var}[X]}\sqrt{a^2\,\text{Var}[X]}} = \frac{a}{|a|}. \tag{2}$$

When $a > 0$, $\rho_{X,Y} = 1$. When $a < 0$, $\rho_{X,Y} = -1$.

Problem 5.5.9 Solution

(a) The joint PDF of X and Y and the region of nonzero probability are



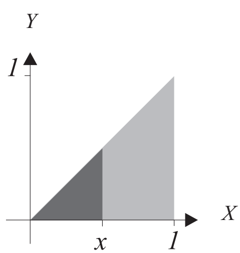
$$f_{X,Y}(x,y) = \begin{cases} cy & 0 \leq y \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(b) To find the value of the constant, c, we integrate the joint PDF over all x and y.

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy &= \int_0^1 \int_0^x cy\, dy\, dx = \int_0^1 \frac{cx^2}{2}\, dx \\ &= \left.\frac{cx^3}{6}\right|_0^1 = \frac{c}{6}. \end{aligned} \tag{2}$$

Thus $c = 6$.

(c) We can find the CDF $F_X(x) = P[X \leq x]$ by integrating the joint PDF over the event $X \leq x$. For $x < 0$, $F_X(x) = 0$. For $x > 1$, $F_X(x) = 1$. For $0 \leq x \leq 1$,

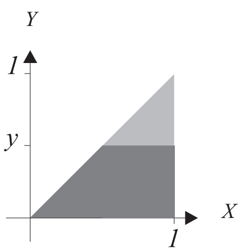


$$\begin{aligned} F_X(x) &= \iint_{x' \leq x} f_{X,Y}(x',y')\, dy'\, dx' \\ &= \int_0^x \int_0^{x'} 6y'\, dy'\, dx' \\ &= \int_0^x 3(x')^2\, dx' = x^3. \end{aligned} \tag{3}$$

The complete expression for the joint CDF is

$$F_X(x) = \begin{cases} 0 & x < 0, \\ x^3 & 0 \le x \le 1, \\ 1 & x \ge 1. \end{cases} \tag{4}$$

(d) Similarly, we find the CDF of Y by integrating $f_{X,Y}(x,y)$ over the event $Y \le y$. For $y < 0$, $F_Y(y) = 0$ and for $y > 1$, $F_Y(y) = 1$. For $0 \le y \le 1$,

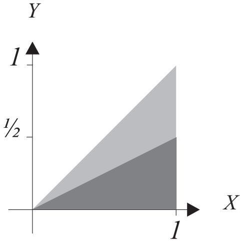


$$\begin{aligned} F_Y(y) &= \iint_{y' \le y} f_{X,Y}(x', y')\, dy'\, dx' \\ &= \int_0^y \int_{y'}^1 6y'\, dx'\, dy' \\ &= \int_0^y 6y'(1-y')\, dy' \\ &= 3(y')^2 - 2(y')^3 \Big|_0^y = 3y^2 - 2y^3. \end{aligned} \tag{5}$$

The complete expression for the CDF of Y is

$$F_Y(y) = \begin{cases} 0 & y < 0, \\ 3y^2 - 2y^3 & 0 \le y \le 1, \\ 1 & y > 1. \end{cases} \tag{6}$$

(e) To find $P[Y \le X/2]$, we integrate the joint PDF $f_{X,Y}(x,y)$ over the region $y \le x/2$.



$$\begin{aligned} P[Y \le X/2] &= \int_0^1 \int_0^{x/2} 6y\, dy\, dx \\ &= \int_0^1 3y^2 \Big|_0^{x/2} dx \\ &= \int_0^1 \frac{3x^2}{4}\, dx = 1/4. \end{aligned} \tag{7}$$

Problem 5.6.1 Solution

The key to this problem is understanding that "small order" and "big order" are synonyms for $W = 1$ and $W = 5$. Similarly, "vanilla", "chocolate", and "strawberry" correspond to the events $D = 20$, $D = 100$ and $D = 300$.

(a) The following table is given in the problem statement.

	vanilla	choc.	strawberry
small order	0.2	0.2	0.2
big order	0.1	0.2	0.1

This table can be translated directly into the joint PMF of W and D.

$P_{W,D}(w,d)$	$d = 20$	$d = 100$	$d = 300$
$w = 1$	0.2	0.2	0.2
$w = 5$	0.1	0.2	0.1

(1)

(b) We find the marginal PMF $P_D(d)$ by summing the columns of the joint PMF. This yields

$$P_D(d) = \begin{cases} 0.3 & d = 20, \\ 0.4 & d = 100, \\ 0.3 & d = 300, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(c) To check independence, we calculate the marginal PMF

$$P_W(w) = \sum_{d=20,100,300} P_{W,D}(w,d) = \begin{cases} 0.6 & w = 1, \\ 0.4 & w = 5, \end{cases} \tag{3}$$

and we check if $P_{W,D}(w,d) = P_W(w)P_D(d)$. In this case, we see that

$$P_{W,D}(1,20) = 0.2 \neq P_W(1)\,P_D(20) = (0.6)(0.3). \tag{4}$$

Hence W and D are dependent.

Problem 5.6.3 Solution

Flip a fair coin 100 times and let X be the number of heads in the first 75 flips and Y be the number of heads in the last 25 flips. We know that X and Y are independent and can find their PMFs easily.

$$P_X(x) = \binom{75}{x}(1/2)^{75}, \qquad P_Y(y) = \binom{25}{y}(1/2)^{25}. \tag{1}$$

The joint PMF of X and N can be expressed as the product of the marginal PMFs because we know that X and Y are independent.

$$P_{X,Y}(x,y) = \binom{75}{x}\binom{25}{y}(1/2)^{100}. \tag{2}$$

Problem 5.6.5 Solution

From the problem statement, X and Y have PDFs

$$f_X(x) = \begin{cases} 1/2 & 0 \le x \le 2, \\ 0 & \text{otherwise}, \end{cases}, \qquad f_Y(y) = \begin{cases} 1/5 & 0 \le y \le 5, \\ 0 & \text{otherwise}. \end{cases} \tag{1}$$

Since X and Y are independent, the joint PDF is

$$f_{X,Y}(x,y) = f_X(x)\,f_Y(y) = \begin{cases} 1/10 & 0 \le x \le 2, 0 \le y \le 5, \\ 0 & \text{otherwise}. \end{cases} \tag{2}$$

Problem 5.6.7 Solution

(a) We find k by the requirement that the joint PDF integrate to 1. That is,

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\;dx\,dy = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} (k+3x^2)\,dx\,dy$$
$$= \left(\int_{-1/2}^{1/2} dy\right)\left(\int_{-1/2}^{1/2} (k+3x^2)\,dx\right)$$
$$= kx + x^3\Big|_{x=-1/2}^{x=1/2} = k + 1/4 \tag{1}$$

Thus k=3/4.

(b) For $-1/2 \le x \le 1/2$, the marginal PDF of X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\;dy = \int_{-1/2}^{1/2} (k+3x^2)\,dy = k + 3x^2. \tag{2}$$

The complete expression for the PDF of X is

$$f_X(x) = \begin{cases} k+3x^2 & -1/2 \le x \le 1/2, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

(c) For $-1/2 \le y \le 1/2$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_X(x)\;dx$$
$$= \int_{-1/2}^{1/2} (k+3x^2)\,dx = kx + x^3\Big|_{x=-1/2}^{x=1/2} = k + 1/4. \tag{4}$$

Since $k = 3/4$, Y is a continuous uniform $(-1/2, 1/2)$ random variable with PDF

$$f_Y(y) = \begin{cases} 1 & -1/2 \le y \le 1/2, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

(d) We need to check whether $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. If you solved for k in part (a), then from (b) and (c) it is obvious that this equality holds and thus X and Y are independent. If you were not able to solve for k in part (a), testing whether $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ yields the requirement $1 = k + 1/4$. With some thought, you should have gone back to check that $k = 3/4$ solves part (a). This would lead to the correct conclusion that X and Y are independent.

Problem 5.6.9 Solution

This problem is quite straightforward. From Theorem 5.5, we can find the joint PDF of X and Y is

$$f_{X,Y}(x,y) = \frac{\partial^2 [F_X(x)\, F_Y(y)]}{\partial x\, \partial y} = \frac{\partial [f_X(x)\, F_Y(y)]}{\partial y} = f_X(x)\, f_Y(y)\,. \qquad (1)$$

Hence, $F_{X,Y}(x,y) = F_X(x)F_Y(y)$ implies that X and Y are independent.

If X and Y are independent, then

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)\,. \qquad (2)$$

By Definition 5.3,

$$\begin{aligned} F_{X,Y}(x,y) &= \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)\; dv\, du \\ &= \left(\int_{-\infty}^{x} f_X(u)\; du \right) \left(\int_{-\infty}^{y} f_Y(v)\; dv \right) \\ &= F_X(x)\, F_X(x)\,. \qquad (3) \end{aligned}$$

Problem 5.7.1 Solution

We recall that the joint PMF of W and D is

$P_{W,D}(w,d)$	$d = 20$	$d = 100$	$d = 300$
$w = 1$	0.2	0.2	0.2
$w = 5$	0.1	0.2	0.1

(1)

In terms of W and D, the cost (in cents) of a shipment is $C = WD$. The expected value of C is

$$\begin{aligned} \mathrm{E}[C] &= \sum_{w,d} wd P_{W,D}(w,d) \\ &= 1(20)(0.2) + 1(100)(0.2) + 1(300)(0.2) \\ &\quad + 5(20)(0.3) + 5(100)(0.4) + 5(300)(0.3) = 764 \text{ cents.} \end{aligned} \tag{2}$$

Problem 5.7.3 Solution

We solve this problem using Theorem 5.9. To do so, it helps to label each pair X, Y with the sum $W = X + Y$:

$P_{X,Y}(x,y)$	$y=1$	$y=2$	$y=3$	$y=4$
$x=5$	0.05 $W{=}6$	0.1 $W{=}7$	0.2 $W{=}8$	0.05 $W{=}9$
$x=6$	0.1 $W{=}7$	0.1 $W{=}8$	0.3 $W{=}9$	0.1 $W{=}10$

It follows that

$$\begin{aligned} \mathrm{E}[X+Y] &= \sum_{x,y} (x+y) P_{X,Y}(x,y) \\ &= 6(0.05) + 7(0.2) + 8(0.3) + 9(0.35) + 10(0.1) = 8.25. \end{aligned} \tag{1}$$

and

$$\begin{aligned} \mathrm{E}\left[(X+Y)^2\right] &= \sum_{x,y} (x+y)^2 P_{X,Y}(x,y) \\ &= 6^2(0.05) + 7^2(0.2) + 8^2(0.3) + 9^2(0.35) + 10^2(0.1) \\ &= 69.15. \end{aligned} \tag{2}$$

It follows that

$$\begin{aligned} \mathrm{Var}[X+Y] &= \mathrm{E}\left[(X+Y)^2\right] - (\mathrm{E}[X+Y])^2 \\ &= 69.15 - (8.25)^2 = 1.0875. \end{aligned} \tag{3}$$

An alternate approach would be to find the marginals $P_X(x)$ and $P_Y(y)$ and use these to calculate $\mathrm{E}[X]$, $\mathrm{E}[Y]$, $\mathrm{Var}[X]$ and $\mathrm{Var}[Y]$. However, we would still need to find the covariance of X and Y to find the variance of $X + Y$.

Problem 5.7.5 Solution

We start by observing that

$$\text{Cov}[X,Y] = \rho\sqrt{\text{Var}[X]\,\text{Var}[Y]} = 1.$$

This implies

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X,Y] = 1 + 4 + 2(1) = 7.$$

Problem 5.7.7 Solution

We will solve this problem when the probability of heads is p. For the fair coin, $p = 1/2$. The number X_1 of flips until the first heads and the number X_2 of additional flips for the second heads both have the geometric PMF

$$P_{X_1}(x) = P_{X_2}(x) = \begin{cases} (1-p)^{x-1}p & x = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Thus, $\text{E}[X_i] = 1/p$ and $\text{Var}[X_i] = (1-p)/p^2$. By Theorem 5.11,

$$\text{E}[Y] = \text{E}[X_1] - \text{E}[X_2] = 0. \tag{2}$$

Since X_1 and X_2 are independent, Theorem 5.17 says

$$\text{Var}[Y] = \text{Var}[X_1] + \text{Var}[-X_2] = \text{Var}[X_1] + \text{Var}[X_2] = \frac{2(1-p)}{p^2}. \tag{3}$$

Problem 5.7.9 Solution

(a) Since X and Y have zero expected value, $\text{Cov}[X,Y] = \text{E}[XY] = 3$, $\text{E}[U] = a\,\text{E}[X] = 0$ and $\text{E}[V] = b\,\text{E}[Y] = 0$. It follows that

$$\begin{aligned} \text{Cov}[U,V] &= \text{E}[UV] \\ &= \text{E}[abXY] \\ &= ab\,\text{E}[XY] = ab\,\text{Cov}[X,Y] = 3ab. \end{aligned} \tag{1}$$

(b) We start by observing that $\text{Var}[U] = a^2 \text{Var}[X]$ and $\text{Var}[V] = b^2 \text{Var}[Y]$. It follows that

$$\begin{aligned}
\rho_{U,V} &= \frac{\text{Cov}\left[U, V\right]}{\sqrt{\text{Var}[U]\,\text{Var}[V]}} \\
&= \frac{ab\,\text{Cov}\left[X, Y\right]}{\sqrt{a^2\,\text{Var}[X] b^2\,\text{Var}[Y]}} = \frac{ab}{\sqrt{a^2 b^2}}\rho_{X,Y} = \frac{1}{2}\frac{ab}{|ab|}. \qquad (2)
\end{aligned}$$

Note that $ab/|ab|$ is 1 if a and b have the same sign or is -1 if they have opposite signs.

(c) Since $\text{E}[X] = 0$,

$$\begin{aligned}
\text{Cov}\left[X, W\right] &= \text{E}\left[XW\right] - \text{E}\left[X\right]\text{E}\left[U\right] \\
&= \text{E}\left[XW\right] \\
&= \text{E}\left[X(aX + bY)\right] \\
&= a\,\text{E}\left[X^2\right] + b\,\text{E}\left[XY\right] \\
&= a\,\text{Var}[X] + b\,\text{Cov}\left[X, Y\right]. \qquad (3)
\end{aligned}$$

Since X and Y are identically distributed, $\text{Var}[X] = \text{Var}[Y]$ and

$$\frac{1}{2} = \rho_{X,Y} = \frac{\text{Cov}\left[X, Y\right]}{\sqrt{\text{Var}[X]\,\text{Var}[Y]}} = \frac{\text{Cov}\left[X, Y\right]}{\text{Var}[X]} = \frac{3}{\text{Var}[X]}. \qquad (4)$$

This implies $\text{Var}[X] = 6$. From (3), $\text{Cov}[X, W] = 6a + 3b$. Thus X and W are uncorrelated if $6a + 3b = 0$, or $b = -2a$.

Problem 5.7.11 Solution

(a) Since $\text{E}[V] = \text{E}[X] - \text{E}[Y] = 0$,

$$\begin{aligned}
\text{Var}[V] = \text{E}\left[V^2\right] &= \text{E}\left[X^2 - 2XY + Y^2\right] \\
&= \text{Var}[X] + \text{Var}[Y] - 2\,\text{Cov}\left[X, Y\right] \\
&= \text{Var}[X] + \text{Var}[Y] - 2\sigma_X \sigma_Y \rho_{X,Y} \\
&= 20 - 16\rho_{X,Y}. \qquad (1)
\end{aligned}$$

This is minimized when $\rho_{X,Y} = 1$. The minimum possible variance is 4. On other hand, $\mathrm{Var}[V]$ is maximized when $\rho_{X,Y} = -1$; the maximum possible variance is 36.

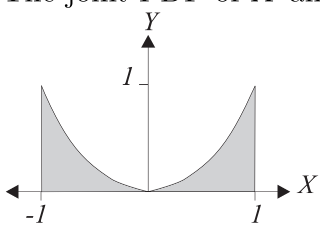
(b) Since $\mathrm{E}[W] = \mathrm{E}[X] - \mathrm{E}[2Y] = 0$,

$$\begin{aligned}\mathrm{Var}[W] = \mathrm{E}\left[W^2\right] &= \mathrm{E}\left[X^2 - 4XY + 4Y^2\right] \\ &= \mathrm{Var}[X] + 4\,\mathrm{Var}[Y] - 4\,\mathrm{Cov}[X,Y] \\ &= \mathrm{Var}[X] + 4\,\mathrm{Var}[Y] - 4\sigma_X\sigma_Y\rho_{X,Y} \\ &= 68 - 32\rho_{X,Y}. \end{aligned} \tag{2}$$

This is is minimized when $\rho_{X,Y} = 1$ and maximized when $\rho_{X,Y} = -1$. The minimum and maximum possible variances are 36 and 100.

Problem 5.7.13 Solution

The joint PDF of X and Y and the region of nonzero probability are



$$f_{X,Y}(x,y) = \begin{cases} 5x^2/2 & -1 \leq x \leq 1, 0 \leq y \leq x^2, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) The first moment of X is

$$\mathrm{E}[X] = \int_{-1}^{1}\int_{0}^{x^2} x\frac{5x^2}{2}\,dy\,dx = \int_{-1}^{1}\frac{5x^5}{2}\,dx = \left.\frac{5x^6}{12}\right|_{-1}^{1} = 0. \tag{2}$$

Since $\mathrm{E}[X] = 0$, the variance of X and the second moment are both

$$\mathrm{Var}[X] = \mathrm{E}\left[X^2\right] = \int_{-1}^{1}\int_{0}^{x^2} x^2\frac{5x^2}{2}\,dy\,dx = \left.\frac{5x^7}{14}\right|_{-1}^{1} = \frac{10}{14}. \tag{3}$$

(b) The first and second moments of Y are

$$\mathrm{E}[Y] = \int_{-1}^{1}\int_{0}^{x^2} y\frac{5x^2}{2}\,dy\,dx = \frac{5}{14}, \tag{4}$$

$$\mathrm{E}\left[Y^2\right] = \int_{-1}^{1}\int_{0} x^2y^2\frac{5x^2}{2}\,dy\,dx = \frac{5}{26}. \tag{5}$$

Therefore, $\mathrm{Var}[Y] = 5/26 - (5/14)^2 = .0576$.

(c) Since $\mathrm{E}[X] = 0$, $\mathrm{Cov}[X, Y] = \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y] = \mathrm{E}[XY]$. Thus,

$$\mathrm{Cov}[X, Y] = \mathrm{E}[XY] = \int_{1}^{1}\int_{0}^{x^2} xy\frac{5x^2}{2}\,dy\,dx = \int_{-1}^{1}\frac{5x^7}{4}\,dx = 0. \tag{6}$$

(d) The expected value of the sum $X + Y$ is

$$\mathrm{E}[X+Y] = \mathrm{E}[X] + \mathrm{E}[Y] = \frac{5}{14}. \tag{7}$$

(e) By Theorem 5.12, the variance of $X + Y$ is

$$\begin{aligned}\mathrm{Var}[X+Y] &= \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X, Y] \\ &= 5/7 + 0.0576 = 0.7719.\end{aligned} \tag{8}$$

Problem 5.7.15 Solution

Since $\mathrm{E}[Y] = \mathrm{E}[X] = \mathrm{E}[Z] = 0$, we know that

$$\mathrm{Var}[Y] = \mathrm{E}\left[Y^2\right], \qquad \mathrm{Var}[X] = \mathrm{E}\left[X^2\right], \qquad \mathrm{Var}[Z] = \mathrm{E}\left[Z^2\right], \tag{1}$$

and

$$\mathrm{Cov}[X, Y] = \mathrm{E}[XY] = \mathrm{E}[X(X+Z)] = \mathrm{E}\left[X^2\right] + \mathrm{E}[XZ]. \tag{2}$$

Since X and Z are independent, $\mathrm{E}[XZ] = \mathrm{E}[X]\,\mathrm{E}[Z] = 0$, implying

$$\mathrm{Cov}[X, Y] = \mathrm{E}\left[X^2\right]. \tag{3}$$

Independence of X and Z also implies $\text{Var}[Y] = \text{Var}[X] + \text{Var}[Z]$, or equivalently, since the signals are all zero-mean,

$$\text{E}\left[Y^2\right] = \text{E}\left[X^2\right] + \text{E}\left[Z^2\right]. \tag{4}$$

These facts imply that the correlation coefficient is

$$\begin{aligned}\rho_{X,Y} = \frac{\text{E}\left[X^2\right]}{\sqrt{\text{E}\left[X^2\right]\text{E}\left[Y^2\right]}} &= \frac{\text{E}\left[X^2\right]}{\sqrt{\text{E}\left[X^2\right]\left(\text{E}\left[X^2\right] + \text{E}\left[Z^2\right]\right)}} \\ &= \frac{1}{\sqrt{1 + \frac{\text{E}[Z^2]}{\text{E}[X^2]}}}. \end{aligned} \tag{5}$$

In terms of the signal to noise ratio, we have

$$\rho_{X,Y} = \frac{1}{\sqrt{1 + \frac{1}{\Gamma}}}. \tag{6}$$

We see in (6) that $\rho_{X,Y} \to 1$ as $\Gamma \to \infty$.

Problem 5.8.1 Solution

Independence of X and Z implies

$$\text{Var}[Y] = \text{Var}[X] + \text{Var}[Z] = 1^2 + 4^2 = 17. \tag{1}$$

Since $\text{E}[X] = \text{E}[Y] = 0$, the covariance of X and Y is

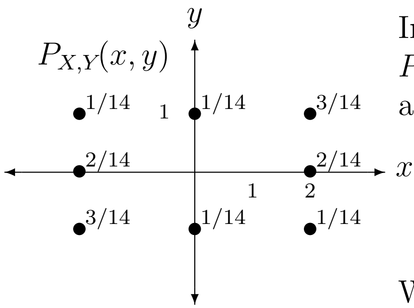
$$\text{Cov}\left[X, Y\right] = \text{E}\left[XY\right] = \text{E}\left[X(X + Z)\right] = \text{E}\left[X^2\right] + \text{E}\left[XZ\right]. \tag{2}$$

Since X and Z are independent, $\text{E}[XZ] = \text{E}[X]\,\text{E}[Z] = 0$. Since $\text{E}[X] = 0$, $\text{E}[X^2] = \text{Var}[X] = 1$. Thus $\text{Cov}[X, Y] = 1$. Finally, the correlation coefficient is

$$\rho_{X,Y} = \frac{\text{Cov}\left[X, Y\right]}{\sqrt{\text{Var}[X]}\sqrt{\text{Var}[Y]}} = \frac{1}{\sqrt{17}} = 0.243. \tag{3}$$

Since $\rho_{X,Y} \neq 0$, we conclude that X and Y are dependent.

Problem 5.8.3 Solution



In Problem 5.2.1, we found the joint PMF $P_{X,Y}(x,y)$ shown here. The expected values and variances were found to be

$$E[X] = 0, \qquad \text{Var}[X] = 24/7, \tag{1}$$

$$E[Y] = 0, \qquad \text{Var}[Y] = 5/7. \tag{2}$$

We need these results to solve this problem.

(a) Random variable $W = 2^{XY}$ has expected value

$$\begin{aligned} E\left[2^{XY}\right] &= \sum_{x=-2,0,2} \sum_{y=-1,0,1} 2^{xy} P_{X,Y}(x,y) \\ &= 2^{-2(-1)}\frac{3}{14} + 2^{-2(0)}\frac{2}{14} + 2^{-2(1)}\frac{1}{14} + 2^{0(-1)}\frac{1}{14} + 2^{0(1)}\frac{1}{14} \\ &\quad + 2^{2(-1)}\frac{1}{14} + 2^{2(0)}\frac{2}{14} + 2^{2(1)}\frac{3}{14} \\ &= 61/28. \end{aligned} \tag{3}$$

(b) The correlation of X and Y is

$$\begin{aligned} r_{X,Y} &= \sum_{x=-2,0,2} \sum_{y=-1,0,1} xy P_{X,Y}(x,y) \\ &= \frac{-2(-1)(3)}{14} + \frac{-2(0)(2)}{14} + \frac{-2(1)(1)}{14} \\ &\quad + \frac{2(-1)(1)}{14} + \frac{2(0)(2)}{14} + \frac{2(1)(3)}{14} \\ &= 4/7. \end{aligned} \tag{4}$$

(c) The covariance of X and Y is

$$\text{Cov}[X,Y] = \text{E}[XY] - \text{E}[X]\,\text{E}[Y] = 4/7. \tag{5}$$

(d) The correlation coefficient is

$$\rho_{X,Y} = \frac{\text{Cov}[X,Y]}{\sqrt{\text{Var}[X]\,\text{Var}[Y]}} = \frac{2}{\sqrt{30}}. \tag{6}$$

(e) By Theorem 5.16,

$$\begin{aligned}\text{Var}[X+Y] &= \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X,Y]\\ &= \frac{24}{7} + \frac{5}{7} + 2\frac{4}{7} = \frac{37}{7}.\end{aligned} \tag{7}$$

Problem 5.8.5 Solution

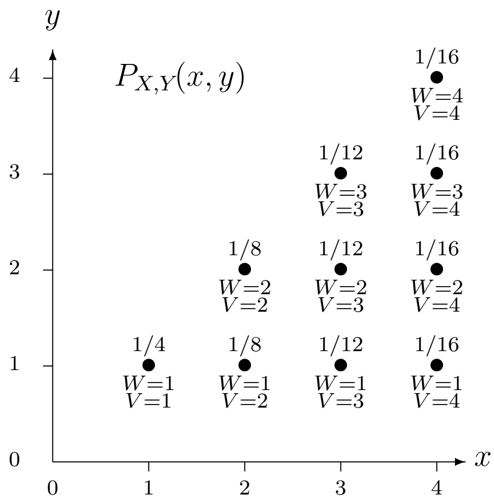
X and Y are independent random variables with PDFs

$$f_X(x) = \begin{cases} \frac{1}{3}e^{-x/3} & x \geq 0,\\ 0 & \text{otherwise,}\end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{2}e^{-y/2} & y \geq 0,\\ 0 & \text{otherwise.}\end{cases} \tag{1}$$

(a) Since X and Y are exponential random variables with parameters $\lambda_X = 1/3$ and $\lambda_Y = 1/2$, Appendix A tells us that $\text{E}[X] = 1/\lambda_X = 3$ and $\text{E}[Y] = 1/\lambda_Y = 2$. Since X and Y are independent, the correlation is $\text{E}[XY] = \text{E}[X]\,\text{E}[Y] = 6$.

(b) Since X and Y are independent, $\text{Cov}[X,Y] = 0$.

Problem 5.8.7 Solution



To solve this problem, we identify the values of $W = \min(X, Y)$ and $V = \max(X, Y)$ for each possible pair x, y. Here we observe that $W = Y$ and $V = X$. This is a result of the underlying experiment in that given $X = x$, each $Y \in \{1, 2, \ldots, x\}$ is equally likely. Hence $Y \leq X$. This implies $\min(X, Y) = Y$ and $\max(X, Y) = X$.

Using the results from Problem 5.8.6, we have the following answers.

(a) The expected values are

$$E[W] = E[Y] = 7/4, \qquad E[V] = E[X] = 5/2. \tag{1}$$

(b) The variances are

$$\text{Var}[W] = \text{Var}[Y] = 41/48, \qquad \text{Var}[V] = \text{Var}[X] = 5/4. \tag{2}$$

(c) The correlation is

$$r_{W,V} = E[WV] = E[XY] = r_{X,Y} = 5. \tag{3}$$

(d) The covariance of W and V is

$$\text{Cov}[W, V] = \text{Cov}[X, Y] = 10/16. \tag{4}$$

(e) The correlation coefficient is

$$\rho_{W,V} = \rho_{X,Y} = \frac{10/16}{\sqrt{(41/48)(5/4)}}. \approx 0.605. \tag{5}$$

Problem 5.8.9 Solution

(a) Since $\hat{X} = aX$, $\mathrm{E}[\hat{X}] = a\,\mathrm{E}[X]$ and

$$\begin{aligned} \mathrm{E}\left[\hat{X} - \mathrm{E}\left[\hat{X}\right]\right] &= \mathrm{E}\left[aX - a\,\mathrm{E}\left[X\right]\right] \\ &= \mathrm{E}\left[a(X - \mathrm{E}\left[X\right])\right] = a\,\mathrm{E}\left[X - \mathrm{E}\left[X\right]\right]. \end{aligned} \tag{1}$$

In the same way, since $\hat{Y} = cY$, $\mathrm{E}[\hat{Y}] = c\,\mathrm{E}[Y]$ and

$$\begin{aligned} \mathrm{E}\left[\hat{Y} - \mathrm{E}\left[\hat{Y}\right]\right] &= \mathrm{E}\left[aY - a\,\mathrm{E}\left[Y\right]\right] \\ &= \mathrm{E}\left[a(Y - \mathrm{E}\left[Y\right])\right] = a\,\mathrm{E}\left[Y - \mathrm{E}\left[Y\right]\right]. \end{aligned} \tag{2}$$

(b)

$$\begin{aligned} \mathrm{Cov}\left[\hat{X}, \hat{Y}\right] &= \mathrm{E}\left[(\hat{X} - \mathrm{E}\left[\hat{X}\right])(\hat{Y} - \mathrm{E}\left[\hat{Y}\right])\right] \\ &= \mathrm{E}\left[ac(X - \mathrm{E}\left[X\right])(Y - \mathrm{E}\left[Y\right])\right] \\ &= ac\,\mathrm{E}\left[(X - \mathrm{E}\left[X\right])(Y - \mathrm{E}\left[Y\right])\right] = ac\,\mathrm{Cov}\left[X, Y\right]. \end{aligned} \tag{3}$$

(c) Since $\hat{X} = aX$,

$$\begin{aligned} \mathrm{Var}[\hat{X}] &= \mathrm{E}\left[(\hat{X} - \mathrm{E}\left[\hat{X}\right])^2\right] \\ &= \mathrm{E}\left[(aX - a\,\mathrm{E}\left[X\right])^2\right] \\ &= \mathrm{E}\left[a^2(X - \mathrm{E}\left[X\right])^2\right] = a^2\,\mathrm{Var}[X]. \end{aligned} \tag{4}$$

In the very same way,

$$\begin{aligned}\operatorname{Var}[\hat{Y}] &= \mathrm{E}\left[(\hat{Y} - \mathrm{E}\left[\hat{Y}\right])^2\right] \\ &= \mathrm{E}\left[(cY - c\,\mathrm{E}\left[Y\right])^2\right] \\ &= \mathrm{E}\left[c^2(Y - \mathrm{E}\left[Y\right])^2\right] = c^2 \operatorname{Var}[Y]. \end{aligned} \tag{5}$$

(d)

$$\begin{aligned}\rho_{\hat{X},\hat{Y}} &= \frac{\operatorname{Cov}\left[\hat{X},\hat{Y}\right]}{\sqrt{\operatorname{Var}[\hat{X}]\operatorname{Var}[\hat{Y}]}} \\ &= \frac{ac\operatorname{Cov}\left[X,Y\right]}{\sqrt{a^2c^2\operatorname{Var}[X]\operatorname{Var}[Y]}} = \frac{\operatorname{Cov}\left[X,Y\right]}{\sqrt{\operatorname{Var}[X]\operatorname{Var}[Y]}} = \rho_{X,Y}. \end{aligned} \tag{6}$$

Problem 5.9.1 Solution

X and Y have joint PDF

$$f_{X,Y}(x,y) = ce^{-(x^2/8)-(y^2/18)}. \tag{1}$$

The omission of any limits for the PDF indicates that it is defined over all x and y. We know that $f_{X,Y}(x,y)$ is in the form of the bivariate Gaussian distribution so we look to Definition 5.10 and attempt to find values for σ_Y, σ_X, $\mathrm{E}[X]$, $\mathrm{E}[Y]$ and ρ. First, we know that the constant is

$$c = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}. \tag{2}$$

Because the exponent of $f_{X,Y}(x,y)$ doesn't contain any cross terms we know that ρ must be zero, and we are left to solve the following for $\mathrm{E}[X]$, $\mathrm{E}[Y]$, σ_X, and σ_Y:

$$\left(\frac{x - \mathrm{E}\left[X\right]}{\sigma_X}\right)^2 = \frac{x^2}{8}, \qquad \left(\frac{y - \mathrm{E}\left[Y\right]}{\sigma_Y}\right)^2 = \frac{y^2}{18}. \tag{3}$$

From which we can conclude that

$$\mathrm{E}[X] = \mathrm{E}[Y] = 0, \tag{4}$$

$$\sigma_X = \sqrt{8}, \tag{5}$$

$$\sigma_Y = \sqrt{18}. \tag{6}$$

Putting all the pieces together, we find that $c = \frac{1}{24\pi}$. Since $\rho = 0$, we also find that X and Y are independent.

Problem 5.9.3 Solution

FALSE: Let $Y = X_1 + aX_2$. If $\mathrm{E}[X_2] = 0$, then $\mathrm{E}[Y] = \mathrm{E}[X_1]$ for all a. Since Y is Gaussian (by Theorem 5.21), $\mathrm{P}[Y \leq y] = 1/2$ if and only if $\mathrm{E}[Y] = \mathrm{E}[X_1] = y$. We obtain a simple counterexample when $y = \mathrm{E}[X_1] - 1$.

Note that the answer would be true if we knew that $\mathrm{E}[X_2] \neq 0$. Also note that the variance of W will depend on the correlation between X_1 and X_2, but the correlation is irrelevant in the above argument.

Problem 5.9.5 Solution

For the joint PDF

$$f_{X,Y}(x,y) = ce^{-(2x^2-4xy+4y^2)}, \tag{1}$$

we proceed as in Problem 5.9.1 to find values for σ_Y, σ_X, $\mathrm{E}[X]$, $\mathrm{E}[Y]$ and ρ.

(a) First, we try to solve the following equations

$$\left(\frac{x - \mathrm{E}[X]}{\sigma_X}\right)^2 = 4(1-\rho^2)x^2, \tag{2}$$

$$\left(\frac{y - \mathrm{E}[Y]}{\sigma_Y}\right)^2 = 8(1-\rho^2)y^2, \tag{3}$$

$$\frac{2\rho}{\sigma_X\sigma_Y} = 8(1-\rho^2). \tag{4}$$

The first two equations yield $\mathrm{E}[X] = \mathrm{E}[Y] = 0$.

(b) To find the correlation coefficient ρ, we observe that

$$\sigma_X = 1/\sqrt{4(1-\rho^2)}, \qquad \sigma_Y = 1/\sqrt{8(1-\rho^2)}. \tag{5}$$

Using σ_X and σ_Y in the third equation yields $\rho = 1/\sqrt{2}$.

(c) Since $\rho = 1/\sqrt{2}$, now we can solve for σ_X and σ_Y.

$$\sigma_X = 1/\sqrt{2}, \qquad \sigma_Y = 1/2. \tag{6}$$

(d) From here we can solve for c.

$$c = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} = \frac{2}{\pi}. \tag{7}$$

(e) X and Y are dependent because $\rho \neq 0$.

Problem 5.9.7 Solution

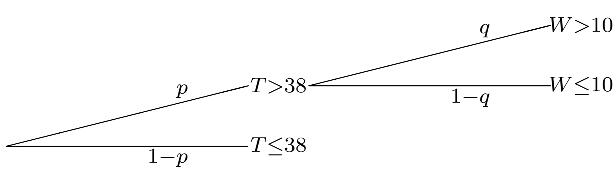
(a) The person's temperature is high with probability

$$p = P[T > 38] = P[T - 37 > 38 - 37] = 1 - \Phi(1) = 0.159. \tag{1}$$

Given that the temperature is high, then W is measured. Since $\rho = 0$, W and T are independent and

$$q = P[W > 10] = P\left[\frac{W-7}{2} > \frac{10-7}{2}\right] = 1 - \Phi(1.5) = 0.067. \tag{2}$$

The tree for this experiment is



The probability the person is ill is

$$\begin{aligned}\mathrm{P}[I] &= \mathrm{P}[T > 38, W > 10] \\ &= \mathrm{P}[T > 38]\,\mathrm{P}[W > 10] = pq = 0.0107. \end{aligned} \tag{3}$$

(b) The general form of the bivariate Gaussian PDF is

$$f_{W,T}(w,t) = \frac{\exp\left[-\frac{\left(\frac{w-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(w-\mu_1)(t-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{t-\mu_2}{\sigma_2}\right)^2}{2(1-\rho^2)}\right]}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}. \tag{4}$$

With $\mu_1 = \mathrm{E}[W] = 7$, $\sigma_1 = \sigma_W = 2$, $\mu_2 = \mathrm{E}[T] = 37$ and $\sigma_2 = \sigma_T = 1$ and $\rho = 1/\sqrt{2}$, we have

$$f_{W,T}(w,t) = \frac{\exp\left[-\frac{(w-7)^2}{4} - \frac{\sqrt{2}(w-7)(t-37)}{2} + (t-37)^2\right]}{2\pi\sqrt{2}}. \tag{5}$$

To find the conditional probability $\mathrm{P}[I|T = t]$, we need to find the conditional PDF of W given $T = t$. The direct way is simply to use algebra to find

$$f_{W|T}(w|t) = \frac{f_{W,T}(w,t)}{f_T(t)}. \tag{6}$$

The required algebra is essentially the same as that needed to prove Theorem 7.15. Its easier just to apply Theorem 7.15 which says that given $T = t$, the conditional distribution of W is Gaussian with

$$\mathrm{E}[W|T = t] = \mathrm{E}[W] + \rho\frac{\sigma_W}{\sigma_T}(t - \mathrm{E}[T]), \tag{7}$$

$$\mathrm{Var}[W|T = t] = \sigma_W^2(1-\rho^2). \tag{8}$$

Plugging in the various parameters gives

$$\mathrm{E}[W|T=t] = 7 + \sqrt{2}(t-37), \quad \text{and} \quad \mathrm{Var}[W|T=t] = 2. \tag{9}$$

Using this conditional mean and variance, we obtain the conditional Gaussian PDF

$$f_{W|T}(w|t) = \frac{1}{\sqrt{4\pi}} e^{-\left(w-(7+\sqrt{2}(t-37))\right)^2/4}. \tag{10}$$

Given $T=t$, the conditional probability the person is declared ill is

$$\begin{aligned}
\mathrm{P}[I|T=t] &= \mathrm{P}[W > 10|T=t] \\
&= \mathrm{P}\left[\frac{W-(7+\sqrt{2}(t-37))}{\sqrt{2}} > \frac{10-(7+\sqrt{2}(t-37))}{\sqrt{2}}\right] \\
&= \mathrm{P}\left[Z > \frac{3-\sqrt{2}(t-37)}{\sqrt{2}}\right] \\
&= Q\left(\frac{3\sqrt{2}}{2} - (t-37)\right).
\end{aligned} \tag{11}$$

Problem 5.9.9 Solution

The key to this problem is knowing that a sum of independent Gaussian random variables is Gaussian.

(a) First we calculate the mean and variance of Y. Since the expectation of the sum is always the sum of the expectations,

$$\mathrm{E}[Y] = \frac{1}{2}\mathrm{E}[X_1] + \frac{1}{2}\mathrm{E}[X_2] = 74. \tag{1}$$

Since X_1 and X_2 are independent, the variance of the sum is the sum of the variances:

$$\begin{aligned}
\mathrm{Var}[Y] &= \mathrm{Var}[X_1/2] + \mathrm{Var}[X_2/2] \\
&= \frac{1}{4}\mathrm{Var}[X_1] + \frac{1}{4}\mathrm{Var}[X_2] = 16^2/2 = 128.
\end{aligned} \tag{2}$$

Thus, Y has standard deviation $\sigma_Y = 8\sqrt{2}$. Since we know Y is Gaussian,

$$P[A] = P[Y \geq 90] = P\left[\frac{Y-74}{8\sqrt{2}} \geq \frac{90-74}{8\sqrt{2}}\right] = Q(\sqrt{2}) = 1 - \Phi(\sqrt{2}). \tag{3}$$

(b) With weight factor w, Y is still Gaussian, but with

$$E[Y] = w\,E[X_1] + (1-w)\,E[X_2] = 74, \tag{4}$$

$$\begin{aligned}\text{Var}[Y] &= \text{Var}[wX_1] + \text{Var}[(1-w)X_2] \\ &= w^2\,\text{Var}[X_1] + (1-w)^2\,\text{Var}[X_2] = 16^2[w^2 + (1-w)^2].\end{aligned} \tag{5}$$

Thus, Y has standard deviation $\sigma_Y = 16\sqrt{w^2 + (1-w)^2}$. Since we know Y is Gaussian,

$$P[A] = P[Y \geq 90] = P\left[\frac{Y-74}{\sigma_Y} \geq \frac{90-74}{\sigma_Y}\right] = 1 - \Phi\left(\frac{1}{\sqrt{w^2 + (1-w)^2}}\right). \tag{6}$$

Since $\Phi(x)$ is increasing in x, $1 - \Phi(x)$ is decreasing in x. To maximize $P[A]$, we want the argument of the Φ function to be as small as possible. Thus we want $w^2 + (1-w)^2$ to be as large as possible. Note that $w^2 + (1-w)^2$ is a parabola with a minimum at $w = 1/2$; it is maximized at $w = 1$ or $w = 0$.

That is, if you need to get exam scores around 74, and you need 90 to get an A, then you need to get lucky to get an A. With $w = 0$, you just need to be lucky on exam 1. With $w = 0$, you need only be lucky on exam 2. It's more likely that you are lucky on one exam rather than two.

(c) With the maximum of the two exams,

$$\begin{aligned}
P[A] &= P[\max(X_1, X_2) > 90] \\
&= 1 - P[\max(X_1, X_2) \le 90] \\
&= 1 - P[X_1 \le 90, X_2 \le 90] \\
&= 1 - P[X_1 \le 90]\, P[X_2 \le 90] \\
&= 1 - (P[X_1 \le 90])^2 \\
&= 1 - \left[\Phi\left(\frac{90-74}{16}\right)\right]^2 = 1 - \Phi^2(1).
\end{aligned} \tag{7}$$

(d) Let N_c and N_a denote the number of A's awarded under the rules in part (c) and part (a). The expected additional number of A's is

$$\begin{aligned}
E[N_c - N_a] &= 100[1 - \Phi^2(1)] - 100[1 - \Phi(\sqrt{2})] \\
&= 100[\Phi(\sqrt{2}) - \Phi^2(1)] = 21.3.
\end{aligned} \tag{8}$$

Problem 5.9.11 Solution

From Equation (5.68), we can write the bivariate Gaussian PDF as

$$f_{X,Y}(x,y) = \frac{1}{\sigma_X\sqrt{2\pi}} e^{-(x-\mu_X)^2/2\sigma_X^2} \frac{1}{\tilde{\sigma}_Y\sqrt{2\pi}} e^{-(y-\tilde{\mu}_Y(x))^2/2\tilde{\sigma}_Y^2}, \tag{1}$$

where $\tilde{\mu}_Y(x) = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)$ and $\tilde{\sigma}_Y = \sigma_Y\sqrt{1-\rho^2}$. However, the definitions of $\tilde{\mu}_Y(x)$ and $\tilde{\sigma}_Y$ are not particularly important for this exercise. When we integrate the joint PDF over all x and y, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy \\
&\quad = \int_{-\infty}^{\infty} \frac{1}{\sigma_X\sqrt{2\pi}} e^{-(x-\mu_X)^2/2\sigma_X^2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\tilde{\sigma}_Y\sqrt{2\pi}} e^{-(y-\tilde{\mu}_Y(x))^2/2\tilde{\sigma}_Y^2}\,dy}_{1}\,dx \\
&\quad = \int_{-\infty}^{\infty} \frac{1}{\sigma_X\sqrt{2\pi}} e^{-(x-\mu_X)^2/2\sigma_X^2}\,dx.
\end{aligned} \tag{2}$$

The marked integral equals 1 because for each value of x, it is the integral of a Gaussian PDF of one variable over all possible values. In fact, it is the integral of the conditional PDF $f_{Y|X}(y|x)$ over all possible y. To complete the proof, we see that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = \int_{-\infty}^{\infty} \frac{1}{\sigma_X\sqrt{2\pi}} e^{-(x-\mu_X)^2/2\sigma_X^2}\,dx = 1. \tag{3}$$

since the remaining integral is the integral of the marginal Gaussian PDF $f_X(x)$ over all possible x.

Problem 5.10.1 Solution

The repair of each laptop can be viewed as an independent trial with four possible outcomes corresponding to the four types of needed repairs.

(a) Since the four types of repairs are mutually exclusive choices and since 4 laptops are returned for repair, the joint distribution of $N_1, \ldots, N_4$ is the multinomial PMF

$$\begin{aligned} P_{N_1,\ldots,N_4}(n_1,\ldots,n_4) &= \binom{4}{n_1,n_2,n_3,n_4} p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4} \\ &= \binom{4}{n_1,n_2,n_3,n_4} \left(\frac{8}{15}\right)^{n_1} \left(\frac{4}{15}\right)^{n_2} \left(\frac{2}{15}\right)^{n_3} \left(\frac{1}{15}\right)^{n_4}. \end{aligned} \tag{1}$$

(b) Let L_2 denote the event that exactly two laptops need LCD repairs. Thus $P[L_2] = P_{N_1}(2)$. Since each laptop requires an LCD repair with probability $p_1 = 8/15$, the number of LCD repairs, N_1, is a binomial $(4, 8/15)$ random variable with PMF

$$P_{N_1}(n_1) = \binom{4}{n_1}(8/15)^{n_1}(7/15)^{4-n_1}. \tag{2}$$

The probability that two laptops need LCD repairs is

$$P_{N_1}(2) = \binom{4}{2}(8/15)^2(7/15)^2 = 0.3717. \tag{3}$$

(c) A repair is type (2) with probability $p_2 = 4/15$. A repair is type (3) with probability $p_3 = 2/15$; otherwise a repair is type "other" with probability $p_o = 9/15$. Define X as the number of "other" repairs needed. The joint PMF of X, N_2, N_3 is the multinomial PMF

$$P_{N_2,N_3,X}(n_2,n_3,x) = \binom{4}{n_2, n_3, x}\left(\frac{4}{15}\right)^{n_2}\left(\frac{2}{15}\right)^{n_3}\left(\frac{9}{15}\right)^{x}. \tag{4}$$

However, Since $X + 4 - N_2 - N_3$, we observe that

$$\begin{aligned} P_{N_2,N_3}(n_2,n_3) &= P_{N_2,N_3,X}(n_2,n_3,4-n_2-n_3) \\ &= \binom{4}{n_2, n_3, 4-n_2-n_3}\left(\frac{4}{15}\right)^{n_2}\left(\frac{2}{15}\right)^{n_3}\left(\frac{9}{15}\right)^{4-n_2-n_3} \\ &= \left(\frac{9}{15}\right)^{4}\binom{4}{n_2, n_3, 4-n_2-n_3}\left(\frac{4}{9}\right)^{n_2}\left(\frac{2}{9}\right)^{n_3} \end{aligned} \tag{5}$$

Similarly, since each repair is a motherboard repair with probability $p_2 = 4/15$, the number of motherboard repairs has binomial PMF

$$P_{N_2}(n_2)\, n_2 = \binom{4}{n_2}\left(\frac{4}{15}\right)^{n_2}\left(\frac{11}{15}\right)^{4-n_2} \tag{6}$$

Finally, the probability that more laptops require motherboard repairs than keyboard repairs is

$$\begin{aligned} \text{P}[N_2 > N_3] &= P_{N_2,N_3}(1,0) + P_{N_2,N_3}(2,0) + P_{N_2,N_3}(2,1) \\ &\quad + P_{N_2}(3) + P_{N_2}(4). \end{aligned} \tag{7}$$

where we use the fact that if $N_2 = 3$ or $N_2 = 4$, then we must have $N_2 > N_3$. Inserting the various probabilities, we obtain

$$\begin{aligned} \text{P}[N_2 > N_3] &= \left(\frac{9}{15}\right)^{4}\left(4\left(\frac{4}{9}\right) + 6\left(\frac{16}{81}\right) + 12\left(\frac{32}{81}\right)\right) \\ &\quad + 4\left(\frac{4}{15}\right)^{3}\left(\frac{11}{15}\right) + \left(\frac{4}{15}\right)^{4} \\ &= \frac{8{,}656}{16{,}875} \approx 0.5129. \end{aligned} \tag{8}$$

Problem 5.10.3 Solution

(a) In terms of the joint PDF, we can write the joint CDF as

$$F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_{X_1,\ldots,X_n}(y_1,\ldots,y_n)\, dy_1 \cdots dy_n. \tag{1}$$

However, simplifying the above integral depends on the values of each x_i. In particular, $f_{X_1,\ldots,X_n}(y_1,\ldots,y_n) = 1$ if and only if $0 \leq y_i \leq 1$ for each i. Since $F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = 0$ if any $x_i < 0$, we limit, for the moment, our attention to the case where $x_i \geq 0$ for all i. In this case, some thought will show that we can write the limits in the following way:

$$\begin{aligned} F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) &= \int_0^{\max(1,x_1)} \cdots \int_0^{\min(1,x_n)} dy_1 \cdots dy_n \\ &= \min(1,x_1)\min(1,x_2)\cdots\min(1,x_n). \end{aligned} \tag{2}$$

A complete expression for the CDF of $X_1,\ldots,X_n$ is

$$F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \begin{cases} \prod_{i=1}^n \min(1,x_i) & 0 \leq x_i, i = 1,2,\ldots,n, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

(b) For $n = 3$,

$$\begin{aligned} 1 - P\left[\min_i X_i \leq 3/4\right] &= P\left[\min_i X_i > 3/4\right] \\ &= P\left[X_1 > 3/4, X_2 > 3/4, X_3 > 3/4\right] \\ &= \int_{3/4}^1 \int_{3/4}^1 \int_{3/4}^1 dx_1\, dx_2\, dx_3 \\ &= (1 - 3/4)^3 = 1/64. \end{aligned} \tag{4}$$

Thus $P[\min_i X_i \leq 3/4] = 63/64$.

Problem 5.10.5 Solution

The value of each byte is an independent experiment with 255 possible outcomes. Each byte takes on the value b_i with probability $p_i = p = 1/255$. The joint PMF of $N_0, \ldots, N_{255}$ is the multinomial PMF

$$\begin{aligned} P_{N_0,\ldots,N_{255}}(n_0,\ldots,n_{255}) &= \binom{10000}{n_0, n_1, \ldots, n_{255}} p^{n_0} p^{n_1} \cdots p^{n_{255}} \\ &= \binom{10000}{n_0, n_1, \ldots, n_{255}} (1/255)^{10000}. \end{aligned} \tag{1}$$

Keep in mind that the multinomial coefficient is defined for nonnegative integers n_i such that

$$\binom{10000}{n_0, n_1, \ldots, n_{255}} = \begin{cases} \frac{10000}{n_0! n_1! \cdots n_{255}!} & n_0 + \cdots + n_{255} = 10000, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

To evaluate the joint PMF of N_0 and N_1, we define a new experiment with three categories: b_0, b_1 and "other." Let $\hat{N}$ denote the number of bytes that are "other." In this case, a byte is in the "other" category with probability $\hat{p} = 253/255$. The joint PMF of N_0, N_1, and $\hat{N}$ is

$$P_{N_0,N_1,\hat{N}}(n_0,n_1,\hat{n}) = \binom{10000}{n_0, n_1, \hat{n}} \left(\frac{1}{255}\right)^{n_0} \left(\frac{1}{255}\right)^{n_1} \left(\frac{253}{255}\right)^{\hat{n}}. \tag{3}$$

Now we note that the following events are one in the same:

$$\{N_0 = n_0, N_1 = n_1\} = \left\{N_0 = n_0, N_1 = n_1, \hat{N} = 10000 - n_0 - n_1\right\}. \tag{4}$$

Hence, for non-negative integers n_0 and n_1 satisfying $n_0 + n_1 \leq 10000$,

$$\begin{aligned} &P_{N_0,N_1}(n_0,n_1) \\ &\quad = P_{N_0,N_1,\hat{N}}(n_0,n_1,10000 - n_0 - n_1) \\ &\quad = \frac{10000!}{n_0! n_1! (10000 - n_0 - n_1)!} \left(\frac{1}{255}\right)^{n_0+n_1} \left(\frac{253}{255}\right)^{10000-n_0-n_1}. \end{aligned} \tag{5}$$

Problem 5.10.7 Solution

We could use Theorem 8.2 to skip several of the steps below. However, it is also nice to solve the problem from first principles.

(a) We first derive the CDF of V. Since each X_i is non-negative, V is non-negative, thus $F_V(v) = 0$ for $v < 0$. For $v \geq 0$, independence of the X_i yields

$$\begin{aligned} F_V(v) = P[V \leq v] &= P[\min(X_1, X_2, X_3) \leq v] \\ &= 1 - P[\min(X_1, X_2, X_3) > v] \\ &= 1 - P[X_1 > v, X_2 > v, X_3 > v] \\ &= 1 - P[X_1 > v] P[X_2 > v] P[X_3 > v]. \end{aligned} \tag{1}$$

Note that independence of the X_i was used in the final step. Since each X_i is an exponential (λ) random variable, for $v \geq 0$,

$$P[X_i > v] = P[X > v] = 1 - F_X(v) = e^{-\lambda v}. \tag{2}$$

Thus,

$$F_V(v) = 1 - \left(e^{-\lambda v}\right)^3 = 1 - e^{-3\lambda v}. \tag{3}$$

The complete expression for the CDF of V is

$$F_V(v) = \begin{cases} 0 & v < 0, \\ 1 - e^{-3\lambda v} & v \geq 0. \end{cases} \tag{4}$$

By taking the derivative of the CDF, we obtain the PDF

$$f_V(v) = \begin{cases} 0 & v < 0, \\ 3\lambda e^{-3\lambda v} & v \geq 0. \end{cases} \tag{5}$$

We see that V is an exponential (3λ) random variable.

(b) The CDF of W is found in a similar way. Since each X_i is non-negative, W is non-negative, thus $F_W(w) = 0$ for $w < 0$. For $w \geq 0$, independence of the X_i yields

$$\begin{aligned} F_W(w) = \mathrm{P}[W \leq w] &= \mathrm{P}[\max(X_1, X_2, X_3) \leq w] \\ &= \mathrm{P}[X_1 \leq w, X_2 \leq w, X_3 \leq w] \\ &= \mathrm{P}[X_1 \leq w]\,\mathrm{P}[X_2 \leq w]\,\mathrm{P}[X_3 \leq w]. \end{aligned} \tag{6}$$

Since each X_i is an exponential (λ) random variable, for $w \geq 0$,

$$\mathrm{P}[X_i \leq w] = 1 - e^{-\lambda w}. \tag{7}$$

Thus, $F_W(w) = (1 - e^{-\lambda w})^3$ for $w \geq 0$. The complete expression for the CDF of Y is

$$F_W(w) = \begin{cases} 0 & w < 0, \\ \left(1 - e^{-\lambda w}\right)^3 & w \geq 0. \end{cases} \tag{8}$$

By taking the derivative of the CDF, we obtain the PDF

$$f_W(w) = \begin{cases} 0 & w < 0, \\ 3(1 - e^{-\lambda w})^2 e^{-\lambda w} & w \geq 0. \end{cases} \tag{9}$$

Problem 5.10.9 Solution

(a) This is straightforward:

$$\begin{aligned} F_{U_n}(u) &= \mathrm{P}[\max(X_1, \ldots, X_n) \leq u] \\ &= \mathrm{P}[X_1 \leq u, \ldots, X_n \leq u] \\ &= \mathrm{P}[X_1 \leq u]\,\mathrm{P}[X_2 \leq u] \cdots \mathrm{P}[X_n \leq u] = (F_X(u))^n. \end{aligned} \tag{1}$$

(b) This is also straightforward.

$$
\begin{aligned}
F_{L_n}(l) &= 1 - \mathrm{P}\left[\min(X_1, \ldots, X_n) > l\right] \\
&= 1 - \mathrm{P}\left[X_1 > l, \ldots, X_n > l\right] \\
&= 1 - \mathrm{P}\left[X_1 > l\right] \mathrm{P}\left[X_2 > l\right] \cdots \mathrm{P}\left[X_n > l\right] \\
&= 1 - (1 - F_X(l))^n . \qquad (2)
\end{aligned}
$$

(c) This part is a little more difficult. The key is to identify the "easy" joint probability

$$
\begin{aligned}
&\mathrm{P}\left[L_n > l, U_n \leq u\right] \\
&\quad = \mathrm{P}\left[\min(X_1, \ldots, X_n) \geq l, \max(X_1, \ldots, X_n) \leq u\right] \\
&\quad = \mathrm{P}\left[l < X_i \leq u, i = 1, 2, \ldots, n\right] \\
&\quad = \mathrm{P}\left[l < X_1 \leq u\right] \cdots \mathrm{P}\left[l < X_n \leq u\right] \\
&\quad = \left[F_X(u) - F_X(l)\right]^n . \qquad (3)
\end{aligned}
$$

Next we observe by the law of total probability that

$$
\mathrm{P}\left[U_n \leq u\right] = \mathrm{P}\left[L_n > l, U_n \leq u\right] + \mathrm{P}\left[L_n \leq l, U_n \leq u\right] . \qquad (4)
$$

The final term on the right side is the joint CDF we desire and using the expressions we derived for the first two terms, we obtain

$$
\begin{aligned}
F_{L_n,U_n}(l, u) &= \mathrm{P}\left[U_n \leq u\right] - \mathrm{P}\left[L_n > l, U_n \leq u\right] \\
&= \left[F_X(u)\right]^n - \left[F_X(u) - F_X(l)\right]^n . \qquad (5)
\end{aligned}
$$

Problem 5.10.11 Solution

Since $U_1, \ldots, U_n$ are iid uniform $(0, 1)$ random variables,

$$
f_{U_1,\ldots,U_n}(u_1, \ldots, u_n) = \begin{cases} 1/T^n & 0 \leq u_i \leq 1; i = 1, 2, \ldots, n, \\ 0 & \text{otherwise.} \end{cases} \qquad (1)
$$

Since $U_1, \ldots, U_n$ are continuous, $\mathrm{P}[U_i = U_j] = 0$ for all $i \neq j$. For the same reason, $\mathrm{P}[X_i = X_j] = 0$ for $i \neq j$. Thus we need only to consider the case when $x_1 < x_2 < \cdots < x_n$.

To understand the claim, it is instructive to start with the $n = 2$ case. In this case, $(X_1, X_2) = (x_1, x_2)$ (with $x_1 < x_2$) if either $(U_1, U_2) = (x_1, x_2)$ or $(U_1, U_2) = (x_2, x_1)$. For infinitesimal Δ,

$$\begin{aligned} f_{X_1,X_2}(x_1, x_2)\,\Delta^2 &= \mathrm{P}\left[x_1 < X_1 \leq x_1 + \Delta, x_2 < X_2 \leq x_2 + \Delta\right] \\ &= \mathrm{P}\left[x_1 < U_1 \leq x_1 + \Delta, x_2 < U_2 \leq x_2 + \Delta\right] \\ &\qquad + \mathrm{P}\left[x_2 < U_1 \leq x_2 + \Delta, x_1 < U_2 \leq x_1 + \Delta\right] \\ &= f_{U_1,U_2}(x_1, x_2)\,\Delta^2 + f_{U_1,U_2}(x_2, x_1)\,\Delta^2. \end{aligned} \tag{2}$$

We see that for $0 \leq x_1 < x_2 \leq 1$ that

$$f_{X_1,X_2}(x_1, x_2) = 2/T^n. \tag{3}$$

For the general case of n uniform random variables, we define

$$\boldsymbol{\pi} = \begin{bmatrix}\pi(1) & \ldots & \pi(n)\end{bmatrix}'. \tag{4}$$

as a permutation vector of the integers $1, 2, \ldots, n$ and Π as the set of $n!$ possible permutation vectors. In this case, the event

$$\{X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\} \tag{5}$$

occurs if

$$U_1 = x_{\pi(1)}, U_2 = x_{\pi(2)}, \ldots, U_n = x_{\pi(n)} \tag{6}$$

for any permutation $\boldsymbol{\pi} \in \Pi$. Thus, for $0 \leq x_1 < x_2 < \cdots < x_n \leq 1$,

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)\,\Delta^n = \sum_{\boldsymbol{\pi} \in \Pi} f_{U_1,\ldots,U_n}\left(x_{\pi(1)}, \ldots, x_{\pi(n)}\right)\Delta^n. \tag{7}$$

Since there are $n!$ permutations and $f_{U_1,\ldots,U_n}(x_{\pi(1)}, \ldots, x_{\pi(n)}) = 1/T^n$ for each permutation $\boldsymbol{\pi}$, we can conclude that

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = n!/T^n. \tag{8}$$

Since the order statistics are necessarily ordered, $f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = 0$ unless $x_1 < \cdots < x_n$.

Problem 5.11.1 Solution

The script `imagepmf` in Example 5.26 generates the grid variables `SX`, `SY`, and `PXY`. Recall that for each entry in the grid, `SX`. `SY` and `PXY` are the corresponding values of x, y and $P_{X,Y}(x,y)$. Displaying them as adjacent column vectors forms the list of all possible pairs x, y and the probabilities $P_{X,Y}(x,y)$. Since any MATLAB vector or matrix `x` is reduced to a column vector with the command `x(:)`, the following simple commands will generate the list:

```
>> format rat;
>> imagepmf;
>> [SX(:) SY(:) PXY(:)]
ans =
     800          400          1/5
    1200          400          1/20
    1600          400           0
     800          800          1/20
    1200          800          1/5
    1600          800          1/10
     800         1200          1/10
    1200         1200          1/10
    1600         1200          1/5
>>
```

Note that the command `format rat` wasn't necessary; it just formats the output as rational numbers, i.e., ratios of integers, which you may or may not find esthetically pleasing.

Problem 5.11.3 Solution

In this problem `randn(1,2)` generates a 1×2 array of independent Gaussian $(0,1)$ random variables. If this array is $\begin{bmatrix} X & Y \end{bmatrix}$, then $W = 4(X+Y)$ and

$$\text{Var}[W] = \text{Var}[4(X+Y)] = 16(\text{Var}[X] + \text{Var}[Y]) = 16(1+1) = 32.$$

Problem 5.11.5 Solution

By following the formulation of Problem 5.2.6, the code to set up the sample grid is reasonably straightforward:

```
function [SX,SY,PXY]=circuits(n,p);
%Usage: [SX,SY,PXY]=circuits(n,p);
%   (See Problem 4.12.4)
[SX,SY]=ndgrid(0:n,0:n);
PXY=0*SX;
PXY(find((SX==n) & (SY==n)))=p^n;
for y=0:(n-1),
    I=find((SY==y) &(SX>=SY) &(SX<n));
    PXY(I)=(p^y)*(1-p)* ...
        binomialpmf(n-y-1,p,SX(I)-y);
end;
```

The only catch is that for a given value of y, we need to calculate the binomial probability of $x - y$ successes in $(n - y - 1)$ trials. We can do this using the function call

```
binomialpmf(n-y-1,p,x-y)
```

However, this function expects the argument `n-y-1` to be a scalar. As a result, we must perform a separate call to `binomialpmf` for each value of `y`.

An alternate solution is direct calculation of the PMF $P_{X,Y}(x, y)$ in Problem 5.2.6. In this case, we calculate $m!$ using the MATLAB function `gamma(m+1)`. Because, `gamma(x)` function will calculate the gamma function for each element in a vector `x`, we can calculate the PMF without any loops:

```
function [SX,SY,PXY]=circuits2(n,p);
%Usage: [SX,SY,PXY]=circuits2(n,p);
%   (See Problem 4.12.4)
[SX,SY]=ndgrid(0:n,0:n);
PXY=0*SX;
PXY(find((SX==n) & (SY==n)))=p^n;
I=find((SY<=SX) &(SX<n));
PXY(I)=(gamma(n-SY(I))./(gamma(SX(I)-SY(I)+1)...
      .*gamma(n-SX(I)))).*(p.^SX(I)).*((1-p).^(n-SX(I)));
```

Some experimentation with `cputime` will show that `circuits2(n,p)` runs much faster than `circuits(n,p)`. As is typical, the `for` loop in `circuit` results in time wasted running the MATLAB interpretor and in regenerating the binomial PMF in each cycle.

To finish the problem, we need to calculate the correlation coefficient

$$\rho_{X,Y} = \frac{\text{Cov}[X,Y]}{\sigma_X \sigma_Y}. \tag{1}$$

In fact, this is one of those problems where a general solution is better than a specific solution. The general problem is that given a pair of finite random variables described by the grid variables `SX`, `SY` and PMF `PXY`, we wish to calculate the correlation coefficient

This problem is solved in a few simple steps. First we write a function that calculates the expected value of a finite random variable.

```
function ex=finiteexp(sx,px);
%Usage: ex=finiteexp(sx,px)
%returns the expected value E[X]
%of finite random variable X described
%by samples sx and probabilities px
ex=sum((sx(:)).*(px(:)));
```

Note that `finiteexp` performs its calculations on the sample values `sx` and probabilities `px` using the column vectors `sx(:)` and `px(:)`. As a result, we can use the same `finiteexp` function when the random variable is represented by grid variables.We can build on `finiteexp` to calculate the variance using `finitevar`:

```
function v=finitevar(sx,px);
%Usage: ex=finitevar(sx,px)
%   returns the variance Var[X]
%   of finite random variables X described by
%   samples sx and probabilities px
ex2=finiteexp(sx.^2,px);
ex=finiteexp(sx,px);
v=ex2-(ex^2);
```

Putting these pieces together, we can calculate the correlation coefficient.

```
function rho=finitecoeff(SX,SY,PXY);
%Usage:  rho=finitecoeff(SX,SY,PXY)
%Calculate the correlation coefficient rho of
%finite random variables X and Y
ex=finiteexp(SX,PXY); vx=finitevar(SX,PXY);
ey=finiteexp(SY,PXY); vy=finitevar(SY,PXY);
R=finiteexp(SX.*SY,PXY);
rho=(R-ex*ey)/sqrt(vx*vy);
```

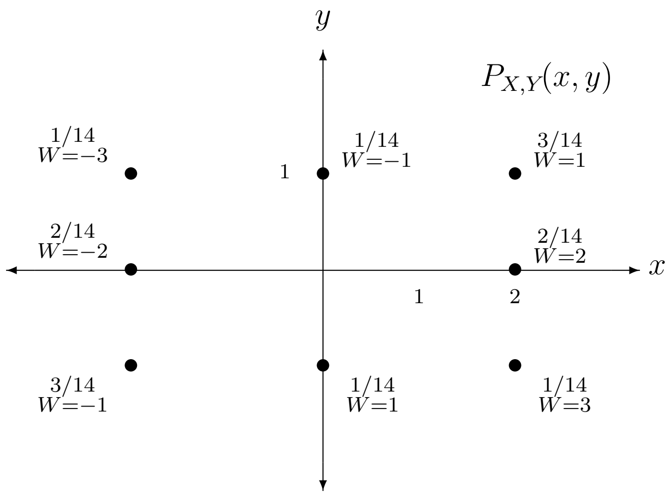
Calculating the correlation coefficient of X and Y, is now a two line exercise..

```
>> [SX,SY,PXY]=circuits2(50,0.9);
>> rho=finitecoeff(SX,SY,PXY)
rho =
    0.4451
>>
```

Problem Solutions – Chapter 6

Problem 6.1.1 Solution

In this problem, it is helpful to label possible points X, Y along with the corresponding values of $W = X - Y$. From the statement of Problem 6.1.1,



To find the PMF of W, we simply add the probabilities associated with each possible value of W:

$$P_W(-3) = P_{X,Y}(-2,1) = 1/14, \tag{1}$$
$$P_W(-2) = P_{X,Y}(-2,0) = 2/14, \tag{2}$$
$$P_W(-1) = P_{X,Y}(-2,-1) + P_{X,Y}(0,1) = 4/14, \tag{3}$$
$$P_W(1) = P_{X,Y}(0,-1) + P_{X,Y}(2,1) = 4/14, \tag{4}$$
$$P_W(2) = P_{X,Y}(2,0) = 2/14, \tag{5}$$
$$P_W(3) = P_{X,Y}(2,1) = 1/14. \tag{6}$$

For all other values of w, $P_W(w) = 0$. A table for the PMF of W is

w	-3	-2	-1	1	2	3
$P_W(w)$	$1/14$	$2/14$	$4/14$	$4/14$	$2/14$	$1/14$

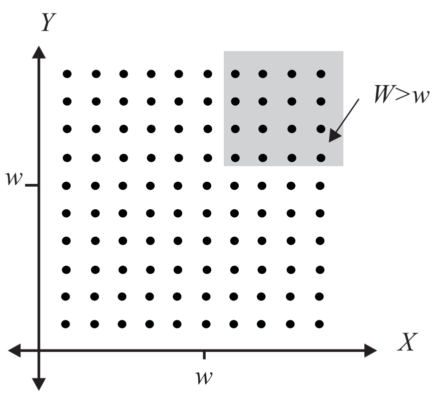
Problem 6.1.3 Solution

This is basically a trick problem. It looks like this problem should be in Section 6.5 since we have to find the PMF of the sum $L = N + M$. However, this problem is an special case since N and M are both binomial with the same success probability $p = 0.4$.

In this case, N is the number of successes in 100 independent trials with success probability $p = 0.4$. M is the number of successes in 50 independent trials with success probability $p = 0.4$. Thus $L = M + N$ is the number of successes in 150 independent trials with success probability $p = 0.4$. We conclude that L has the binomial $(n = 150, p = 0.4)$ PMF

$$P_L(l) = \binom{150}{l} (0.4)^l (0.6)^{150-l}. \tag{1}$$

Problem 6.1.5 Solution



The x, y pairs with nonzero probability are shown in the figure. For $w = 0, 1, \ldots, 10$, we observe that

$$\begin{aligned} P[W > w] &= P[\min(X, Y) > w] \\ &= P[X > w, Y > w] \\ &= 0.01(10 - w)^2. \end{aligned} \tag{1}$$

To find the PMF of W, we observe that for $w = 1, \ldots, 10$,

$$\begin{aligned} P_W(w) &= \mathrm{P}[W > w-1] - \mathrm{P}[W > w] \\ &= 0.01[(10-w-1)^2 - (10-w)^2] = 0.01(21-2w). \end{aligned} \tag{2}$$

The complete expression for the PMF of W is

$$P_W(w) = \begin{cases} 0.01(21-2w) & w = 1, 2, \ldots, 10, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Problem 6.2.1 Solution

Since $0 \leq X \leq 1$, $Y = X^2$ satisfies $0 \leq Y \leq 1$. We can conclude that $F_Y(y) = 0$ for $y < 0$ and that $F_Y(y) = 1$ for $y \geq 1$. For $0 \leq y < 1$,

$$F_Y(y) = \mathrm{P}\left[X^2 \leq y\right] = \mathrm{P}\left[X \leq \sqrt{y}\right]. \tag{1}$$

Since $f_X(x) = 1$ for $0 \leq x \leq 1$, we see that for $0 \leq y < 1$,

$$\mathrm{P}\left[X \leq \sqrt{y}\right] = \int_0^{\sqrt{y}} dx = \sqrt{y} \tag{2}$$

Hence, the CDF of Y is

$$F_Y(y) = \begin{cases} 0 & y < 0, \\ \sqrt{y} & 0 \leq y < 1, \\ 1 & y \geq 1. \end{cases} \tag{3}$$

By taking the derivative of the CDF, we obtain the PDF

$$f_Y(y) = \begin{cases} 1/(2\sqrt{y}) & 0 \leq y < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Problem 6.2.3 Solution

Note that T has the continuous uniform PDF

$$f_T(t) = \begin{cases} 1/15 & 60 \leq t \leq 75, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The rider's maximum possible speed is $V = 3000/60 = 50$ km/hr while the rider's minimum speed is $V = 3000/75 = 40$ km/hr. For $40 \leq v \leq 50$,

$$\begin{aligned} F_V(v) &= P\left[\frac{3000}{T} \leq v\right] = P\left[T \geq \frac{3000}{v}\right] \\ &= \int_{3000/v}^{75} \frac{1}{15}\, dt = \left.\frac{t}{15}\right|_{3000/v}^{75} = 5 - \frac{200}{v}. \end{aligned} \tag{2}$$

Thus the CDF, and via a derivative, the PDF are

$$F_V(v) = \begin{cases} 0 & v < 40, \\ 5 - 200/v & 40 \leq v \leq 50, \\ 1 & v > 50, \end{cases} \quad f_V(v) = \begin{cases} 0 & v < 40, \\ 200/v^2 & 40 \leq v \leq 50, \\ 0 & v > 50. \end{cases} \tag{3}$$

Problem 6.2.5 Solution

Since X is non-negative, $W = X^2$ is also non-negative. Hence for $w < 0$, $f_W(w) = 0$. For $w \geq 0$,

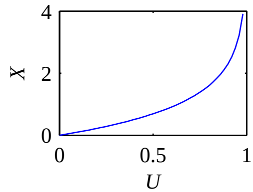
$$\begin{aligned} F_W(w) = P[W \leq w] &= P\left[X^2 \leq w\right] \\ &= P[X \leq w] \\ &= 1 - e^{-\lambda\sqrt{w}}. \end{aligned} \tag{1}$$

Taking the derivative with respect to w yields $f_W(w) = \lambda e^{-\lambda\sqrt{w}}/(2\sqrt{w})$. The complete expression for the PDF is

$$f_W(w) = \begin{cases} \frac{\lambda e^{-\lambda\sqrt{w}}}{2\sqrt{w}} & w \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Problem 6.2.7 Solution

Before solving for the PDF, it is helpful to have a sketch of the function $X = -\ln(1-U)$.



(a) From the sketch, we observe that X will be nonnegative. Hence $F_X(x) = 0$ for $x < 0$. Since U has a uniform distribution on $[0,1]$, for $0 \le u \le 1$, $P[U \le u] = u$. We use this fact to find the CDF of X. For $x \ge 0$,

$$\begin{aligned} F_X(x) &= P[-\ln(1-U) \le x] \\ &= P\left[1-U \ge e^{-x}\right] = P\left[U \le 1-e^{-x}\right]. \end{aligned} \tag{1}$$

For $x \ge 0$, $0 \le 1 - e^{-x} \le 1$ and so

$$F_X(x) = F_U\left(1-e^{-x}\right) = 1 - e^{-x}. \tag{2}$$

The complete CDF can be written as

$$F_X(x) = \begin{cases} 0 & x < 0, \\ 1 - e^{-x} & x \ge 0. \end{cases} \tag{3}$$

(b) By taking the derivative, the PDF is

$$f_X(x) = \begin{cases} e^{-x} & x \ge 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Thus, X has an exponential PDF. In fact, since most computer languages provide uniform $[0,1]$ random numbers, the procedure outlined in this problem provides a way to generate exponential random variables from uniform random variables.

(c) Since X is an exponential random variable with parameter $a = 1$, $E[X] = 1$.

Problem 6.2.9 Solution

Since X is constrained to the interval $[-1, 1]$, we see that $20 \leq Y \leq 35$. Thus , $F_Y(y) = 0$ for $y < 20$ and $F_Y(y) = 1$ fpr $y > 35$. For $20 \leq y \leq 35$,

$$\begin{aligned} F_Y(y) &= P\left[20 + 15X^2 \leq y\right] \\ &= P\left[X^2 \leq \frac{y-20}{15}\right] \\ &= P\left[-\sqrt{\frac{y-20}{15}} \leq X \leq \sqrt{\frac{y-20}{15}}\right] \\ &= \int_{-\sqrt{\frac{y-20}{15}}}^{\sqrt{\frac{y-20}{15}}} \frac{1}{2}\, dx = \sqrt{\frac{y-20}{15}}. \end{aligned} \tag{1}$$

The complete expression for the CDF and, by taking the derivative, the PDF are

$$F_Y(y) = \begin{cases} 0 & y < 20, \\ \sqrt{\frac{y-20}{15}} & 20 \leq y \leq 35, \\ 1 & y > 35, \end{cases} \quad f_Y(y) = \begin{cases} \frac{1}{\sqrt{60(y-20)}} & 20 \leq y \leq 35, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Problem 6.2.11 Solution

If X has a uniform distribution from 0 to 1 then the PDF and corresponding CDF of X are

$$f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1, \\ 0 & \text{otherwise,} \end{cases} \qquad F_X(x) = \begin{cases} 0 & x < 0, \\ x & 0 \leq x \leq 1, \\ 1 & x > 1. \end{cases} \tag{1}$$

For $b - a > 0$, we can find the CDF of the function $Y = a + (b - a)X$

$$F_Y(y) = P[Y \leq y] = P[a + (b - a)X \leq y] \\ = P\left[X \leq \frac{y - a}{b - a}\right] \\ = F_X\left(\frac{y - a}{b - a}\right) = \frac{y - a}{b - a}. \tag{2}$$

Therefore the CDF of Y is

$$F_Y(y) = \begin{cases} 0 & y < a, \\ \frac{y-a}{b-a} & a \leq y \leq b, \\ 1 & y \geq b. \end{cases} \tag{3}$$

By differentiating with respect to y we arrive at the PDF

$$f_Y(y) = \begin{cases} 1/(b - a) & a \leq x \leq b, \\ 0 & \text{otherwise}, \end{cases} \tag{4}$$

which we recognize as the PDF of a uniform (a, b) random variable.

Problem 6.2.13 Solution

We can prove the assertion by considering the cases where $a > 0$ and $a < 0$, respectively. For the case where $a > 0$ we have

$$F_Y(y) = P[Y \leq y] = P\left[X \leq \frac{y - b}{a}\right] = F_X\left(\frac{y - b}{a}\right). \tag{1}$$

Therefore by taking the derivative we find that

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y - b}{a}\right), \qquad a > 0. \tag{2}$$

Similarly for the case when $a < 0$ we have

$$F_Y(y) = P[Y \leq y] = P\left[X \geq \frac{y - b}{a}\right] = 1 - F_X\left(\frac{y - b}{a}\right). \tag{3}$$

And by taking the derivative, we find that for negative a,

$$f_Y(y) = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right), \qquad a < 0. \tag{4}$$

A valid expression for both positive and negative a is

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right). \tag{5}$$

Therefore the assertion is proved.

Problem 6.3.1 Solution

From Problem 4.7.1, random variable X has CDF

$$F_X(x) = \begin{cases} 0 & x < -1, \\ x/3 + 1/3 & -1 \le x < 0, \\ x/3 + 2/3 & 0 \le x < 1, \\ 1 & 1 \le x. \end{cases} \tag{1}$$

(a) We can find the CDF of Y, $F_Y(y)$ by noting that Y can only take on two possible values, 0 and 100. And the probability that Y takes on these two values depends on the probability that $X < 0$ and $X \ge 0$, respectively. Therefore

$$F_Y(y) = \mathrm{P}[Y \le y] = \begin{cases} 0 & y < 0, \\ \mathrm{P}[X < 0] & 0 \le y < 100, \\ 1 & y \ge 100. \end{cases} \tag{2}$$

The probabilities concerned with X can be found from the given CDF $F_X(x)$. This is the general strategy for solving problems of this type: to express the CDF of Y in terms of the CDF of X. Since $\mathrm{P}[X < 0] = F_X(0^-) = 1/3$, the CDF of Y is

$$F_Y(y) = \mathrm{P}[Y \le y] = \begin{cases} 0 & y < 0, \\ 1/3 & 0 \le y < 100, \\ 1 & y \ge 100. \end{cases} \tag{3}$$

(b) The CDF $F_Y(y)$ has jumps of 1/3 at $y = 0$ and 2/3 at $y = 100$. The corresponding PDF of Y is

$$f_Y(y) = \delta(y)/3 + 2\delta(y-100)/3. \tag{4}$$

(c) The expected value of Y is

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = 0 \cdot \frac{1}{3} + 100 \cdot \frac{2}{3} = 66.66. \tag{5}$$

Problem 6.3.3 Solution

Since the microphone voltage V is uniformly distributed between -1 and 1 volts, V has PDF and CDF

$$f_V(v) = \begin{cases} 1/2 & -1 \le v \le 1, \\ 0 & \text{otherwise,} \end{cases} \qquad F_V(v) = \begin{cases} 0 & v < -1, \\ (v+1)/2 & -1 \le v \le 1, \\ 1 & v > 1. \end{cases} \tag{1}$$

The voltage is processed by a limiter whose output magnitude is given by below

$$L = \begin{cases} |V| & |V| \le 0.5, \\ 0.5 & \text{otherwise.} \end{cases} \tag{2}$$

(a)

$$\begin{aligned} P[L = 0.5] &= P[|V| \ge 0.5] = P[V \ge 0.5] + P[V \le -0.5] \\ &= 1 - F_V(0.5) + F_V(-0.5) \\ &= 1 - 1.5/2 + 0.5/2 = 1/2. \end{aligned} \tag{3}$$

(b) For $0 \le l \le 0.5$,

$$\begin{aligned} F_L(l) = P[|V| \le l] &= P[-l \le v \le l] \\ &= F_V(l) - F_V(-l) \\ &= 1/2(l+1) - 1/2(-l+1) = l. \end{aligned} \tag{4}$$

So the CDF of L is

$$F_L(l) = \begin{cases} 0 & l < 0, \\ l & 0 \leq l < 0.5, \\ 1 & l \geq 0.5. \end{cases} \tag{5}$$

(c) By taking the derivative of $F_L(l)$, the PDF of L is

$$f_L(l) = \begin{cases} 1 + (0.5)\delta(l - 0.5) & 0 \leq l \leq 0.5, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

The expected value of L is

$$\begin{aligned} \mathrm{E}[L] &= \int_{-\infty}^{\infty} l f_L(l)\, dl \\ &= \int_0^{0.5} l\, dl + 0.5 \int_0^{0.5} l(0.5)\delta(l - 0.5)\, dl = 0.375. \end{aligned} \tag{7}$$

Problem 6.3.5 Solution

Given the following function of random variable X,

$$Y = g(X) = \begin{cases} 10 & X < 0, \\ -10 & X \geq 0. \end{cases} \tag{1}$$

we follow the same procedure as in Problem 6.3.1. We attempt to express the CDF of Y in terms of the CDF of X. We know that Y is always less than -10. We also know that $-10 \leq Y < 10$ when $X \geq 0$, and finally, that $Y = 10$ when $X < 0$. Therefore

$$F_Y(y) = \mathrm{P}[Y \leq y] = \begin{cases} 0 & y < -10, \\ \mathrm{P}[X \geq 0] = 1 - F_X(0) & -10 \leq y < 10, \\ 1 & y \geq 10. \end{cases} \tag{2}$$

Problem 6.3.7 Solution

The PDF of U is

$$f_U(u) = \begin{cases} 1/2 & -1 \leq u \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Since $W \geq 0$, we see that $F_W(w) = 0$ for $w < 0$. Next, we observe that the rectifier output W is a mixed random variable since

$$P[W = 0] = P[U < 0] = \int_{-1}^{0} f_U(u)\, du = 1/2. \tag{2}$$

The above facts imply that

$$F_W(0) = P[W \leq 0] = P[W = 0] = 1/2. \tag{3}$$

Next, we note that for $0 < w < 1$,

$$F_W(w) = P[U \leq w] = \int_{-1}^{w} f_U(u)\, du = (w+1)/2. \tag{4}$$

Finally, $U \leq 1$ implies $W \leq 1$, which implies $F_W(w) = 1$ for $w \geq 1$. Hence, the complete expression for the CDF is

$$F_W(w) = \begin{cases} 0 & w < 0, \\ (w+1)/2 & 0 \leq w \leq 1, \\ 1 & w > 1. \end{cases} \tag{5}$$

By taking the derivative of the CDF, we find the PDF of W; however, we must keep in mind that the discontinuity in the CDF at $w = 0$ yields a corresponding impulse in the PDF.

$$f_W(w) = \begin{cases} (\delta(w)+1)/2 & 0 \leq w \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

From the PDF, we can calculate the expected value

$$E[W] = \int_0^1 w(\delta(w)+1)/2\,dw = 0 + \int_0^1 (w/2)\,dw = 1/4. \tag{7}$$

Perhaps an easier way to find the expected value is to use Theorem 3.10. In this case,

$$E[W] = \int_{-\infty}^{\infty} g(u) f_W(w)\ du = \int_0^1 u(1/2)\,du = 1/4. \tag{8}$$

As we expect, both approaches give the same answer.

Problem 6.3.9 Solution

You may find it helpful to plot W as a function of V for the following calculations. We start by finding the CDF $F_W(w) = P[W \le w]$. Since $0 \le W \le 10$, we know that

$$F_W(w) = 0 \qquad (w < 0) \tag{1}$$

and that

$$F_W(w) = 1 \qquad (w \ge 10). \tag{2}$$

Next we recall that continuous uniform V has the CDF

$$F_V(v) = \begin{cases} 0 & v < -15, \\ (v+15)/30 & -15 \le v \le 15, \\ 1 & v > 15. \end{cases} \tag{3}$$

Now we can write for $w = 0$ that

$$F_W(0) = P[V \le 0] = F_V(0) = 15/30 = 1/2. \tag{4}$$

For $0 < w < 10$,

$$F_W(w) = P[W \le w] = P[V \le w] = F_V(w) = \frac{w+15}{30}. \tag{5}$$

Thus the complete CDF of W is

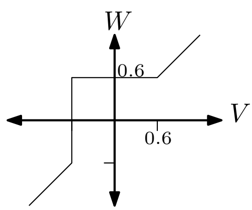
$$F_W(w) = \begin{cases} 0 & w < 0, \\ (w+15)/30 & 0 \leq w < 10, \\ 1 & w \geq 10. \end{cases} \tag{6}$$

If you study (and perhaps plot) $F_W(w)$, you'll see that it has a jump discontinuity of height 1/2 at $w = 0$ and also has a second jump of height 1/6 at $w = 10$. Thus when we take a derivative of the CDF, we obtain the PDF

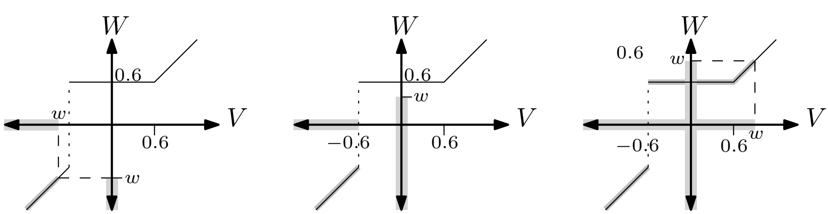
$$f_W(w) = \begin{cases} (1/2)\delta(w) & w = 0, \\ 1/30 & 0 < w < 10, \\ (1/6)\delta(w-10) & w = 10, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Problem 6.3.11 Solution

A key to this problem is recognizing that W is a mixed random variable. Here is the mapping from V to W:



To find the CDF $F_W(w) = P[W \leq w]$, careful study of the function shows there are three different cases for w that must be considered:



(a) $w < -0.6$ **(b)** $-0.6 \leq w < 0.6$ **(c)** $w \geq 0.6$

(a) If $w < -0.6$, then the event $\{W \leq w\}$, shown as the highlighted range on the vertical axis of graph (a) corresponds to the event that the the pair (V, W) is on the gray highlighted segment of the function $W = g(V)$, which corresponds to the event $\{V \leq w\}$. In this case, $F_W(w) = \mathrm{P}[V \leq w] = F_V(w)$.

(b) If $-0.6 < w < 0.6$, then the event $\{W \leq w\}$, shown as the highlighted range on the vertical axis of graph (b) corresponds to the event that the pair (V, W) is on the gray highlighted segment of the function $W = g(V)$, which corresponds to the event $\{V < 0.6\}$. In this case, $F_W(w) = \mathrm{P}[V < 0.6] = F_V(0.6^-)$.

(c) If $w \geq 0.6$, then the event $\{W \leq w\}$, shown as the highlighted range on the vertical axis of graph (c) corresponds to the event that the pair (V, W) is on the gray highlighted segment of the function $W = g(V)$, which now includes pairs v, w on the horizontal segment such that $w = 0.6$, and this corresponds to the event $\{V \leq w\}$. In this case, $F_W(w) = \mathrm{P}[V \leq w] = F_V(w)$.

We combine these three cases in the CDF

$$F_W(w) = \begin{cases} F_V(w) & w < -0.6, \\ F_V(0.6^-) & -0.6 \leq w < 0.6, \\ F_V(w) & w \geq 0.6. \end{cases} \tag{1}$$

Thus far, our answer is valid for any CDF $F_V(v)$. Now we specialize the result to the given CDF for V. Since V is a continuous uniform $(-5, 5)$ random variable, V has CDF

$$F_V(v) = \begin{cases} 0 & v < -5, \\ (v+5)/10 & -5 \leq v \leq 5, \\ 1 & \text{otherwise.} \end{cases} \tag{2}$$

The given V causes the case $w < 0.6$ to split into two cases: $w < -5$ and $-5 \leq w < 0.6$. Similarly, the $w \geq 0.6$ case gets split into two cases. Applying this CDF to Equation (1), we obtain

$$F_W(w) = \begin{cases} 0 & w < -5, \\ (w+5)/10 & -5 \le w < -0.6, \\ 0.44 & -0.6 \le w < 0.6, \\ (w+5)/10 & 0.6 \le w < 5, \\ 1 & w > 1. \end{cases} \tag{3}$$

In this CDF, there is a jump from 0.44 to 0.56 at $w = 0.6$. This jump of height 0.12 corresponds precisely to $P[W = 0.6] = 0.12$.

Since the CDF has a jump at $w = 0.6$, we have an impulse at $w = 0$ when we take the derivative:

$$f_W(w) = \begin{cases} 0.1 & -5 \le w < -0.6, \\ 0 & -0.6 \le w < 0.6, \\ 0.12\delta(w - 0.6) & w = 0.6, \\ 0.1 & 0.6 < w \le 5, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Problem 6.3.13 Solution

(a) Given $F_X(x)$ is a continuous function, there exists x_0 such that $F_X(x_0) = u$. For each value of u, the corresponding x_0 is unique. To see this, suppose there were also x_1 such that $F_X(x_1) = u$. Without loss of generality, we can assume $x_1 > x_0$ since otherwise we could exchange the points x_0 and x_1. Since $F_X(x_0) = F_X(x_1) = u$, the fact that $F_X(x)$ is nondecreasing implies $F_X(x) = u$ for all $x \in [x_0, x_1]$, i.e., $F_X(x)$ is flat over the interval $[x_0, x_1]$, which contradicts the assumption that $F_X(x)$ has no flat intervals. Thus, for any $u \in (0, 1)$, there is a unique x_0 such that $F_X(x) = u$. Moreiver, the same x_0 is the minimum of all x' such that $F_X(x') \ge u$. The uniqueness of x_0 such that $F_X(x)x_0 = u$ permits us to define $\tilde{F}(u) = x_0 = F_X^{-1}(u)$.

(b) In this part, we are given that $F_X(x)$ has a jump discontinuity at x_0. That is, there exists $u_0^- = F_X(x_0^-)$ and $u_0^+ = F_X(x_0^+)$ with $u_0^- < u_0^+$. Consider any u in the interval $[u_0^-, u_0^+]$. Since $F_X(x_0) = F_X(x_0^+)$ and $F_X(x)$ is nondecreasing,

$$F_X(x) \geq F_X(x_0) = u_0^+, \qquad x \geq x_0. \tag{1}$$

Moreover,

$$F_X(x) < F_X\left(x_0^-\right) = u_0^-, \qquad x < x_0. \tag{2}$$

Thus for any u satisfying $u_o^- \leq u \leq u_0^+$, $F_X(x) < u$ for $x < x_0$ and $F_X(x) \geq u$ for $x \geq x_0$. Thus, $\tilde{F}(u) = \min\{x|F_X(x) \geq u\} = x_0$.

(c) We note that the first two parts of this problem were just designed to show the properties of $\tilde{F}(u)$. First, we observe that

$$\mathrm{P}\left[\hat{X} \leq x\right] = \mathrm{P}\left[\tilde{F}(U) \leq x\right] = \mathrm{P}\left[\min\left\{x'|F_X(x') \geq U\right\} \leq x\right]. \tag{3}$$

To prove the claim, we define, for any x, the events

$$A: \quad \min\left\{x'|F_X(x') \geq U\right\} \leq x, \tag{4}$$

$$B: \quad U \leq F_X(x). \tag{5}$$

Note that $\mathrm{P}[A] = \mathrm{P}[\hat{X} \leq x]$. In addition, $\mathrm{P}[B] = \mathrm{P}[U \leq F_X(x)] = F_X(x)$ since $\mathrm{P}[U \leq u] = u$ for any $u \in [0,1]$.

We will show that the events A and B are the same. This fact implies

$$\mathrm{P}\left[\hat{X} \leq x\right] = \mathrm{P}[A] = \mathrm{P}[B] = \mathrm{P}[U \leq F_X(x)] = F_X(x). \tag{6}$$

All that remains is to show A and B are the same. As always, we need to show that $A \subset B$ and that $B \subset A$.

- To show $A \subset B$, suppose A is true and $\min\{x'|F_X(x') \geq U\} \leq x$. This implies there exists $x_0 \leq x$ such that $F_X(x_0) \geq U$. Since $x_0 \leq$

x, it follows from $F_X(x)$ being nondecreasing that $F_X(x_0) \le F_X(x)$. We can thus conclude that

$$U \le F_X(x_0) \le F_X(x)\,. \tag{7}$$

That is, event B is true.

- To show $B \subset A$, we suppose event B is true so that $U \le F_X(x)$. We define the set

$$L = \{x' | F_X(x') \ge U\}\,. \tag{8}$$

We note $x \in L$. It follows that the minimum element $\min\{x'|x' \in L\} \le x$. That is,

$$\min\{x'|F_X(x') \ge U\} \le x, \tag{9}$$

which is simply event A.

Problem 6.4.1 Solution

Since $0 \le X \le 1$, and $0 \le Y \le 1$, we have $0 \le V \le 1$. This implies $F_V(v) = 0$ for $v < 0$ and $F_V(v) = 1$ for $v \ge 1$. For $0 \le v \le 1$,

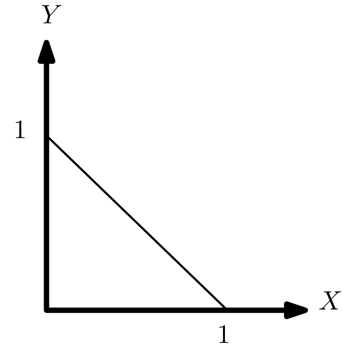
$$\begin{aligned} F_V(v) = \mathrm{P}[\max(X,Y) \le v] &= \mathrm{P}[X \le v, Y \le v] \\ &= \int_0^v \int_0^v 6xy^2\,dx\,dy \\ &= \left(\int_0^v 2x\,dx\right)\left(\int_0^v 3y^2\,dy\right) \\ &= (v^2)(v^3) = v^5. \end{aligned} \tag{1}$$

The CDF and (by taking the derivative) PDF of V are

$$F_V(v) = \begin{cases} 0 & v < 0, \\ v^5 & 0 \le v \le 1, \\ 1 & v > 1, \end{cases} \qquad f_V(v) = \begin{cases} 5v^4 & 0 \le v \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Problem 6.4.3 Solution

The key to the solution is to draw the triangular region where the PDF is nonzero:

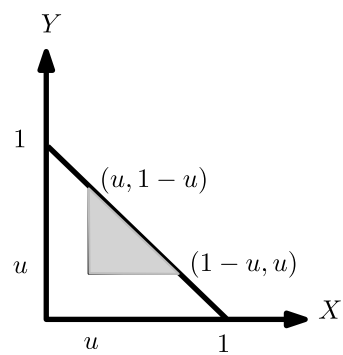


(a) X and Y are not independent. For example it is easy to see that $f_X(3/4) = f_Y(3/4) > 0$ and thus $f_X(3/4)f_Y(3/4) > 0$. However, $f_{X,Y}(3/4, 3/$ 0.

(b) First we find the CDF. Since $X \geq 0$ and $Y \geq 0$, we know that $F_U(u) = 0$ for $u < 0$. Next, for non-negative u, we see that

$$\begin{aligned} F_U(u) = \mathrm{P}[\min(X,Y) \leq u] &= 1 - \mathrm{P}[\min(X,Y) > u] \\ &= 1 - \mathrm{P}[X > u, Y > u]. \end{aligned} \tag{1}$$

At this point is is instructive to draw the region for small u:



We see that this area exists as long as $u \le 1-u$, or $u \le 1/2$. This is because if both $X > 1/2$ and $Y > 1/2$ then $X+Y > 1$ which violates the constraint $X+Y \le 1$. For $0 \le u \le 1/2$,

$$\begin{aligned} F_U(u) &= 1 - \int_u^{1-u} \int_u^{1-x} 2\,dy\,dx \\ &= 1 - 2\frac{1}{2}[(1-u)-u]^2 = 1 - [1-2u]^2. \end{aligned} \tag{2}$$

Note that we wrote the integral expression but we calculated the integral as c times the area of integration. Thus the CDF of U is

$$F_U(u) = \begin{cases} 0 & u < 0, \\ 1 - [1-2u]^2 & 0 \le u \le 1/2, \\ 1 & u > 1/2. \end{cases} \tag{3}$$

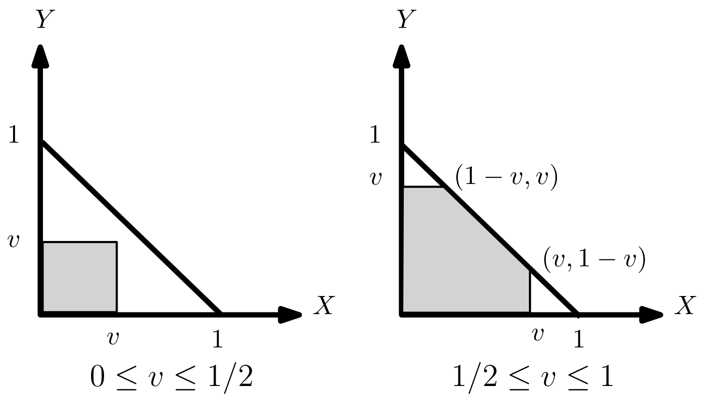
Taking the derivative, we find the PDF of U is

$$F_U(u) = \begin{cases} 4(1-2u) & 0 \le u \le 1/2, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

(c) For the CDF of V, we can write

$$\begin{aligned} F_V(v) = \mathrm{P}[V \le v] &= \mathrm{P}[\max(X,Y) \le v] \\ &= \mathrm{P}[X \le v, Y \le v] \\ &= \int_0^v \int_0^v f_{X,Y}(x,y)\; dx, dy. \end{aligned} \tag{5}$$

This is tricky because there are two distinct cases:



For $0 \leq v \leq 1/2$,

$$F_V(v) = \int_0^v \int_0^v 2\,dx\,dy = 2v^2. \tag{6}$$

For $1/2 \leq v \leq 1$, you can write the integral as

$$\begin{aligned} F_V(v) &= \int_0^{1-v} \int_0^v 2\,dy\,dx + \int_{1-v}^v \int_0^{1-x} 2\,dy\,dx \\ &= 2\left[v^2 - \frac{1}{2}[v - (1-v)]^2\right] \\ &= 2v^2 - (2v-1)^2 = 4v - 2v^2 - 1, \end{aligned} \tag{7}$$

where we skipped the steps of the integral by observing that the shaded area of integration is a square of area v^2 minus the cutoff triangle on the upper right corner. The full expression for the CDF of V is

$$F_V(v) = \begin{cases} 0 & v < 0, \\ 2v^2 & 0 \leq v \leq 1/2, \\ 4v - 2v^2 - 1 & 1/2 \leq v \leq 1, \\ 1 & v > 1. \end{cases} \tag{8}$$

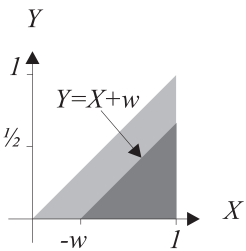
Taking a derivative, the PDF of V is

$$f_V(v) = \begin{cases} 4v & 0 \leq v \leq 1/2, \\ 4(1-v) & 1/2 \leq v \leq 1. \end{cases} \tag{9}$$

Problem 6.4.5 Solution

(a) Since the joint PDF $f_{X,Y}(x,y)$ is nonzero only for $0 \leq y \leq x \leq 1$, we observe that $W = Y - X \leq 0$ since $Y \leq X$. In addition, the most negative value of W occurs when $Y = 0$ and $X = 1$ and $W = -1$. Hence the range of W is $S_W = \{w | -1 \leq w \leq 0\}$.

(b) For $w < -1$, $F_W(w) = 0$. For $w > 0$, $F_W(w) = 1$. For $-1 \leq w \leq 0$, the CDF of W is



$$
\begin{aligned}
F_W(w) &= \mathrm{P}[Y - X \leq w] \\
&= \int_{-w}^{1} \int_{0}^{x+w} 6y \, dy \, dx \\
&= \int_{-w}^{1} 3(x+w)^2 \, dx \\
&= (x+w)^3 \Big|_{-w}^{1} = (1+w)^3. \qquad (1)
\end{aligned}
$$

Therefore, the complete CDF of W is

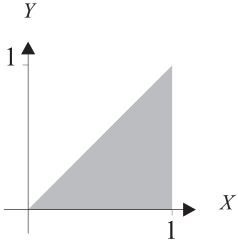
$$
F_W(w) = \begin{cases} 0 & w < -1, \\ (1+w)^3 & -1 \leq w \leq 0, \\ 1 & w > 0. \end{cases} \qquad (2)
$$

By taking the derivative of $f_W(w)$ with respect to w, we obtain the PDF

$$
f_W(w) = \begin{cases} 3(w+1)^2 & -1 \leq w \leq 0, \\ 0 & \text{otherwise.} \end{cases} \qquad (3)
$$

Problem 6.4.7 Solution

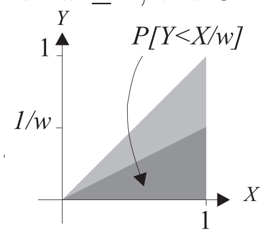
Random variables X and Y have joint PDF



$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) Since $f_{X,Y}(x,y) = 0$ for $y > x$, we can conclude that $Y \le X$ and that $W = X/Y \ge 1$. Since Y can be arbitrarily small but positive, W can be arbitrarily large. Hence the range of W is $S_W = \{w | w \ge 1\}$.

(b) For $w \ge 1$, the CDF of W is



$$\begin{aligned} F_W(w) &= \mathrm{P}[X/Y \le w] \\ &= 1 - \mathrm{P}[X/Y > w] \\ &= 1 - \mathrm{P}[Y < X/w] \\ &= 1 - 1/w. \end{aligned} \tag{2}$$

Note that we have used the fact that $\mathrm{P}[Y < X/w]$ equals 1/2 times the area of the corresponding triangle. The complete CDF is

$$F_W(w) = \begin{cases} 0 & w < 1, \\ 1 - 1/w & w \ge 1. \end{cases} \tag{3}$$

The PDF of W is found by differentiating the CDF.

$$f_W(w) = \frac{dF_W(w)}{dw} = \begin{cases} 1/w^2 & w \ge 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

To find the expected value $\mathrm{E}[W]$, we write

$$\mathrm{E}[W] = \int_{-\infty}^{\infty} w f_W(w)\, dw = \int_1^{\infty} \frac{dw}{w}. \tag{5}$$

However, the integral diverges and $\mathrm{E}[W]$ is undefined.

Problem 6.4.9 Solution

Since X_1 and X_2 are iid Gaussian $(0, 1)$, each has PDF

$$f_{X_i}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \tag{1}$$

For $w < 0$, $F_W(w) = 0$. For $w \geq 0$, we define the disc

$$\mathcal{R}(w) = \left\{(x_1, x_2) | x_1^2 + x_2^2 \leq w\right\}. \tag{2}$$

and we write

$$\begin{aligned} F_W(w) = \mathrm{P}\left[X_1^2 + X_2^2 \leq w\right] &= \iint_{\mathcal{R}(w)} f_{X_1}(x_1) f_{X_2}(x_2)\, dx_1\, dx_2 \\ &= \iint_{\mathcal{R}(w)} \frac{1}{2\pi} e^{-(x_1^2+x_2^2)/2}\, dx_1\, dx_2. \end{aligned} \tag{3}$$

Changing to polar coordinates with $r = \sqrt{x_1^2 + x_2^2}$ yields

$$\begin{aligned} F_W(w) &= \int_0^{2\pi} \int_0^{\sqrt{w}} \frac{1}{2\pi} e^{-r^2/2} r\, dr\, d\theta \\ &= \int_0^{\sqrt{w}} r e^{-r^2/2}\, dr = -e^{-r^2/2} \Big|_0^{\sqrt{w}} = 1 - e^{-w/2}. \end{aligned} \tag{4}$$

Taking the derivative of $F_W(w)$, the complete expression for the PDF of W is

$$f_W(w) = \begin{cases} 0 & w < 0, \\ \frac{1}{2} e^{-w/2} & w \geq 0. \end{cases} \tag{5}$$

Thus W is an exponential $(\lambda = 1/2)$ random variable.

Problem 6.4.11 Solution

Although Y is a function of two random variables X and Z, it is not similar to other problems of the form $Y = g(X, Z)$ because Z is discrete. However, we can still use the same approach to find the CDF $F_Y(y)$ by identifying those pairs (X, Z) that belong to the event $\{Y \leq y\}$. In particular, since $Y = ZX$, we can write the event $\{Y \leq y\}$ as the disjoint union

$$\{Y \leq y\} = \{X \leq y, Z = 1\} \cup \{X \geq -y, Z = -1\}. \tag{1}$$

In particular, we note that if $X \geq -y$ and $Z = -1$, then $Y = ZX = -X \leq y$. It follows that

$$\begin{aligned} F_Y(y) &= \mathrm{P}[Y \leq y] \\ &= \mathrm{P}[X \leq y, Z = 1] + \mathrm{P}[X \geq -y, Z = -1] \\ &= \mathrm{P}[X \leq y]\,\mathrm{P}[Z = 1] + \mathrm{P}[X \geq -y]\,\mathrm{P}[Z = -1] \qquad (2) \\ &= p\,\mathrm{P}[X \leq y] + (1-p)\,\mathrm{P}[-X \leq y] \\ &= p\,\mathrm{P}[X \leq y] + (1-p)\,\mathrm{P}[-X \leq y] \\ &= p\Phi(y) + (1-p)\Phi(y) = \Phi(y). \qquad (3) \end{aligned}$$

Note that we use independence of X and Z to write (2). It follows that Y is Gaussian $(0, 1)$ and has PDF

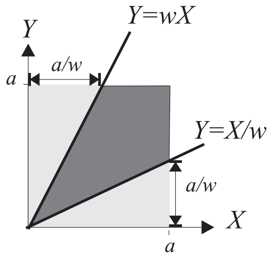
$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}. \tag{4}$$

Note that what has happened here is that as often as Z turns a negative X into a positive $Y = -X$, it also turns a positive X into a negative $Y = -X$. Because the PDF of X is an even function, these switches probabilistically cancel each other out.

Problem 6.4.13 Solution

Following the hint, we observe that either $Y \geq X$ or $X \geq Y$, or, equivalently, $(Y/X) \geq 1$ or $(X/Y) \geq 1$. Hence, $W \geq 1$. To find the CDF $F_W(w)$, we know that $F_W(w) = 0$ for $w < 1$. For $w \geq 1$, we solve

$$\begin{aligned} F_W(w) &= \mathrm{P}[\max[(X/Y), (Y/X)] \leq w] \\ &= \mathrm{P}[(X/Y) \leq w, (Y/X) \leq w] \\ &= \mathrm{P}[Y \geq X/w, Y \leq wX] \\ &= \mathrm{P}[X/w \leq Y \leq wX]. \qquad (1) \end{aligned}$$



We note that in the middle of the above steps, nonnegativity of X and Y was essential. We can depict the given set $\{X/w \leq Y \leq wX\}$ as the dark region on the X, Y plane. Because the PDF is uniform over the square, it is easier to use geometry to calculate the probability. In particular, each of the lighter triangles that are not part of the region of interest has area $a^2/2w$.

This implies

$$\mathrm{P}\left[X/w \leq Y \leq wX\right] = 1 - \frac{a^2/2w + a^2/2w}{a^2} = 1 - \frac{1}{w}. \tag{2}$$

The final expression for the CDF of W is

$$F_W(w) = \begin{cases} 0 & w < 1, \\ 1 - 1/w & w \geq 1. \end{cases} \tag{3}$$

By taking the derivative, we obtain the PDF

$$f_W(w) = \begin{cases} 0 & w < 1, \\ 1/w^2 & w \geq 1. \end{cases} \tag{4}$$

Problem 6.4.15 Solution

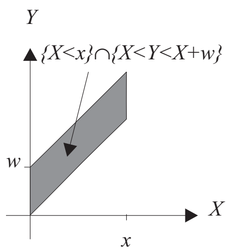
(a) To find if W and X are independent, we must be able to factor the joint density function $f_{X,W}(x, w)$ into the product $f_X(x) f_W(w)$ of marginal density functions. To verify this, we must find the joint PDF of X and W. First we find the joint CDF.

$$\begin{aligned} F_{X,W}(x, w) &= \mathrm{P}\left[X \leq x, W \leq w\right] \\ &= \mathrm{P}\left[X \leq x, Y - X \leq w\right] = \mathrm{P}\left[X \leq x, Y \leq X + w\right]. \end{aligned} \tag{1}$$

Since $Y \geq X$, the CDF of W satisfies

$$F_{X,W}(x,w) = \mathrm{P}[X \leq x, X \leq Y \leq X + w]. \tag{2}$$

Thus, for $x \geq 0$ and $w \geq 0$,



$$\begin{aligned} F_{X,W}(x,w) &= \int_0^x \int_{x'}^{x'+w} \lambda^2 e^{-\lambda y}\, dy\, dx' \\ &= \int_0^x \left(-\lambda e^{-\lambda y} \Big|_{x'}^{x'+w} \right) dx' \\ &= \int_0^x \left(-\lambda e^{-\lambda(x'+w)} + \lambda e^{-\lambda x'} \right) dx' \\ &= e^{-\lambda(x'+w)} - e^{-\lambda x'} \Big|_0^x \\ &= (1 - e^{-\lambda x})(1 - e^{-\lambda w}) \end{aligned} \tag{3}$$

We see that $F_{X,W}(x,w) = F_X(x)F_W(w)$. Moreover, by applying Theorem 5.5,

$$f_{X,W}(x,w) = \frac{\partial^2 F_{X,W}(x,w)}{\partial x\, \partial w} = \lambda e^{-\lambda x} \lambda e^{-\lambda w} = f_X(x)\, f_W(w). \tag{4}$$

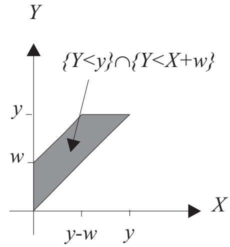
Since we have our desired factorization, W and X are independent.

(b) Following the same procedure, we find the joint CDF of Y and W.

$$\begin{aligned} F_{W,Y}(w,y) = \mathrm{P}[W \leq w, Y \leq y] &= \mathrm{P}[Y - X \leq w, Y \leq y] \\ &= \mathrm{P}[Y \leq X + w, Y \leq y]. \end{aligned} \tag{5}$$

The region of integration corresponding to the event $\{Y \leq x + w, Y \leq y\}$ depends on whether $y < w$ or $y \geq w$. Keep in mind that although $W = Y - X \leq Y$, the dummy arguments y and w of $f_{W,Y}(w,y)$ need not obey the same constraints. In any case, we must consider each case separately.

For $y > w$, the integration is

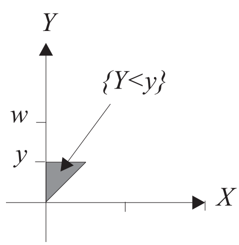


$$F_{W,Y}(w,y) = \int_0^{y-w} \int_u^{u+w} \lambda^2 e^{-\lambda v}\, dv\, du + \int_{y-w}^{y} \int_u^{y} \lambda^2 e^{-\lambda v}\, dv\, du$$
$$= \lambda \int_0^{y-w} \left[e^{-\lambda u} - e^{-\lambda(u+w)}\right] du + \lambda \int_{y-w}^{y} \left[e^{-\lambda u} - e^{-\lambda y}\right] du. \tag{6}$$

It follows that

$$F_{W,Y}(w,y) = \left[-e^{-\lambda u} + e^{-\lambda(u+w)}\right]\Big|_0^{y-w} + \left[-e^{-\lambda u} - u\lambda e^{-\lambda y}\right]\Big|_{y-w}^{y}$$
$$= 1 - e^{-\lambda w} - \lambda w e^{-\lambda y}. \tag{7}$$

For $y \leq w$,



$$F_{W,Y}(w,y) = \int_0^{y} \int_u^{y} \lambda^2 e^{-\lambda v}\, dv\, du$$
$$= \int_0^{y} \left[-\lambda e^{-\lambda y} + \lambda e^{-\lambda u}\right] du$$
$$= -\lambda u e^{-\lambda y} - e^{-\lambda u}\Big|_0^{y}$$
$$= 1 - (1 + \lambda y) e^{-\lambda y}. \tag{8}$$

The complete expression for the joint CDF is

$$F_{W,Y}(w,y) = \begin{cases} 1 - e^{-\lambda w} - \lambda w e^{-\lambda y} & 0 \leq w \leq y, \\ 1 - (1 + \lambda y) e^{-\lambda y} & 0 \leq y \leq w, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Applying Theorem 5.5 yields

$$f_{W,Y}(w,y) = \frac{\partial^2 F_{W,Y}(w,y)}{\partial w\,\partial y} = \begin{cases} 2\lambda^2 e^{-\lambda y} & 0 \le w \le y \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

The joint PDF $f_{W,Y}(w,y)$ doesn't factor and thus W and Y are dependent.

Problem 6.5.1 Solution

Since X and Y are take on only integer values, $W = X + Y$ is integer valued as well. Thus for an integer w,

$$P_W(w) = \mathrm{P}[W = w] = \mathrm{P}[X + Y = w]. \tag{1}$$

Suppose $X = k$, then $W = w$ if and only if $Y = w - k$. To find all ways that $X + Y = w$, we must consider each possible integer k such that $X = k$. Thus

$$P_W(w) = \sum_{k=-\infty}^{\infty} \mathrm{P}[X = k, Y = w - k] = \sum_{k=-\infty}^{\infty} P_{X,Y}(k, w-k). \tag{2}$$

Since X and Y are independent, $P_{X,Y}(k, w-k) = P_X(k)P_Y(w-k)$. It follows that for any integer w,

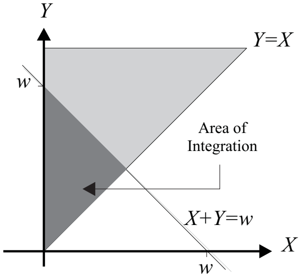
$$P_W(w) = \sum_{k=-\infty}^{\infty} P_X(k)\,P_Y(w-k). \tag{3}$$

Problem 6.5.3 Solution

The joint PDF of X and Y is

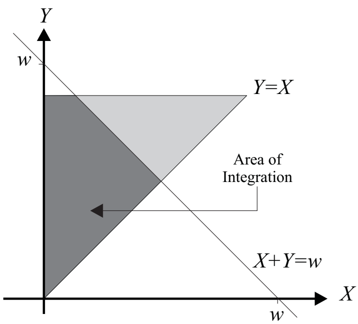
$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le x \le y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We wish to find the PDF of W where $W = X + Y$. First we find the CDF of W, $F_W(w)$, but we must realize that the CDF will require different integrations for different values of w.



For values of $0 \leq w \leq 1$ we look to integrate the shaded area in the figure to the right.

$$F_W(w) = \int_0^{\frac{w}{2}} \int_x^{w-x} 2\,dy\,dx = \frac{w^2}{2}. \tag{2}$$



For values of w in the region $1 \leq w \leq 2$ we look to integrate over the shaded region in the graph to the right. From the graph we see that we can integrate with respect to x first, ranging y from 0 to $w/2$, thereby covering the lower right triangle of the shaded region and leaving the upper trapezoid, which is accounted for in the second term of the following expression:

$$\begin{aligned} F_W(w) &= \int_0^{\frac{w}{2}} \int_0^{y} 2\,dx\,dy + \int_{\frac{w}{2}}^{1} \int_0^{w-y} 2\,dx\,dy \\ &= 2w - 1 - \frac{w^2}{2}. \end{aligned} \tag{3}$$

Putting all the parts together gives the CDF

$$F_W(w) = \begin{cases} 0 & w < 0, \\ \frac{w^2}{2} & 0 \leq w \leq 1, \\ 2w - 1 - \frac{w^2}{2} & 1 \leq w \leq 2, \\ 1 & w > 2, \end{cases} \tag{4}$$

and (by taking the derivative) the PDF

$$f_W(w) = \begin{cases} w & 0 \le w \le 1, \\ 2 - w & 1 \le w \le 2, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Problem 6.5.5 Solution

By using Theorem 6.9, we can find the PDF of $W = X + Y$ by convolving the two exponential distributions. For $\mu \neq \lambda$,

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f_X(x) f_Y(w - x)\, dx \\ &= \int_0^w \lambda e^{-\lambda x} \mu e^{-\mu(w-x)}\, dx \\ &= \lambda\mu e^{-\mu w} \int_0^w e^{-(\lambda-\mu)x}\, dx \\ &= \begin{cases} \frac{\lambda\mu}{\lambda-\mu}\left(e^{-\mu w} - e^{-\lambda w}\right) & w \ge 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{1}$$

When $\mu = \lambda$, the previous derivation is invalid because of the denominator term $\lambda - \mu$. For $\mu = \lambda$, we have

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f_X(x) f_Y(w - x)\, dx \\ &= \int_0^w \lambda e^{-\lambda x} \lambda e^{-\lambda(w-x)}\, dx \\ &= \lambda^2 e^{-\lambda w} \int_0^w dx \\ &= \begin{cases} \lambda^2 w e^{-\lambda w} & w \ge 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2}$$

Note that when $\mu = \lambda$, W is the sum of two iid exponential random variables and has a second order Erlang PDF.

Problem 6.5.7 Solution

We first find the CDF of W following the same procedure as in the proof of Theorem 6.8.

$$F_W(w) = P[X \leq Y + w] = \int_{-\infty}^{\infty} \int_{-\infty}^{y+w} f_{X,Y}(x,y)\, dx\, dy. \tag{1}$$

By taking the derivative with respect to w, we obtain

$$\begin{aligned} f_W(w) = \frac{dF_W(w)}{dw} &= \int_{-\infty}^{\infty} \frac{d}{dw} \left(\int_{-\infty}^{y+w} f_{X,Y}(x,y)\, dx \right) dy \\ &= \int_{-\infty}^{\infty} f_{X,Y}(w+y,y)\, dy. \end{aligned} \tag{2}$$

With the variable substitution $y = x - w$, we have $dy = dx$ and

$$f_W(w) = \int_{-\infty}^{\infty} f_{X,Y}(x, x-w)\, dx. \tag{3}$$

Problem 6.6.1 Solution

Given $0 \leq u \leq 1$, we need to find the "inverse" function that finds the value of w satisfying $u = F_W(w)$. The problem is that for $u = 1/4$, any w in the interval $[-3,3]$ satisfies $F_W(w) = 1/4$. However, in terms of generating samples of random variable W, this doesn't matter. For a uniform $(0,1)$ random variable U, $P[U = 1/4] = 0$. Thus we can choose any $w \in [-3,3]$. In particular, we define the inverse CDF as

$$w = F_W^{-1}(u) = \begin{cases} 8u - 5 & 0 \leq u \leq 1/4, \\ (8u+7)/3 & 1/4 < u \leq 1. \end{cases} \tag{1}$$

Note that because $0 \leq F_W(w) \leq 1$, the inverse $F_W^{-1}(u)$ is defined only for $0 \leq u \leq 1$. Careful inspection will show that $u = (w+5)/8$ for $-5 \leq w < -3$ and that $u = 1/4 + 3(w-3)/8$ for $-3 \leq w \leq 5$. Thus, for a uniform $(0,1)$ random variable U, the function $W = F_W^{-1}(U)$ produces a random variable with CDF $F_W(w)$. To implement this solution in MATLAB, we define

```
function w=iwcdf(u);
w=((u>=0).*(u <= 0.25).*(8*u-5))+...
    ((u > 0.25).*(u<=1).*((8*u+7)/3));
```

so that the MATLAB code `W=icdfrv(@iwcdf,m)` generates m samples of random variable W.

Problem 6.6.3 Solution

In the first approach X is an exponential (λ) random variable, Y is an independent exponential (μ) random variable, and $W = Y/X$. we implement this approach in the function `wrv1.m` shown below.

In the second approach, we use Theorem 6.5 and generate samples of a uniform $(0, 1)$ random variable U and calculate $W = F_W^{-1}(U)$. In this problem,

$$F_W(w) = 1 - \frac{\lambda/\mu}{\lambda/\mu + w}. \tag{1}$$

Setting $u = F_W(w)$ and solving for w yields

$$w = F_W^{-1}(u) = \frac{\lambda}{\mu}\left(\frac{u}{1-u}\right). \tag{2}$$

We implement this solution in the function `wrv2`. Here are the two solutions:

```
function w=wrv1(lambda,mu,m)
%Usage: w=wrv1(lambda,mu,m)
%Return m samples of W=Y/X
%X is exponential (lambda)
%Y is exponential (mu)

x=exponentialrv(lambda,m);
y=exponentialrv(mu,m);
w=y./x;
```

```
function w=wrv2(lambda,mu,m)
%Usage: w=wrv1(lambda,mu,m)
%Return m samples of W=Y/X
%X is exponential (lambda)
%Y is exponential (mu)
%Uses CDF of F_W(w)

u=rand(m,1);
w=(lambda/mu)*u./(1-u);
```

We would expect that `wrv2` would be faster simply because it does less work. In fact, its instructive to account for the work each program does.

- `wrv1` Each exponential random sample requires the generation of a uniform random variable, and the calculation of a logarithm. Thus, we generate $2m$ uniform random variables, calculate $2m$ logarithms, and perform m floating point divisions.

- `wrv2` Generate m uniform random variables and perform m floating points divisions.

This quickie analysis indicates that `wrv1` executes roughly $5m$ operations while `wrv2` executes about $2m$ operations. We might guess that `wrv2` would be faster by a factor of 2.5. Experimentally, we calculated the execution time associated with generating a million samples:

```
>> t2=cputime;w2=wrv2(1,1,1000000);t2=cputime-t2
t2 =
    0.2500
>> t1=cputime;w1=wrv1(1,1,1000000);t1=cputime-t1
t1 =
    0.7610
>>
```

We see in our simple experiments that `wrv2` is faster by a rough factor of 3. (Note that repeating such trials yielded qualitatively similar results.)

Problem Solutions – Chapter 7

Problem 7.1.1 Solution

Given the CDF $F_X(x)$, we can write

$$\begin{aligned} F_{X|X>0}(x) &= \mathrm{P}[X \leq x | X > 0] \\ &= \frac{\mathrm{P}[X \leq x, X > 0]}{\mathrm{P}[X > 0]} \\ &= \frac{\mathrm{P}[0 < X \leq x]}{\mathrm{P}[X > 0]} = \begin{cases} 0 & x \leq 0, \\ \frac{F_X(x) - F_X(0)}{\mathrm{P}[X>0]}. & x > 0. \end{cases} \end{aligned} \tag{1}$$

From $F_X(x)$, we know that $F_X(0) = 0.4$ and $\mathrm{P}[X > 0] = 1 - F_X(0) = 0.6$. Thus

$$\begin{aligned} F_{X|X>0}(x) &= \begin{cases} 0 & x \leq 0, \\ \frac{F_X(x) - 0.4}{0.6} & x > 0, \end{cases} \\ &= \begin{cases} 0 & x < 5, \\ \frac{0.8-0.4}{0.6} = \frac{2}{3} & 5 \leq x < 7, \\ 1 & x \geq 7. \end{cases} \end{aligned} \tag{2}$$

From the jumps in the conditional CDF $F_{X|X>0}(x)$, we can write down the conditional PMF

$$P_{X|X>0}(x) = \begin{cases} 2/3 & x = 5, \\ 1/3 & x = 7, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Alternatively, we can start with the jumps in $F_X(x)$ and read off the PMF of X as

$$P_X(x) = \begin{cases} 0.4 & x = -3, \\ 0.4 & x = 5, \\ 0.2 & x = 7, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The event $\{X > 0\}$ has probability $\mathrm{P}[X > 0] = P_X(5) + P_X(7) = 0.6$. From Theorem 7.1, the conditional PMF of X given $X > 0$ is

$$P_{X|X>0}(x) = \begin{cases} \frac{P_X(x)}{\mathrm{P}[X>0]} & x \in B, \\ 0 & \text{otherwise}, \end{cases} = \begin{cases} 2/3 & x = 5, \\ 1/3 & x = 7, \\ 0 & \text{otherwise}. \end{cases} \tag{5}$$

Problem 7.1.3 Solution

The event $B = \{X \neq 0\}$ has probability $\mathrm{P}[B] = 1 - \mathrm{P}[X = 0] = 15/16$. The conditional PMF of X given B is

$$P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{\mathrm{P}[B]} & x \in B, \\ 0 & \text{otherwise}, \end{cases} = \binom{4}{x}\frac{1}{15}. \tag{1}$$

Problem 7.1.5 Solution

(a) You run $M = m$ miles if (with probability $(1-q)^m$) you choose to run the first m miles and then (with probability q) you choose to quite just prior to mile $m+1$. The PMF of M, the number of miles run on an arbitrary day is

$$P_M(m) = \begin{cases} q(1-q)^m & m = 0, 1, \ldots, \\ 0 & \text{otherwise}. \end{cases} \tag{1}$$

(b) The probability that we run a marathon on any particular day is the probability that $M \geq 26$.

$$r = \mathrm{P}\left[M \geq 26\right] = \sum_{m=26}^{\infty} q(1-q)^m = (1-q)^{26}. \tag{2}$$

(c) We run a marathon on each day with probability equal to r, and we do not run a marathon with probability $1-r$. Therefore in a year we have 365 tests of our jogging resolve, and thus 365 chances to run a marathon. So the PMF of the number of marathons run in a year, J, can be expressed as

$$P_J(j) = \binom{365}{j} r^j (1-r)^{365-j}. \tag{3}$$

(d) The random variable K is defined as the number of miles we run above that required for a marathon, $K = M - 26$. Given the event, A, that we have run a marathon, we wish to know how many miles in excess of 26 we in fact ran. So we want to know the conditional PMF $P_{K|A}(k)$.

$$P_{K|A}(k) = \frac{\mathrm{P}[K=k, A]}{\mathrm{P}[A]} = \frac{\mathrm{P}[M = 26+k]}{\mathrm{P}[A]}. \tag{4}$$

Since $\mathrm{P}[A] = r$, for $k = 0, 1, \ldots,$

$$P_{K|A}(k) = \frac{(1-q)^{26+k} q}{(1-q)^{26}} = (1-q)^k q. \tag{5}$$

The complete expression of for the conditional PMF of K is

$$P_{K|A}(k) = \begin{cases} (1-q)^k q & k = 0, 1, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Problem 7.1.7 Solution

(a) Given that a person is healthy, X is a Gaussian ($\mu = 90, \sigma = 20$) random variable. Thus,

$$f_{X|H}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} = \frac{1}{20\sqrt{2\pi}} e^{-(x-90)^2/800}. \tag{1}$$

(b) Given the event H, we use the conditional PDF $f_{X|H}(x)$ to calculate the required probabilities

$$\begin{aligned} \mathrm{P}\left[T^+|H\right] = \mathrm{P}\left[X \geq 140|H\right] &= \mathrm{P}\left[X - 90 \geq 50|H\right] \\ &= \mathrm{P}\left[\frac{X-90}{20} \geq 2.5|H\right] \\ &= 1 - \Phi(2.5) = 0.006. \end{aligned} \tag{2}$$

Similarly,

$$\begin{aligned} \mathrm{P}\left[T^-|H\right] = \mathrm{P}\left[X \leq 110|H\right] &= \mathrm{P}\left[X - 90 \leq 20|H\right] \\ &= \mathrm{P}\left[\frac{X-90}{20} \leq 1|H\right] \\ &= \Phi(1) = 0.841. \end{aligned} \tag{3}$$

(c) Using Bayes Theorem, we have

$$\mathrm{P}\left[H|T^-\right] = \frac{\mathrm{P}\left[T^-|H\right]\mathrm{P}\left[H\right]}{\mathrm{P}\left[T^-\right]} = \frac{\mathrm{P}\left[T^-|H\right]\mathrm{P}\left[H\right]}{\mathrm{P}\left[T^-|D\right]\mathrm{P}\left[D\right] + \mathrm{P}\left[T^-|H\right]\mathrm{P}\left[H\right]}. \tag{4}$$

In the denominator, we need to calculate

$$\begin{aligned} \mathrm{P}\left[T^-|D\right] = \mathrm{P}\left[X \leq 110|D\right] &= \mathrm{P}\left[X - 160 \leq -50|D\right] \\ &= \mathrm{P}\left[\frac{X-160}{40} \leq -1.25|D\right] \\ &= \Phi(-1.25) = 1 - \Phi(1.25) = 0.106. \end{aligned} \tag{5}$$

Thus,

$$\begin{aligned} \mathrm{P}\left[H|T^-\right] &= \frac{\mathrm{P}\left[T^-|H\right]\mathrm{P}\left[H\right]}{\mathrm{P}\left[T^-|D\right]\mathrm{P}\left[D\right] + \mathrm{P}\left[T^-|H\right]\mathrm{P}\left[H\right]} \\ &= \frac{0.841(0.9)}{0.106(0.1) + 0.841(0.9)} = 0.986. \end{aligned} \tag{6}$$

(d) Since T^-, T^0, and T^+ are mutually exclusive and collectively exhaustive,

$$\begin{aligned} \mathrm{P}\left[T^0|H\right] &= 1 - \mathrm{P}\left[T^-|H\right] - \mathrm{P}\left[T^+|H\right] \\ &= 1 - 0.841 - 0.006 = 0.153. \end{aligned} \tag{7}$$

We say that a test is a failure if the result is T^0. Thus, given the event H, each test has conditional failure probability of $q = 0.153$, or success probability $p = 1 - q = 0.847$. Given H, the number of trials N until a success is a geometric (p) random variable with PMF

$$P_{N|H}(n) = \begin{cases} (1-p)^{n-1}p & n = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Problem 7.1.9 Solution

For this problem, almost any non-uniform random variable X will yield a non-uniform random variable Z. For example, suppose X has the "triangular" PDF

$$f_X(x) = \begin{cases} 8x/r^2 & 0 \le x \le r/2, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

In this case, the event B_i that $Y = i\Delta + \Delta/2$ occurs if and only if $i\Delta \le X < (i+1)\Delta$. Thus

$$\mathrm{P}\left[B_i\right] = \int_{i\Delta}^{(i+1)\Delta} \frac{8x}{r^2}\,dx = \frac{8\Delta(i\Delta + \Delta/2)}{r^2}. \tag{2}$$

It follows that the conditional PDF of X given B_i is

$$f_{X|B_i}(x) = \begin{cases} \frac{f_X(x)}{\mathrm{P}[B_i]} & x \in B_i, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} \frac{x}{\Delta(i\Delta + \Delta/2)} & i\Delta \le x < (i+1)\Delta, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Given event B_i, $Y = i\Delta + \Delta/2$, so that $Z = X - Y = X - i\Delta - \Delta/2$. This implies

$$f_{Z|B_i}(z) = f_{X|B_i}(z + i\Delta + \Delta/2) = \begin{cases} \frac{z + i\Delta + \Delta/2}{\Delta(i\Delta + \Delta/2)} & -\Delta/2 \le z < \Delta/2, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We observe that the PDF of Z depends on which event B_i occurs. Moreover, $f_{Z|B_i}(z)$ is non-uniform for all B_i.

Problem 7.2.1 Solution

The probability of the event B is

$$\begin{aligned} \mathrm{P}[B] = \mathrm{P}[X \geq \mu_X] = \mathrm{P}[X \geq 3] &= P_X(3) + P_X(4) + P_X(5) \\ &= \frac{\binom{5}{3} + \binom{5}{4} + \binom{5}{5}}{32} = 21/32. \end{aligned} \tag{1}$$

The conditional PMF of X given B is

$$P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{\mathrm{P}[B]} & x \in B, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} \binom{5}{x}\frac{1}{21} & x = 3, 4, 5, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The conditional first and second moments of X are

$$\begin{aligned} \mathrm{E}[X|B] = \sum_{x=3}^{5} x P_{X|B}(x) &= 3\binom{5}{3}\frac{1}{21} + 4\binom{5}{4}\frac{1}{21} + 5\binom{5}{5}\frac{1}{21} \\ &= [30 + 20 + 5]/21 = 55/21, \end{aligned} \tag{3}$$

$$\begin{aligned} \mathrm{E}\left[X^2|B\right] = \sum_{x=3}^{5} x^2 P_{X|B}(x) &= 3^2\binom{5}{3}\frac{1}{21} + 4^2\binom{5}{4}\frac{1}{21} + 5^2\binom{5}{5}\frac{1}{21} \\ &= [90 + 80 + 25]/21 = 195/21 = 65/7. \end{aligned} \tag{4}$$

The conditional variance of X is

$$\begin{aligned} \mathrm{Var}[X|B] &= \mathrm{E}\left[X^2|B\right] - (\mathrm{E}[X|B])^2 \\ &= 65/7 - (55/21)^2 = 1070/441 = 2.43. \end{aligned} \tag{5}$$

Problem 7.2.3 Solution

The PDF of X is

$$f_X(x) = \begin{cases} 1/10 & -5 \leq x \leq 5, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) The event B has probability

$$P[B] = P[-3 \leq X \leq 3] = \int_{-3}^{3} \frac{1}{10}\, dx = \frac{3}{5}. \tag{2}$$

From Definition 7.3, the conditional PDF of X given B is

$$f_{X|B}(x) = \begin{cases} f_X(x) \,/\, P[B] & x \in B, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} 1/6 & |x| \leq 3, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

(b) Given B, we see that X has a uniform PDF over $[a, b]$ with $a = -3$ and $b = 3$. From Theorem 4.6, the conditional expected value of X is $E[X|B] = (a + b)/2 = 0$.

(c) From Theorem 4.6, the conditional variance of X is $\text{Var}[X|B] = (b - a)^2/12 = 3$.

Problem 7.2.5 Solution

The condition *right side of the circle* is $R = [0, 1/2]$. Using the PDF in Example 4.5, we have

$$P[R] = \int_0^{1/2} f_Y(y)\, dy = \int_0^{1/2} 3y^2\, dy = 1/8. \tag{1}$$

Therefore, the conditional PDF of Y given event R is

$$f_{Y|R}(y) = \begin{cases} 24y^2 & 0 \leq y \leq 1/2 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The conditional expected value and mean square value are

$$E[Y|R] = \int_{-\infty}^{\infty} y f_{Y|R}(y)\, dy = \int_0^{1/2} 24y^3\, dy = 3/8 \text{ meter}, \tag{3}$$

$$E\left[Y^2|R\right] = \int_{-\infty}^{\infty} y^2 f_{Y|R}(y)\, dy = \int_0^{1/2} 24y^4\, dy = 3/20 \text{ m}^2. \tag{4}$$

The conditional variance is

$$\text{Var}[Y|R] = \text{E}\left[Y^2|R\right] - (\text{E}[Y|R])^2 = \frac{3}{20} - \left(\frac{3}{8}\right)^2 = 3/320 \text{ m}^2. \tag{5}$$

The conditional standard deviation is $\sigma_{Y|R} = \sqrt{\text{Var}[Y|R]} = 0.0968$ meters.

Problem 7.2.7 Solution

(a) Consider each circuit test as a Bernoulli trial such that a failed circuit is called a success. The number of trials until the first success (i.e. a failed circuit) has the geometric PMF

$$P_N(n) = \begin{cases} (1-p)^{n-1}p & n = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(b) The probability there are at least 20 tests is

$$\text{P}[B] = \text{P}[N \geq 20] = \sum_{n=20}^{\infty} P_N(n) = (1-p)^{19}. \tag{2}$$

Note that $(1-p)^{19}$ is just the probability that the first 19 circuits pass the test, which is what we would expect since there must be at least 20 tests if the first 19 circuits pass. The conditional PMF of N given B is

$$\begin{aligned} P_{N|B}(n) &= \begin{cases} \frac{P_N(n)}{\text{P}[B]} & n \in B, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} (1-p)^{n-20}p & n = 20, 21, \ldots, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{3}$$

(c) Given the event B, the conditional expectation of N is

$$\mathrm{E}[N|B] = \sum_n n P_{N|B}(n) = \sum_{n=20}^{\infty} n(1-p)^{n-20} p. \tag{4}$$

Making the substitution $j = n - 19$ yields

$$\mathrm{E}[N|B] = \sum_{j=1}^{\infty} (j+19)(1-p)^{j-1} p = 1/p + 19. \tag{5}$$

We see that in the above sum, we effectively have the expected value of $J + 19$ where J is geometric random variable with parameter p. This is not surprising since the $N \geq 20$ iff we observed 19 successful tests. After 19 successful tests, the number of additional tests needed to find the first failure is still a geometric random variable with mean $1/p$.

Problem 7.2.9 Solution

(a) We first find the conditional PDF of T. The PDF of T is

$$f_T(t) = \begin{cases} 100e^{-100t} & t \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The conditioning event has probability

$$\mathrm{P}[T > 0.02] = \int_{0.02}^{\infty} f_T(t)\, dt = -e^{-100t}\Big|_{0.02}^{\infty} = e^{-2}. \tag{2}$$

From Definition 7.3, the conditional PDF of T is

$$\begin{aligned} f_{T|T>0.02}(t) &= \begin{cases} \frac{f_T(t)}{\mathrm{P}[T>0.02]} & t \geq 0.02, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} 100e^{-100(t-0.02)} & t \geq 0.02, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{3}$$

The conditional expected value of T is

$$\mathrm{E}\left[T|T > 0.02\right] = \int_{0.02}^{\infty} t(100)e^{-100(t-0.02)}\,dt. \tag{4}$$

The substitution $\tau = t - 0.02$ yields

$$\begin{aligned}
\mathrm{E}\left[T|T > 0.02\right] &= \int_{0}^{\infty} (\tau + 0.02)(100)e^{-100\tau}\,d\tau \\
&= \int_{0}^{\infty} (\tau + 0.02) f_T(\tau)\, d\tau \\
&= \mathrm{E}\left[T + 0.02\right] = 0.03.
\end{aligned} \tag{5}$$

(b) The conditional second moment of T is

$$\mathrm{E}\left[T^2|T > 0.02\right] = \int_{0.02}^{\infty} t^2(100)e^{-100(t-0.02)}\,dt. \tag{6}$$

The substitution $\tau = t - 0.02$ yields

$$\begin{aligned}
\mathrm{E}\left[T^2|T > 0.02\right] &= \int_{0}^{\infty} (\tau + 0.02)^2(100)e^{-100\tau}\,d\tau \\
&= \int_{0}^{\infty} (\tau + 0.02)^2 f_T(\tau)\, d\tau \\
&= \mathrm{E}\left[(T + 0.02)^2\right].
\end{aligned} \tag{7}$$

Now we can calculate the conditional variance.

$$\begin{aligned}
\mathrm{Var}[T|T > 0.02] &= \mathrm{E}\left[T^2|T > 0.02\right] - \left(\mathrm{E}\left[T|T > 0.02\right]\right)^2 \\
&= \mathrm{E}\left[(T + 0.02)^2\right] - \left(\mathrm{E}\left[T + 0.02\right]\right)^2 \\
&= \mathrm{Var}[T + 0.02] \\
&= \mathrm{Var}[T] = 0.01.
\end{aligned} \tag{8}$$

Problem 7.2.11 Solution

(a) In Problem 4.7.8, we found that the PDF of D is

$$f_D(y) = \begin{cases} 0.3\delta(y) & y < 60, \\ 0.07e^{-(y-60)/10} & y \geq 60. \end{cases} \tag{1}$$

First, we observe that $D > 0$ if the throw is good so that $P[D > 0] = 0.7$. A second way to find this probability is

$$P[D > 0] = \int_{0^+}^{\infty} f_D(y)\, dy = 0.7. \tag{2}$$

From Definition 7.3, we can write

$$\begin{aligned} f_{D|D>0}(y) &= \begin{cases} \frac{f_D(y)}{P[D>0]} & y > 0, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} (1/10)e^{-(y-60)/10} & y \geq 60, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{3}$$

(b) If instead we learn that $D \leq 70$, we can calculate the conditional PDF by first calculating

$$\begin{aligned} P[D \leq 70] &= \int_0^{70} f_D(y)\, dy \\ &= \int_0^{60} 0.3\delta(y)\, dy + \int_{60}^{70} 0.07e^{-(y-60)/10}\, dy \\ &= 0.3 + -0.7e^{-(y-60)/10}\Big|_{60}^{70} = 1 - 0.7e^{-1}. \end{aligned} \tag{4}$$

The conditional PDF is

$$f_{D|D\leq 70}(y) = \begin{cases} \frac{f_D(y)}{\mathrm{P}[D\leq 70]} & y \leq 70, \\ 0 & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \frac{0.3}{1-0.7e^{-1}}\delta(y) & 0 \leq y < 60, \\ \frac{0.07}{1-0.7e^{-1}}e^{-(y-60)/10} & 60 \leq y \leq 70, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Problem 7.3.1 Solution

X and Y each have the discrete uniform PMF

$$P_X(x) = P_Y(x) = \begin{cases} 0.1 & x = 1, 2, \ldots, 10, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The joint PMF of X and Y is

$$P_{X,Y}(x, y) = P_X(x)\, P_Y(y)$$

$$= \begin{cases} 0.01 & x = 1, 2, \ldots, 10; y = 1, 2, \ldots, 10, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The event A occurs iff $X > 5$ and $Y > 5$ and has probability

$$\mathrm{P}[A] = \mathrm{P}[X > 5, Y > 5] = \sum_{x=6}^{10}\sum_{y=6}^{10} 0.01 = 0.25. \tag{3}$$

Alternatively, we could have used independence of X and Y to write $\mathrm{P}[A] = \mathrm{P}[X > 5]\,\mathrm{P}[Y > 5] = 1/4$. From Theorem 7.6,

$$P_{X,Y|A}(x, y) = \begin{cases} \frac{P_{X,Y}(x,y)}{\mathrm{P}[A]} & (x, y) \in A, \\ 0 & \text{otherwise,} \end{cases}$$

$$= \begin{cases} 0.04 & x = 6, \ldots, 10; y = 6, \ldots, 20, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Problem 7.3.3 Solution

Given the event $A = \{X + Y \leq 1\}$, we wish to find $f_{X,Y|A}(x,y)$. First we find

$$\mathrm{P}[A] = \int_0^1 \int_0^{1-x} 6e^{-(2x+3y)}\, dy\, dx = 1 - 3e^{-2} + 2e^{-3}. \qquad (1)$$

So then

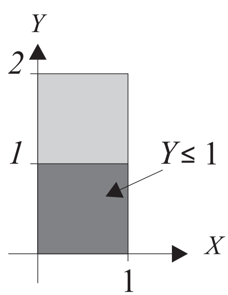
$$f_{X,Y|A}(x,y) = \begin{cases} \frac{6e^{-(2x+3y)}}{1-3e^{-2}+2e^{-3}} & x+y \leq 1, x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \qquad (2)$$

Problem 7.3.5 Solution

The joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} (x+y)/3 & 0 \leq x \leq 1, 0 \leq y \leq 2, \\ 0 & \text{otherwise.} \end{cases} \qquad (1)$$

(a) The probability that $Y \leq 1$ is



$$\begin{aligned} \mathrm{P}[A] = \mathrm{P}[Y \leq 1] &= \iint_{y \leq 1} f_{X,Y}(x,y)\, dx\, dy \\ &= \int_0^1 \int_0^1 \frac{x+y}{3}\, dy\, dx \\ &= \int_0^1 \left(\frac{xy}{3} + \frac{y^2}{6} \Big|_{y=0}^{y=1} \right) dx \\ &= \int_0^1 \frac{2x+1}{6}\, dx \\ &= \frac{x^2}{6} + \frac{x}{6} \Big|_0^1 = \frac{1}{3}. \end{aligned} \qquad (2)$$

(b) By Definition 7.7, the conditional joint PDF of X and Y given A is

$$f_{X,Y|A}(x,y) = \begin{cases} \frac{f_{X,Y}(x,y)}{\mathrm{P}[A]} & (x,y) \in A, \\ 0 & \text{otherwise}, \end{cases}$$
$$= \begin{cases} x+y & 0 \le x \le 1, 0 \le y \le 1, \\ 0 & \text{otherwise}. \end{cases} \tag{3}$$

From $f_{X,Y|A}(x,y)$, we find the conditional marginal PDF $f_{X|A}(x)$. For $0 \le x \le 1$,

$$f_{X|A}(x) = \int_{-\infty}^{\infty} f_{X,Y|A}(x,y)\ dy$$
$$= \int_0^1 (x+y)\,dy = xy + \left.\frac{y^2}{2}\right|_{y=0}^{y=1} = x + \frac{1}{2}. \tag{4}$$

The complete expression is

$$f_{X|A}(x) = \begin{cases} x + 1/2 & 0 \le x \le 1, \\ 0 & \text{otherwise}. \end{cases} \tag{5}$$

For $0 \le y \le 1$, the conditional marginal PDF of Y is

$$f_{Y|A}(y) = \int_{-\infty}^{\infty} f_{X,Y|A}(x,y)\ dx$$
$$= \int_0^1 (x+y)\,dx = \left.\frac{x^2}{2} + xy\right|_{x=0}^{x=1} = y + \frac{1}{2}. \tag{6}$$

The complete expression is

$$f_{Y|A}(y) = \begin{cases} y + 1/2 & 0 \le y \le 1, \\ 0 & \text{otherwise}. \end{cases} \tag{7}$$

Problem 7.3.7 Solution

(a) For a woman, B and T are independent and thus

$$P_{B,T|W}(b,t) = P_{B|W}(b)\, P_{T|W}(t)\,. \tag{1}$$

Expressed in the form of a table, we have

$P_{B,T\|W}(b,t)$	$t=0$	$t=1$	$t=2$
$b=0$	0.36	0.12	0.12
$b=1$	0.18	0.06	0.06
$b=2$	0.06	0.02	0.02

(2)

(b) For a man, B and T are independent and thus

$$P_{B,T|M}(b,t) = P_{B|M}(b)\, P_{T|M}(t)\,. \tag{3}$$

Expressed in the form of a table, we have

$P_{B,T\|M}(b,t)$	$t=0$	$t=1$	$t=2$
$b=0$	0.04	0.04	0.12
$b=1$	0.06	0.06	0.18
$b=2$	0.10	0.10	0.30

(4)

(c) To find the joint PMF, we use the law of total probability to write

$$\begin{aligned} P_{B,T}(b,t) &= \mathrm{P}[W]\, P_{B,T|W}(b,t) + \mathrm{P}[M]\, P_{B,T|M}(b,t) \\ &= \frac{1}{2} P_{B,T|W}(b,t) + \frac{1}{2} P_{B,T|M}(b,t)\,. \end{aligned} \tag{5}$$

Equation (5) amounts to adding the tables for $P_{B,T|W}(b,t)$ and $P_{B,T|M}(b,t)$ and dividing by two. This yields

$P_{B,T}(b,t)$	$t=0$	$t=1$	$t=2$
$b=0$	0.20	0.08	0.12
$b=1$	0.12	0.06	0.12
$b=2$	0.08	0.06	0.16

(6)

(d) To check independence, we compute the marginal PMFs by writing the row and column sums:

$P_{B,T}(b,t)$	$t=0$	$t=1$	$t=2$	$P_B(b)$
$b=0$	0.20	0.08	0.12	0.40
$b=1$	0.12	0.06	0.12	0.30
$b=2$	0.08	0.06	0.16	0.30
$P_T(t)$	0.40	0.20	0.40	

(7)

We see that B and T are dependent since $P_{B,T}(b,t) \neq P_B(b)P_T(t)$. For example, $P_{B,T}(0,0) = 0.20$ but $P_B(0)P_T(0) = (0.40)(0.40) = 0.16$.

Now we calculate the covariance $\text{Cov}[B,T] = \text{E}[BT] - \text{E}[B]\,\text{E}[T]$ via

$$\text{E}[B] = \sum_{b=0}^{2} bP_B(b) = (1)(0.30) + (2)(0.30) = 0.90, \tag{8}$$

$$\text{E}[T] = \sum_{t=0}^{2} tP_T(t) = (1)(0.20) + (2)(0.40) = 1.0, \tag{9}$$

$$\begin{aligned} \text{E}[BT] &= \sum_{b=0}^{2}\sum_{t=0}^{2} btP_{B,T}(b,t) \\ &= (1\cdot 1)(0.06) + (1\cdot 2)(0.12) \\ &\quad + (2\cdot 1)(0.06) + (2\cdot 2)(0.16) \\ &= 1.06. \end{aligned} \tag{10}$$

Thus the covariance is

$$\text{Cov}[B,T] = \text{E}[BT] - \text{E}[B]\,\text{E}[T] = 1.06 - (0.90)(1.0) = 0.16. \tag{11}$$

Thus baldness B and time watching football T are positively correlated. This should seem odd since B and T are uncorrelated for men and also uncorrelated for women. However, men tend to be balder and watch more football while women tend to be not bald and watch less football. Averaged over mean and women, the result is that baldness and football watching are positively correlated because given that a person watches more football, the person is more likely to be a man and thus is more likely to be balder than average (since the average includes women who tend not to be bald.)

Problem 7.3.9 Solution

X and Y are independent random variables with PDFs

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1, \\ 0 & \text{otherwise}, \end{cases} \qquad f_Y(y) = \begin{cases} 3y^2 & 0 \le y \le 1, \\ 0 & \text{otherwise}. \end{cases} \tag{1}$$

For the event $A = \{X > Y\}$, this problem asks us to calculate the conditional expectations $\mathrm{E}[X|A]$ and $\mathrm{E}[Y|A]$. We will do this using the conditional joint PDF $f_{X,Y|A}(x,y)$. Since X and Y are independent, it is tempting to argue that the event $X > Y$ does not alter the probability model for X and Y. Unfortunately, this is not the case. When we learn that $X > Y$, it increases the probability that X is large and Y is small. We will see this when we compare the conditional expectations $\mathrm{E}[X|A]$ and $\mathrm{E}[Y|A]$ to $\mathrm{E}[X]$ and $\mathrm{E}[Y]$.

(a) We can calculate the unconditional expectations, $\mathrm{E}[X]$ and $\mathrm{E}[Y]$, using the marginal PDFs $f_X(x)$ and $f_Y(y)$.

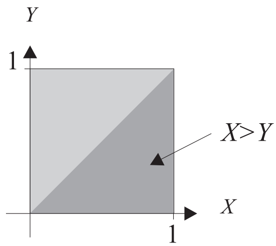
$$\mathrm{E}[X] = \int_{-\infty}^{\infty} f_X(x)\, dx = \int_0^1 2x^2\, dx = 2/3, \tag{2}$$

$$\mathrm{E}[Y] = \int_{-\infty}^{\infty} f_Y(y)\, dy = \int_0^1 3y^3\, dy = 3/4. \tag{3}$$

(b) First, we need to calculate the conditional joint PDF $ipdf X, Y|Ax, y$. The first step is to write down the joint PDF of X and Y:

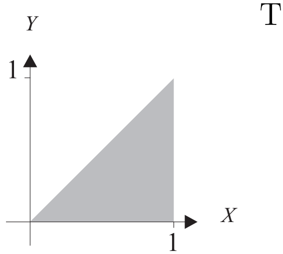
$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) = \begin{cases} 6xy^2 & 0 \le x \le 1, 0 \le y \le 1, \\ 0 & \text{otherwise}. \end{cases} \tag{4}$$

The event A has probability



$$P[A] = \iint_{x>y} f_{X,Y}(x,y)\, dy\, dx = \int_0^1 \int_0^x 6xy^2\, dy\, dx = \int_0^1 2x^4\, dx = 2/5. \tag{5}$$

The conditional joint PDF of X and Y given A is



$$f_{X,Y|A}(x,y) = \begin{cases} \frac{f_{X,Y}(x,y)}{P[A]} & (x,y) \in A, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} 15xy^2 & 0 \le y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

The triangular region of nonzero probability is a signal that given A, X and Y are no longer independent. The conditional expected value of X given A is

$$E[X|A] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X,Y|A}(x,y|a)\, x, y\, dy\, dx = 15 \int_0^1 x^2 \int_0^x y^2\, dy\, dx = 5 \int_0^1 x^5\, dx = 5/6. \tag{7}$$

The conditional expected value of Y given A is

$$\begin{aligned}
\mathrm{E}[Y|A] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{X,Y|A}(x,y)\,dy\,dx \\
&= 15\int_0^1 x\int_0^x y^3\,dy\,dx \\
&= \frac{15}{4}\int_0^1 x^5\,dx = 5/8. \qquad (8)
\end{aligned}$$

We see that $\mathrm{E}[X|A] > \mathrm{E}[X]$ while $\mathrm{E}[Y|A] < \mathrm{E}[Y]$. That is, learning $X > Y$ gives us a clue that X may be larger than usual while Y may be smaller than usual.

Problem 7.4.1 Solution

These conditional PDFs require no calculation. Straight from the problem statement,

$$f_{Y_1|X}(y_1|1) = \frac{1}{\sqrt{2\pi}} e^{-(y_1-x)^2/2}, \qquad (1)$$

$$f_{Y_2|X}(y_2|x) = \frac{1}{\sqrt{2\pi x^2}} e^{-(y_2-x)^2/2x^2}. \qquad (2)$$

Conditional PDFs like $f_{Y_1|X}(y_1|x)$ occur often. Conditional PDFs resembling $f_{Y_2|X}(y_2|x)$ are fairly uncommon.

Problem 7.4.3 Solution

This problem is mostly about translating words to math. From the words, we learned that

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1, \\ 0 & \text{otherwise}, \end{cases} \qquad (1)$$

$$f_{Y|X}(y|x) = \begin{cases} 1/(1+x) & 0 \le y \le 1+x, \\ 0 & \text{otherwise}. \end{cases} \qquad (2)$$

It follows that the joint PDF is

$$
\begin{aligned}
f_{X,Y}(x,y) &= f_{Y|X}(y|x)\, f_X(x) \\
&= \begin{cases} 1/(1+x) & 0 \le x \le 1, 0 \le y \le 1+x, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{3}
$$

Problem 7.4.5 Solution

The main part of this problem is just interpreting the problem statement. No calculations are necessary. Since a trip is equally likely to last 2, 3 or 4 days,

$$
P_D(d) = \begin{cases} 1/3 & d = 2, 3, 4, \\ 0 & \text{otherwise.} \end{cases} \tag{1}
$$

Given a trip lasts d days, the weight change is equally likely to be any value between $-d$ and d pounds. Thus,

$$
P_{W|D}(w|d) = \begin{cases} 1/(2d+1) & w = -d, -d+1, \ldots, d, \\ 0 & \text{otherwise.} \end{cases} \tag{2}
$$

The joint PMF is simply

$$
\begin{aligned}
P_{D,W}(d,w) &= P_{W|D}(w|d)\, P_D(d) \\
&= \begin{cases} 1/(6d+3) & d = 2, 3, 4; w = -d, \ldots, d, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{3}
$$

Problem 7.4.7 Solution

We are told in the problem statement that if we know r, the number of feet a student sits from the blackboard, then we also know that that student's grade is a Gaussian random variable with mean $80 - r$ and standard deviation r. This is exactly

$$
f_{X|R}(x|r) = \frac{1}{\sqrt{2\pi r^2}} e^{-(x-[80-r])^2/2r^2}. \tag{1}
$$

Problem 7.4.9 Solution

This problem is fairly easy when we use conditional PMF's. In particular, given that $N = n$ pizzas were sold before noon, each of those pizzas has mushrooms with probability 1/3. The conditional PMF of M given N is the binomial distribution

$$P_{M|N}(m|n) = \binom{n}{m}(1/3)^m(2/3)^{n-m}. \tag{1}$$

Since $P_{M|N}(m|n)$ depends on the event $N = n$, we see that M and N are dependent.

The other fact we know is that for each of the 100 pizzas sold, the pizza is sold before noon with probability 1/2. Hence, N has the binomial PMF

$$P_N(n) = \binom{100}{n}(1/2)^n(1/2)^{100-n}. \tag{2}$$

The joint PMF of N and M is for integers m, n,

$$\begin{aligned} P_{M,N}(m,n) &= P_{M|N}(m|n)\, P_N(n) \\ &= \binom{n}{m}\binom{100}{n}(1/3)^m(2/3)^{n-m}(1/2)^{100}. \end{aligned} \tag{3}$$

Problem 7.4.11 Solution

We can make a table of the possible outcomes and the corresponding values of W and Y

outcome	$P[\cdot]$	W	Y
hh	p^2	0	2
ht	$p(1-p)$	1	1
th	$p(1-p)$	-1	1
tt	$(1-p)^2$	0	0

(1)

In the following table, we write the joint PMF $P_{W,Y}(w,y)$ along with the marginal PMFs $P_Y(y)$ and $P_W(w)$.

$$
\begin{array}{c|cccc}
P_{W,Y}(w,y) & w=-1 & w=0 & w=1 & P_Y(y) \\
\hline
y=0 & 0 & (1-p)^2 & 0 & (1-p)^2 \\
y=1 & p(1-p) & 0 & p(1-p) & 2p(1-p) \\
y=2 & 0 & p^2 & 0 & p^2 \\
P_W(w) & p(1-p) & 1-2p+2p^2 & p(1-p) &
\end{array}
\tag{2}
$$

Using the definition $P_{W|Y}(w|y) = P_{W,Y}(w,y)/P_Y(y)$, we can find the conditional PMFs of W given Y:

$$
P_{W|Y}(w|0) = \begin{cases} 1 & w = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3}
$$

$$
P_{W|Y}(w|1) = \begin{cases} 1/2 & w = -1, 1, \\ 0 & \text{otherwise,} \end{cases} \tag{4}
$$

$$
P_{W|Y}(w|2) = \begin{cases} 1 & w = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5}
$$

Similarly, the conditional PMFs of Y given W are

$$
P_{Y|W}(y|-1) = \begin{cases} 1 & y = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{6}
$$

$$
P_{Y|W}(y|0) = \begin{cases} \frac{(1-p)^2}{1-2p+2p^2} & y = 0, \\ \frac{p^2}{1-2p+2p^2} & y = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{7}
$$

$$
P_{Y|W}(y|1) = \begin{cases} 1 & y = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8}
$$

Problem 7.4.13 Solution

The key to solving this problem is to find the joint PMF of M and N. Note that $N \geq M$. For $n > m$, the joint event $\{M = m, N = n\}$ has probability

$$
\begin{aligned}
\mathrm{P}[M=m, N=n] &= \mathrm{P}[\overbrace{dd\cdots d}^{\substack{m-1 \\ \text{calls}}}\, v\, \overbrace{dd\cdots d}^{\substack{n-m-1 \\ \text{calls}}}\, v] \\
&= (1-p)^{m-1} p (1-p)^{n-m-1} p \\
&= (1-p)^{n-2} p^2. \qquad (1)
\end{aligned}
$$

A complete expression for the joint PMF of M and N is

$$
P_{M,N}(m,n) = \begin{cases} (1-p)^{n-2}p^2 & m = 1,2,\ldots,n-1; \\ & n = m+1, m+2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \qquad (2)
$$

The marginal PMF of N satisfies

$$
P_N(n) = \sum_{m=1}^{n-1} (1-p)^{n-2}p^2 = (n-1)(1-p)^{n-2}p^2, \qquad n = 2,3,\ldots. \qquad (3)
$$

Similarly, for $m = 1,2,\ldots$, the marginal PMF of M satisfies

$$
\begin{aligned}
P_M(m) &= \sum_{n=m+1}^{\infty} (1-p)^{n-2}p^2 \\
&= p^2[(1-p)^{m-1} + (1-p)^m + \cdots] \\
&= (1-p)^{m-1}p. \qquad (4)
\end{aligned}
$$

The complete expressions for the marginal PMF's are

$$
P_M(m) = \begin{cases} (1-p)^{m-1}p & m = 1,2,\ldots, \\ 0 & \text{otherwise,} \end{cases} \qquad (5)
$$

$$
P_N(n) = \begin{cases} (n-1)(1-p)^{n-2}p^2 & n = 2,3,\ldots, \\ 0 & \text{otherwise.} \end{cases} \qquad (6)
$$

Not surprisingly, if we view each voice call as a successful Bernoulli trial, M has a geometric PMF since it is the number of trials up to and including the first success. Also, N has a Pascal PMF since it is the number of trials required to see 2 successes. The conditional PMF's are now easy to find.

$$P_{N|M}(n|m) = \frac{P_{M,N}(m,n)}{P_M(m)} = \begin{cases} (1-p)^{n-m-1}p & n = m+1, m+2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

The interpretation of the conditional PMF of N given M is that given $M = m$, $N = m + N'$ where N' has a geometric PMF with mean $1/p$. The conditional PMF of M given N is

$$P_{M|N}(m|n) = \frac{P_{M,N}(m,n)}{P_N(n)} = \begin{cases} 1/(n-1) & m = 1, \ldots, n-1, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Given that call $N = n$ was the second voice call, the first voice call is equally likely to occur in any of the previous $n-1$ calls.

Problem 7.4.15 Solution

If you construct a tree describing what type of packet (if any) that arrived in any 1 millisecond period, it will be apparent that an email packet arrives with probability $\alpha = pqr$ or no email packet arrives with probability $1 - \alpha$. That is, whether an email packet arrives each millisecond is a Bernoulli trial with success probability α. Thus, the time required for the first success has the geometric PMF

$$P_T(t) = \begin{cases} (1-\alpha)^{t-1}\alpha & t = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Note that N is the number of trials required to observe 100 successes. Moreover, the number of trials needed to observe 100 successes is $N = T + N'$ where N' is the number of trials needed to observe successes 2 through 100. Since N' is just the number of trials needed to observe 99 successes, it has the Pascal $(k = 99, p)$ PMF

$$P_{N'}(n) = \binom{n-1}{98} \alpha^{99}(1-\alpha)^{n-99}. \tag{2}$$

Since the trials needed to generate successes 2 though 100 are independent of the trials that yield the first success, N' and T are independent. Hence

$$P_{N|T}(n|t) = P_{N'|T}(n-t|t) = P_{N'}(n-t). \tag{3}$$

Applying the PMF of N' found above, we have

$$P_{N|T}(n|t) = \binom{n-t-1}{98} \alpha^{99}(1-\alpha)^{n-t-99}. \tag{4}$$

Finally the joint PMF of N and T is

$$\begin{aligned} P_{N,T}(n,t) &= P_{N|T}(n|t)\, P_T(t) \\ &= \binom{n-t-1}{98} \alpha^{100}(1-\alpha)^{n-100}. \end{aligned} \tag{5}$$

This solution can also be found a consideration of the sample sequence of Bernoulli trials in which we either observe or do not observe an email packet.

To find the conditional PMF $P_{T|N}(t|n)$, we first must recognize that N is simply the number of trials needed to observe 100 successes and thus has the Pascal PMF

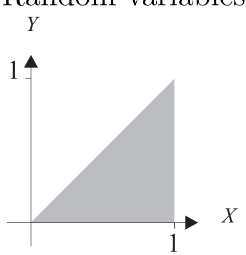
$$P_N(n) = \binom{n-1}{99} \alpha^{100}(1-\alpha)^{n-100}. \tag{6}$$

Hence for any integer $n \geq 100$, the conditional PMF is

$$P_{T|N}(t|n) = \frac{P_{N,T}(n,t)}{P_N(n)}. = \frac{\binom{n-t-1}{98}}{\binom{n-1}{99}}. \tag{7}$$

Problem 7.5.1 Solution

Random variables X and Y have joint PDF



$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

For $0 \le y \le 1$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \int_y^1 2\, dx = 2(1-y). \tag{2}$$

Also, for $y < 0$ or $y > 1$, $f_Y(y) = 0$. The complete expression for the marginal PDF is

$$f_Y(y) = \begin{cases} 2(1-y) & 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

By Theorem 7.10, the conditional PDF of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{1-y} & y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

That is, since $Y \le X \le 1$, X is uniform over $[y,1]$ when $Y = y$. The conditional expectation of X given $Y = y$ can be calculated as

$$\mathrm{E}\left[X|Y=y\right] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx \tag{5}$$

$$= \int_y^1 \frac{x}{1-y}\, dx = \left.\frac{x^2}{2(1-y)}\right|_y^1 = \frac{1+y}{2}. \tag{6}$$

In fact, since we know that the conditional PDF of X is uniform over $[y,1]$ when $Y = y$, it wasn't really necessary to perform the calculation.

Problem 7.5.3 Solution

(a) First we observe that A takes on the values $S_A = \{-1, 1\}$ while B takes on values from $S_B = \{0, 1\}$. To construct a table describing $P_{A,B}(a,b)$ we build a table for all possible values of pairs (A,B). The general form of the entries is

$P_{A,B}(a,b)$	$b=0$	$b=1$
$a=-1$	$P_{B\|A}(0\|-1)\, P_A(-1)$	$P_{B\|A}(1\|-1)\, P_A(-1)$
$a=1$	$P_{B\|A}(0\|1)\, P_A(1)$	$P_{B\|A}(1\|1)\, P_A(1)$

(1)

Now we fill in the entries using the conditional PMFs $P_{B|A}(b|a)$ and the marginal PMF $P_A(a)$. This yields

$$\begin{array}{c|cc} P_{A,B}(a,b) & b=0 & b=1 \\ \hline a=-1 & (1/3)(1/3) & (2/3)(1/3) \\ a=1 & (1/2)(2/3) & (1/2)(2/3) \end{array}, \tag{2}$$

which simplifies to

$$\begin{array}{c|cc} P_{A,B}(a,b) & b=0 & b=1 \\ \hline a=-1 & 1/9 & 2/9 \\ a=1 & 1/3 & 1/3 \end{array}. \tag{3}$$

(b) Since $P_A(1) = P_{A,B}(1,0) + P_{A,B}(1,1) = 2/3$,

$$P_{B|A}(b|1) = \frac{P_{A,B}(1,b)}{P_A(1)} = \begin{cases} 1/2 & b=0,1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

If $A = 1$, the conditional expectation of B is

$$\mathrm{E}[B|A=1] = \sum_{b=0}^{1} b P_{B|A}(b|1) = P_{B|A}(1|1) = 1/2. \tag{5}$$

(c) Before finding the conditional PMF $P_{A|B}(a|1)$, we first sum the columns of the joint PMF table to find

$$P_B(b) = \begin{cases} 4/9 & b=0, \\ 5/9 & b=1, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

The conditional PMF of A given $B = 1$ is

$$P_{A|B}(a|1) = \frac{P_{A,B}(a,1)}{P_B(1)} = \begin{cases} 2/5 & a=-1, \\ 3/5 & a=1, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

(d) Now that we have the conditional PMF $P_{A|B}(a|1)$, calculating conditional expectations is easy.

$$\mathrm{E}[A|B=1] = \sum_{a=-1,1} a P_{A|B}(a|1) = -1(2/5) + (3/5) = 1/5, \tag{8}$$

$$\mathrm{E}\left[A^2|B=1\right] = \sum_{a=-1,1} a^2 P_{A|B}(a|1) = 2/5 + 3/5 = 1. \tag{9}$$

The conditional variance is then

$$\begin{aligned}\mathrm{Var}[A|B=1] &= \mathrm{E}\left[A^2|B=1\right] - (\mathrm{E}[A|B=1])^2 \\ &= 1 - (1/5)^2 = 24/25.\end{aligned} \tag{10}$$

(e) To calculate the covariance, we need

$$\mathrm{E}[A] = \sum_{a=-1,1} a P_A(a) = -1(1/3) + 1(2/3) = 1/3, \tag{11}$$

$$\mathrm{E}[B] = \sum_{b=0}^{1} b P_B(b) = 0(4/9) + 1(5/9) = 5/9, \tag{12}$$

$$\begin{aligned}\mathrm{E}[AB] &= \sum_{a=-1,1} \sum_{b=0}^{1} ab P_{A,B}(a,b) \\ &= -1(0)(1/9) + -1(1)(2/9) + 1(0)(1/3) + 1(1)(1/3) \\ &= 1/9.\end{aligned} \tag{13}$$

The covariance is just

$$\begin{aligned}\mathrm{Cov}[A,B] &= \mathrm{E}[AB] - \mathrm{E}[A]\,\mathrm{E}[B] \\ &= 1/9 - (1/3)(5/9) = -2/27.\end{aligned} \tag{14}$$

Problem 7.5.5 Solution

Random variables N and K have the joint PMF

$$P_{N,K}(n,k) = \begin{cases} \frac{100^n e^{-100}}{(n+1)!} & k = 0,1,\ldots,n; \\ & n = 0,1,\ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) We can find the marginal PMF for N by summing over all possible K. For $n \geq 0$,

$$P_N(n) = \sum_{k=0}^{n} \frac{100^n e^{-100}}{(n+1)!} = \frac{100^n e^{-100}}{n!}. \tag{2}$$

We see that N has a Poisson PMF with expected value 100. For $n \geq 0$, the conditional PMF of K given $N = n$ is

$$P_{K|N}(k|n) = \frac{P_{N,K}(n,k)}{P_N(n)} = \begin{cases} 1/(n+1) & k = 0, 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

That is, given $N = n$, K has a discrete uniform PMF over $\{0, 1, \ldots, n\}$. Thus,

$$\mathrm{E}[K|N = n] = \sum_{k=0}^{n} k/(n+1) = n/2. \tag{4}$$

(b) Since $\mathrm{E}[K|N = n] = n/2$, we can conclude that $\mathrm{E}[K|N] = N/2$. Thus, by Theorem 7.13,

$$\mathrm{E}[K] = \mathrm{E}[\mathrm{E}[K|N]] = \mathrm{E}[N/2] = 50, \tag{5}$$

since N is Poisson with $\mathrm{E}[N] = 100$.

Problem 7.5.7 Solution

We are given that the joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} 1/(\pi r^2) & 0 \leq x^2 + y^2 \leq r^2, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) The marginal PDF of X is

$$f_X(x) = 2\int_0^{\sqrt{r^2-x^2}} \frac{1}{\pi r^2}\, dy = \begin{cases} \frac{2\sqrt{r^2-x^2}}{\pi r^2} & -r \leq x \leq r, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The conditional PDF of Y given X is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} 1/(2\sqrt{r^2-x^2}) & y^2 \leq r^2 - x^2, \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

(b) Given $X = x$, we observe that over the interval $[-\sqrt{r^2-x^2}, \sqrt{r^2-x^2}]$, Y has a uniform PDF. Since the conditional PDF $f_{Y|X}(y|x)$ is symmetric about $y = 0$,

$$\mathrm{E}[Y|X = x] = 0. \quad (4)$$

Problem 7.5.9 Solution

Since 50 cents of each dollar ticket is added to the jackpot,

$$J_{i-1} = J_i + \frac{N_i}{2}. \quad (1)$$

Given $J_i = j$, N_i has a Poisson distribution with mean j. It follows that $\mathrm{E}[N_i|J_i = j] = j$ and that $\mathrm{Var}[N_i|J_i = j] = j$. This implies

$$\begin{aligned} \mathrm{E}\left[N_i^2|J_i = j\right] &= \mathrm{Var}[N_i|J_i = j] + (\mathrm{E}[N_i|J_i = j])^2 \\ &= j + j^2. \end{aligned} \quad (2)$$

In terms of the conditional expectations given J_i, these facts can be written as

$$\mathrm{E}[N_i|J_i] = J_i \qquad \mathrm{E}\left[N_i^2|J_i\right] = J_i + J_i^2. \quad (3)$$

This permits us to evaluate the moments of J_{i-1} in terms of the moments of J_i. Specifically,

$$\mathrm{E}[J_{i-1}|J_i] = \mathrm{E}[J_i|J_i] + \frac{1}{2}\mathrm{E}[N_i|J_i] = J_i + \frac{J_i}{2} = \frac{3J_i}{2}. \quad (4)$$

Using the iterated expectation, this implies

$$\mathrm{E}[J_{i-1}] = \mathrm{E}[\mathrm{E}[J_{i-1}|J_i]] = \frac{3}{2}\mathrm{E}[J_i]. \quad (5)$$

We can use this the calculate $\mathrm{E}[J_i]$ for all i. Since the jackpot starts at 1 million dollars, $J_6 = 10^6$ and $\mathrm{E}[J_6] = 10^6$. This implies

$$\mathrm{E}\left[J_i\right] = (3/2)^{6-i} 10^6 \tag{6}$$

Now we will find the second moment $\mathrm{E}[J_i^2]$. Since

$$J_{i-1}^2 = J_i^2 + N_i J_i + N_i^2/4, \tag{7}$$

we have

$$\begin{aligned}
\mathrm{E}\left[J_{i-1}^2 | J_i\right] &= \mathrm{E}\left[J_i^2 | J_i\right] + \mathrm{E}\left[N_i J_i | J_i\right] + \mathrm{E}\left[N_i^2 | J_i\right]/4 \\
&= J_i^2 + J_i \, \mathrm{E}\left[N_i | J_i\right] + (J_i + J_i^2)/4 \\
&= (3/2)^2 J_i^2 + J_i/4.
\end{aligned} \tag{8}$$

By taking the expectation over J_i we have

$$\mathrm{E}\left[J_{i-1}^2\right] = (3/2)^2 \, \mathrm{E}\left[J_i^2\right] + \mathrm{E}\left[J_i\right]/4 \tag{9}$$

This recursion allows us to calculate $\mathrm{E}[J_i^2]$ for $i = 6, 5, \ldots, 0$. Since $J_6 = 10^6$, $\mathrm{E}[J_6^2] = 10^{12}$. From the recursion, we obtain

$$\begin{aligned}
\mathrm{E}\left[J_5^2\right] &= (3/2)^2 \, \mathrm{E}\left[J_6^2\right] + \mathrm{E}\left[J_6\right]/4 \\
&= (3/2)^2 10^{12} + \frac{1}{4} 10^6, \qquad (10) \\
\mathrm{E}\left[J_4^2\right] &= (3/2)^2 \, \mathrm{E}\left[J_5^2\right] + \mathrm{E}\left[J_5\right]/4 \\
&= (3/2)^4 10^{12} + \frac{1}{4}\left[(3/2)^2 + (3/2)\right] 10^6, \qquad (11) \\
\mathrm{E}\left[J_3^2\right] &= (3/2)^2 \, \mathrm{E}\left[J_4^2\right] + \mathrm{E}\left[J_4\right]/4 \\
&= (3/2)^6 10^{12} + \frac{1}{4}\left[(3/2)^4 + (3/2)^3 + (3/2)^2\right] 10^6. \qquad (12)
\end{aligned}$$

The same recursion will also allow us to show that

$$\mathrm{E}\left[J_2^2\right] = (3/2)^8 10^{12} + \frac{1}{4}\left[(3/2)^6 + (3/2)^5 + (3/2)^4 + (3/2)^3\right]10^6, \tag{13}$$

$$\begin{aligned}\mathrm{E}\left[J_1^2\right] &= (3/2)^{10}10^{12} \\ &\quad + \frac{1}{4}\left[(3/2)^8 + (3/2)^7 + (3/2)^6 + (3/2)^5 + (3/2)^4\right]10^6,\end{aligned} \tag{14}$$

$$\mathrm{E}\left[J_0^2\right] = (3/2)^{12}10^{12} + \frac{1}{4}\left[(3/2)^{10} + (3/2)^9 + \cdots + (3/2)^5\right]10^6. \tag{15}$$

Finally, day 0 is the same as any other day in that $J = J_0 + N_0/2$ where N_0 is a Poisson random variable with mean J_0. By the same argument that we used to develop recursions for $\mathrm{E}[J_i]$ and $\mathrm{E}[J_i^2]$, we can show

$$\mathrm{E}\left[J\right] = (3/2)\,\mathrm{E}\left[J_0\right] = (3/2)^7 10^6 \approx 17 \times 10^6. \tag{16}$$

and

$$\begin{aligned}\mathrm{E}\left[J^2\right] &= (3/2)^2\,\mathrm{E}\left[J_0^2\right] + \mathrm{E}\left[J_0\right]/4 \\ &= (3/2)^{14}10^{12} + \frac{1}{4}\left[(3/2)^{12} + (3/2)^{11} + \cdots + (3/2)^6\right]10^6 \\ &= (3/2)^{14}10^{12} + \frac{10^6}{2}(3/2)^6[(3/2)^7 - 1].\end{aligned} \tag{17}$$

Finally, the variance of J is

$$\mathrm{Var}[J] = \mathrm{E}\left[J^2\right] - (\mathrm{E}\left[J\right])^2 = \frac{10^6}{2}(3/2)^6[(3/2)^7 - 1]. \tag{18}$$

Since the variance is hard to interpret, we note that the standard deviation of J is $\sigma_J \approx 9572$. Although the expected jackpot grows rapidly, the standard deviation of the jackpot is fairly small.

Problem 7.6.1 Solution

This problem is actually easy and short if you think carefully.

(a) Since Z is Gaussian $(0,2)$ and Z and X are independent,

$$f_{Z|X}(z|x) = f_Z(z) = \frac{1}{\sqrt{8\pi}} e^{-z^2/8}. \tag{1}$$

(b) Using the hint, we observe that if $X = 2$, then $Y = 2 + Z$. Furthermore, independence of X and Z implies that given $X = 2$, Z still has the Gaussian PDF $f_Z(z)$. Thus, given $X = x = 2$, $Y = 2 + Z$ is conditionally Gaussian with

$$\mathrm{E}[Y|X=2] = 2 + \mathrm{E}[Z|X=2] = 2, \tag{2}$$
$$\mathrm{Var}[Y|X=2] = \mathrm{Var}[2+Z|X=2] = \mathrm{Var}[Z|X=2] = 2. \tag{3}$$

The conditional PDF of Y is

$$f_{Y|X}(y|2) = \frac{1}{\sqrt{8\pi}} e^{-(y-2)^2/8}. \tag{4}$$

Problem 7.6.3 Solution

We need to calculate

$$\mathrm{Cov}\left[\hat{X}, \hat{Y}\right] = \mathrm{E}\left[\hat{X}\hat{Y}\right] - \mathrm{E}\left[\hat{X}\right]\mathrm{E}\left[\hat{Y}\right]. \tag{1}$$

To do so, we need to condition on whether a cyclist is male (event M) or female (event F):

$$\begin{aligned}\mathrm{E}\left[\hat{X}\right] &= p\,\mathrm{E}\left[\hat{X}|M\right] + (1-p)\,\mathrm{E}\left[\hat{X}|F\right] \\ &= p\,\mathrm{E}[X] + (1-p)\,\mathrm{E}[X'] = 0.8(20) + (0.2)(15) = 16, \qquad (2)\\ \mathrm{E}\left[\hat{Y}\right] &= p\,\mathrm{E}[Y|M] + (1-p)\,\mathrm{E}[Y|F] \\ &= p\,\mathrm{E}[Y] + (1-p)\,\mathrm{E}[Y'] = 0.8(75) + (0.2)(50) = 70. \qquad (3)\end{aligned}$$

Similarly, for the correlation term,

$$\begin{aligned}\mathrm{E}\left[\hat{X}\hat{Y}\right] &= p\,\mathrm{E}\left[\hat{X}\hat{Y}|M\right] + (1-p)\,\mathrm{E}\left[\hat{X}\hat{Y}|F\right] \\ &= 0.8\,\mathrm{E}[XY] + 0.2\,\mathrm{E}[X'Y']. \qquad (4)\end{aligned}$$

However, each of these terms needs some additional calculation:

$$\begin{aligned} \mathrm{E}[XY] &= \mathrm{Cov}[X,Y] + \mathrm{E}[X]\,\mathrm{E}[Y] \\ &= \rho_{XY}\sigma_X\sigma_Y + \mathrm{E}[X]\,\mathrm{E}[Y] \\ &= -0.6(10) + (20)(75) = 1494. \end{aligned} \tag{5}$$

and

$$\begin{aligned} \mathrm{E}[X'Y'] &= \mathrm{Cov}[X',Y'] + \mathrm{E}[X']\,\mathrm{E}[Y'] \\ &= \rho_{X'Y'}\sigma_{X'}\sigma_{Y'} + \mathrm{E}[X']\,\mathrm{E}[Y'] \\ &= -0.6(10) + (15)(50) = 744. \end{aligned} \tag{6}$$

Thus,

$$\begin{aligned} \mathrm{E}\left[\hat{X}\hat{Y}\right] &= 0.8\,\mathrm{E}[XY] + 0.2\,\mathrm{E}[X'Y'] \\ &= 0.8(1494) + 0.2(744) = 1344. \end{aligned} \tag{7}$$

and

$$\begin{aligned} \mathrm{Cov}[X,Y] &= \mathrm{E}\left[\hat{X}\hat{Y}\right] - \mathrm{E}\left[\hat{X}\right]\mathrm{E}\left[\hat{Y}\right] \\ &= 1344 - (70)(19) = 14. \end{aligned} \tag{8}$$

Thus we see that the covariance of $\hat{X}$ and $\hat{Y}$ is positive. It follows that $\rho_{\hat{X}\hat{Y}} > 0$. Hence speed $\hat{X}$ and weight $\hat{Y}$ are positively correlated when we choose a cyclist randomly among men and women even though they are negatively correlated for women and negatively correlated for men. The reason for this is that men are heavier but they also ride faster than women. When we mix the populations, a fast rider is likely to be a male rider who is likely to be a relatively heavy rider (compared to a woman).

Problem 7.6.5 Solution

The key to this problem is to see that the integrals in the given proof of Theorem 5.19 are actually iterated expectation. We start with the definition

$$\rho_{X,Y} = \frac{\mathrm{E}[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X\sigma_Y}. \tag{1}$$

To evaluate this expected value, we use the method of iterated expectation in Theorem 7.14 to write

$$\begin{aligned}
\rho_{X,Y} &= \frac{\mathrm{E}\left[\mathrm{E}\left[(X-\mu_X)(Y-\mu_Y)|Y\right]\right]}{\sigma_X\sigma_Y} \\
&= \frac{\mathrm{E}\left[(Y-\mu_Y)\,\mathrm{E}\left[(X-\mu_X)|Y\right]\right]}{\sigma_X\sigma_Y}. \qquad (2)
\end{aligned}$$

In Equation (2), the "given Y" conditioning allows us to treat $Y-\mu_Y$ as a given that comes outside of the inner expectation. Next, Theorem 7.16 implies

$$\mathrm{E}\left[(X-\mu_X)|Y\right] = \mathrm{E}\left[X|Y\right] - \mu_X = \rho\frac{\sigma_X}{\sigma_Y}(Y-\mu_Y). \qquad (3)$$

Therefore, (2) and (2) imply

$$\begin{aligned}
\rho_{X,Y} &= \frac{\mathrm{E}\left[(Y-\mu_Y)\rho\frac{\sigma_X}{\sigma_Y}(Y-\mu_Y)\right]}{\sigma_X\sigma_Y} \\
&= \frac{\rho\,\mathrm{E}\left[(Y-\mu_Y)^2\right]}{\sigma_Y^2}. = \rho. \qquad (4)
\end{aligned}$$

Problem 7.7.1 Solution

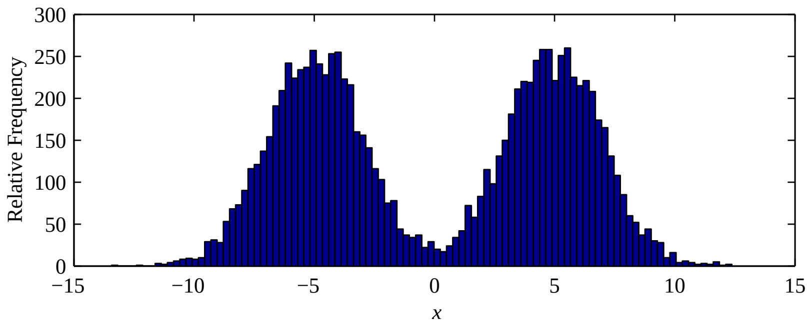
The modem receiver voltage is genrated by taking a ± 5 voltage representing data, and adding to it a Gaussian $(0,2)$ noise variable. Although siuations in which two random variables are added together are not analyzed until Chapter 5, generating samples of the receiver voltage is easy in MATLAB. Here is the code:

```
function x=modemrv(m);
%Usage: x=modemrv(m)
%generates m samples of X, the modem
%receiver voltage in Exampe 3.32.
%X=+-5 + N where N is Gaussian (0,2)
sb=[-5; 5]; pb=[0.5; 0.5];
b=finiterv(sb,pb,m);
noise=gaussrv(0,2,m);
x=b+noise;
```

The commands

```
x=modemrv(10000); hist(x,100);
```

generate 10,000 sample of the modem receiver voltage and plots the relative frequencies using 100 bins. Here is an example plot:



As expected, the result is qualitatively similar ("hills" around $X = -5$ and $X = 5$) to the sketch in Figure 4.3.

Problem Solutions – Chapter 8

Problem 8.1.1 Solution

This problem is very simple. In terms of the vector $\mathbf{X}$, the PDF is

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} 1 & \mathbf{0} \leq \mathbf{x} \leq \mathbf{1}, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

However, just keep in mind that the inequalities $\mathbf{0} \leq \mathbf{x}$ and $\mathbf{x} \leq \mathbf{1}$ are vector inequalities that must hold for every component x_i.

Problem 8.1.3 Solution

Filling in the parameters in Problem 8.1.2, we obtain the vector PDF

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} \frac{2}{3}(x_1 + x_2 + x_3) & 0 \leq x_1, x_2, x_3 \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

In this case, for $0 \leq x_3 \leq 1$, the marginal PDF of X_3 is

$$\begin{aligned} f_{X_3}(x_3) &= \frac{2}{3} \int_0^1 \int_0^1 (x_1 + x_2 + x_3)\, dx_1\, dx_2 \\ &= \frac{2}{3} \int_0^1 \left(\frac{x_1^2}{2} + x_2 x_1 + x_3 x_1 \right) \Bigg|_{x_1=0}^{x_1=1} dx_2 \\ &= \frac{2}{3} \int_0^1 \left(\frac{1}{2} + x_2 + x_3 \right) dx_2 \\ &= \frac{2}{3} \left(\frac{x_2}{2} + \frac{x_2^2}{2} + x_3 x_2 \right) \Bigg|_{x_2=0}^{x_2=1} = \frac{2}{3} \left(\frac{1}{2} + \frac{1}{2} + x_3 \right) \end{aligned} \tag{2}$$

The complete expresion for the marginal PDF of X_3 is

$$f_{X_3}(x_3) = \begin{cases} 2(1 + x_3)/3 & 0 \leq x_3 \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Problem 8.1.5 Solution

Since J_1, J_2 and J_3 are independent, we can write

$$P_{\mathbf{K}}(\mathbf{k}) = P_{J_1}(k_1)\, P_{J_2}(k_2 - k_1)\, P_{J_3}(k_3 - k_2)\,. \tag{1}$$

Since $P_{J_i}(j) > 0$ only for integers $j > 0$, we have that $P_{\mathbf{K}}(\mathbf{k}) > 0$ only for $0 < k_1 < k_2 < k_3$; otherwise $P_{\mathbf{K}}(\mathbf{k}) = 0$. Finally, for $0 < k_1 < k_2 < k_3$,

$$\begin{aligned} P_{\mathbf{K}}(\mathbf{k}) &= (1-p)^{k_1-1}p(1-p)^{k_2-k_1-1}p(1-p)^{k_3-k_2-1}p \\ &= (1-p)^{k_3-3}p^3. \end{aligned} \tag{2}$$

Problem 8.1.7 Solution

In Example 5.21, random variables $N_1, \ldots, N_r$ have the multinomial distribution

$$P_{N_1,\ldots,N_r}(n_1, \ldots, n_r) = \binom{n}{n_1, \ldots, n_r} p_1^{n_1} \cdots p_r^{n_r} \tag{1}$$

where $n > r > 2$.

(a) To evaluate the joint PMF of N_1 and N_2, we define a new experiment with mutually exclusive events: s_1, s_2 and "other" Let $\hat{N}$ denote the number of trial outcomes that are "other". In this case, a trial is in the "other" category with probability $\hat{p} = 1 - p_1 - p_2$. The joint PMF of N_1, N_2, and $\hat{N}$ is

$$P_{N_1,N_2,\hat{N}}(n_1, n_2, \hat{n}) = \binom{n}{n_1, n_2, \hat{n}} p_1^{n_1} p_2^{n_2} (1 - p_1 - p_2)^{\hat{n}}. \tag{2}$$

Now we note that the following events are one in the same:

$$\{N_1 = n_1, N_2 = n_2\} = \left\{N_1 = n_1, N_2 = n_2, \hat{N} = n - n_1 - n_2\right\}. \tag{3}$$

Hence,

$$\begin{aligned} P_{N_1,N_2}(n_1, n_2) &= P_{N_1,N_2,\hat{N}}(n_1, n_2, n - n_1 - n_2) \\ &= \binom{n}{n_1, n_2, n - n_1 - n_2} p_1^{n_1} p_2^{n_2} (1 - p_1 - p_2)^{n-n_1-n_2}. \end{aligned} \tag{4}$$

From the definition of the multinomial coefficient, $P_{N_1,N_2}(n_1, n_2)$ is nonzero only for non-negative integers n_1 and n_2 satisfying $n_1 + n_2 \leq n$.

(b) We could find the PMF of T_i by summing $P_{N_1,\ldots,N_r}(n_1, \ldots, n_r)$. However, it is easier to start from first principles. Suppose we say a success occurs if the outcome of the trial is in the set $\{s_1, s_2, \ldots, s_i\}$ and otherwise a failure occurs. In this case, the success probability is $q_i = p_1 + \cdots + p_i$ and T_i is the number of successes in n trials. Thus, T_i has the binomial PMF

$$P_{T_i}(t) = \begin{cases} \binom{n}{t} q_i^t (1 - q_i)^{n-t} & t = 0, 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

(c) The joint PMF of T_1 and T_2 satisfies

$$\begin{aligned} P_{T_1,T_2}(t_1, t_2) &= \mathrm{P}\left[N_1 = t_1, N_1 + N_2 = t_2\right] \\ &= \mathrm{P}\left[N_1 = t_1, N_2 = t_2 - t_1\right] \\ &= P_{N_1,N_2}(t_1, t_2 - t_1). \end{aligned} \tag{6}$$

By the result of part (a),

$$P_{T_1,T_2}(t_1, t_2) = \binom{n}{t_1, t_2 - t_1, n - t_2} p_1^{t_1} p_2^{t_2 - t_1} (1 - p_1 - p_2)^{n - t_2}. \tag{7}$$

Similar to the previous parts, keep in mind that $P_{T_1,T_2}(t_1, t_2)$ is nonzero only if $0 \leq t_1 \leq t_2 \leq n$.

Problem 8.1.9 Solution

In Problem 8.1.5, we found that the joint PMF of $\mathbf{K} = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}'$ is

$$P_{\mathbf{K}}(\mathbf{k}) = \begin{cases} p^3 (1 - p)^{k_3 - 3} & k_1 < k_2 < k_3, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

In this problem, we generalize the result to n messages.

(a) For $k_1 < k_2 < \cdots < k_n$, the joint event

$$\{K_1 = k_1, K_2 = k_2, \cdots, K_n = k_n\} \tag{2}$$

occurs if and only if all of the following events occur

A_1 $k_1 - 1$ failures, followed by a successful transmission,
A_2 $(k_2 - 1) - k_1$ failures followed by a successful transmission,
A_3 $(k_3 - 1) - k_2$ failures followed by a successful transmission,
$\vdots$
A_n $(k_n - 1) - k_{n-1}$ failures followed by a successful transmission.

Note that the events $A_1, A_2, \ldots, A_n$ are independent and

$$\mathrm{P}[A_j] = (1-p)^{k_j - k_{j-1} - 1} p. \tag{3}$$

Thus

$$\begin{aligned} &P_{K_1,\ldots,K_n}(k_1, \ldots, k_n) \\ &\quad = \mathrm{P}[A_1]\,\mathrm{P}[A_2]\cdots\mathrm{P}[A_n] \\ &\quad = p^n (1-p)^{(k_1-1)+(k_2-k_1-1)+(k_3-k_2-1)+\cdots+(k_n-k_{n-1}-1)} \\ &\quad = p^n (1-p)^{k_n - n}. \end{aligned} \tag{4}$$

To clarify subsequent results, it is better to rename $\mathbf{K}$ as

$$\mathbf{K}_n = \begin{bmatrix} K_1 & K_2 & \cdots & K_n \end{bmatrix}'. \tag{5}$$

We see that

$$P_{\mathbf{K}_n}(\mathbf{k}_n) = \begin{cases} p^n (1-p)^{k_n - n} & 1 \le k_1 < k_2 < \cdots < k_n, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

(b) For $j < n$,

$$P_{K_1,K_2,\ldots,K_j}(k_1, k_2, \ldots, k_j) = P_{\mathbf{K}_j}(\mathbf{k}_j). \tag{7}$$

Since $\mathbf{K}_j$ is just $\mathbf{K}_n$ with $n = j$, we have

$$P_{\mathbf{K}_j}(\mathbf{k}_j) = \begin{cases} p^j(1-p)^{k_j - j} & 1 \leq k_1 < k_2 < \cdots < k_j, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

(c) Rather than try to deduce $P_{K_i}(k_i)$ from the joint PMF $P_{\mathbf{K}_n}(\mathbf{k}_n)$, it is simpler to return to first principles. In particular, K_i is the number of trials up to and including the ith success and has the Pascal (i, p) PMF

$$P_{K_i}(k_i) = \binom{k_i - 1}{i - 1} p^i (1-p)^{k_i - i}. \tag{9}$$

Problem 8.2.1 Solution

For $i \neq j$, X_i and X_j are independent and $\mathrm{E}[X_i X_j] = \mathrm{E}[X_i]\,\mathrm{E}[X_j] = 0$ since $\mathrm{E}[X_i] = 0$. Thus the i,jth entry in the covariance matrix $\mathbf{C}_\mathbf{X}$ is

$$C_\mathbf{X}(i,j) = \mathrm{E}\left[X_i X_j\right] = \begin{cases} \sigma_i^2 & i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Thus for random vector $\mathbf{X} = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}'$, all the off-diagonal entries in the covariance matrix are zero and the covariance matrix is

$$\mathbf{C}_\mathbf{X} = \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_n^2 \end{bmatrix}. \tag{2}$$

Problem 8.2.3 Solution

We will use the PDF

$$f_\mathbf{X}(\mathbf{x}) = \begin{cases} 6e^{-(x_1 + 2x_2 + 3x_3)} & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

to find the marginal PDFs $f_{X_i}(x_i)$. In particular, for $x_1 \geq 0$,

$$\begin{aligned} f_{X_1}(x_1) &= \int_0^\infty \int_0^\infty f_{\mathbf{X}}(\mathbf{x})\, dx_2\, dx_3 \\ &= 6e^{-x_1} \left(\int_0^\infty e^{-2x_2}\, dx_2 \right) \left(\int_0^\infty \right) e^{-3x_3}\, dx_3 \\ &= 6e^{-x_1} \left(-\frac{1}{2} e^{-2x_2} \Big|_0^\infty \right) \left(-\frac{1}{3} e^{-3x_3} \Big|_0^\infty \right) = e^{-x_1}. \end{aligned} \tag{2}$$

Thus,

$$f_{X_1}(x_1) = \begin{cases} e^{-x_1} & x_1 \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Following similar steps, one can show that

$$f_{X_2}(x_2) = \int_0^\infty \int_0^\infty f_{\mathbf{X}}(\mathbf{x})\, dx_1\, dx_3 = \begin{cases} 2^{-2x_2} & x_2 \geq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

$$f_{X_3}(x_3) = \int_0^\infty \int_0^\infty f_{\mathbf{X}}(\mathbf{x})\, dx_1\, dx_2 = \begin{cases} 3^{-3x_3} & x_3 \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Thus

$$f_{\mathbf{X}}(\mathbf{x}) = f_{X_1}(x_1)\, f_{X_2}(x_2)\, f_{X_3}(x_3)\,. \tag{6}$$

We conclude that X_1, X_2, and X_3 are independent.

Problem 8.2.5 Solution

We find the marginal PDFs using Theorem 5.26. First we note that for $x < 0$, $f_{X_i}(x) = 0$. For $x_1 \geq 0$,

$$f_{X_1}(x_1) = \int_{x_1}^\infty \left(\int_{x_2}^\infty e^{-x_3}\, dx_3 \right) dx_2 = \int_{x_1}^\infty e^{-x_2}\, dx_2 = e^{-x_1}. \tag{1}$$

Similarly, for $x_2 \geq 0$, X_2 has marginal PDF

$$f_{X_2}(x_2) = \int_0^{x_2} \left(\int_{x_2}^{\infty} e^{-x_3}\, dx_3 \right) dx_1 = \int_0^{x_2} e^{-x_2}\, dx_1 = x_2 e^{-x_2}. \tag{2}$$

Lastly,

$$\begin{aligned} f_{X_3}(x_3) &= \int_0^{x_3} \left(\int_{x_1}^{x_3} e^{-x_3}\, dx_2 \right) dx_1 \\ &= \int_0^{x_3} (x_3 - x_1) e^{-x_3}\, dx_1 \\ &= -\frac{1}{2}(x_3 - x_1)^2 e^{-x_3} \bigg|_{x_1=0}^{x_1=x_3} = \frac{1}{2} x_3^2 e^{-x_3}. \end{aligned} \tag{3}$$

The complete expressions for the three marginal PDFs are

$$f_{X_1}(x_1) = \begin{cases} e^{-x_1} & x_1 \geq 0, \\ 0 & \text{otherwise}, \end{cases} \tag{4}$$

$$f_{X_2}(x_2) = \begin{cases} x_2 e^{-x_2} & x_2 \geq 0, \\ 0 & \text{otherwise}, \end{cases} \tag{5}$$

$$f_{X_3}(x_3) = \begin{cases} (1/2) x_3^2 e^{-x_3} & x_3 \geq 0, \\ 0 & \text{otherwise}. \end{cases} \tag{6}$$

In fact, each X_i is an Erlang $(n, \lambda) = (i, 1)$ random variable.

Problem 8.3.1 Solution

For discrete random vectors, it is true in general that

$$P_{\mathbf{Y}}(\mathbf{y}) = \mathrm{P}[\mathbf{Y} = \mathbf{y}] = \mathrm{P}[\mathbf{AX} + \mathbf{b} = \mathbf{y}] = \mathrm{P}[\mathbf{AX} = \mathbf{y} - \mathbf{b}]. \tag{1}$$

For an arbitrary matrix $\mathbf{A}$, the system of equations $\mathbf{Ax} = \mathbf{y} - \mathbf{b}$ may have no solutions (if the columns of $\mathbf{A}$ do not span the vector space), multiple solutions (if the columns of $\mathbf{A}$ are linearly dependent), or, when $\mathbf{A}$ is invertible, exactly one solution. In the invertible case,

$$P_{\mathbf{Y}}(\mathbf{y}) = \mathrm{P}\left[\mathbf{AX} = \mathbf{y} - \mathbf{b}\right] = \mathrm{P}\left[\mathbf{X} = \mathbf{A}^{-1}(\mathbf{y} - \mathbf{b})\right] = P_{\mathbf{X}}\left(\mathbf{A}^{-1}(\mathbf{y} - \mathbf{b})\right). \quad (2)$$

As an aside, we note that when $\mathbf{Ax} = \mathbf{y} - \mathbf{b}$ has multiple solutions, we would need to do some bookkeeping to add up the probabilities $P_{\mathbf{X}}(\mathbf{x})$ for all vectors $\mathbf{x}$ satisfying $\mathbf{Ax} = \mathbf{y} - \mathbf{b}$. This can get disagreeably complicated.

Problem 8.3.3 Solution

The response time X_i of the ith truck has PDF $f_{X_i}(x_i)$ and CDF $F_{X_i}(x_i)$ given by

$$f_{X_i}(x) = \begin{cases} \frac{1}{2}e^{-x/2} & x \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$
$$F_{X_i}(x) = \begin{cases} 1 - e^{-x/2} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Let $R = \max(X_1, X_2, \ldots, X_6)$ denote the maximum response time. From Theorem 8.2, R has PDF

$$F_R(r) = (F_X(r))^6. \quad (2)$$

(a) The probability that all six responses arrive within five seconds is

$$\mathrm{P}\left[R \leq 5\right] = F_R(5) = (F_X(5))^6 = (1 - e^{-5/2})^6 = 0.5982. \quad (3)$$

(b) This question is worded in a somewhat confusing way. The “expected response time” refers to $\mathrm{E}[X_i]$, the response time of an individual truck, rather than $\mathrm{E}[R]$. If the expected response time of a truck is τ, then each X_i has CDF

$$F_{X_i}(x) = F_X(x) = \begin{cases} 1 - e^{-x/\tau} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

The goal of this problem is to find the maximum permissible value of τ. When each truck has expected response time τ, the CDF of R is

$$F_R(r) = (F_X(x)\, r)^6 = \begin{cases} (1 - e^{-r/\tau})^6 & r \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

We need to find τ such that

$$\mathrm{P}\left[R \leq 3\right] = (1 - e^{-3/\tau})^6 = 0.9. \tag{6}$$

This implies

$$\tau = \frac{-3}{\ln\left(1 - (0.9)^{1/6}\right)} = 0.7406 \text{ s}. \tag{7}$$

Problem 8.4.1 Solution

(a) The coavariance matrix of $\mathbf{X} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}'$ is

$$\mathbf{C_X} = \begin{bmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}\left[X_1, X_2\right] \\ \mathrm{Cov}\left[X_1, X_2\right] & \mathrm{Var}[X_2] \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix}. \tag{1}$$

(b) From the problem statement,

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \mathbf{X} = \mathbf{AX}. \tag{2}$$

By Theorem 8.8, $\mathbf{Y}$ has covariance matrix

$$\mathbf{C_Y} = \mathbf{AC_XA'} = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 28 & -66 \\ -66 & 252 \end{bmatrix}. \tag{3}$$

Problem 8.4.3 Solution

Since $\mathbf{X}$ and $\mathbf{Y}$ are independent and $E[Y_j] = 0$ for all components Y_j, we observe that $E[X_iY_j] = E[X_i] E[Y_j] = 0$. This implies that the cross-covariance matrix is

$$E[\mathbf{X}\mathbf{Y}'] = E[\mathbf{X}] E[\mathbf{Y}'] = \mathbf{0}. \tag{1}$$

Problem 8.4.5 Solution

From $\mathbf{C_Y}$ we see that

$$\rho_{Y_1Y_2} = \frac{\text{Cov}[Y_1, Y_2]}{\sqrt{\text{Var}[Y_1] \text{Var}[Y_2]}} = \frac{\gamma}{\sqrt{(25)(4)}} = \gamma/10. \tag{1}$$

The requirement $|\rho_{Y_1Y_2}| \leq 1$ implies $|\gamma| \leq 10$. Note that you can instead require that the eigenvalues of $\mathbf{C_Y}$ are non-negative. This will lead to the same condition.

Problem 8.4.7 Solution

This problem is quite difficult unless one uses the observation that the vector $\mathbf{K}$ can be expressed in terms of the vector $\mathbf{J} = \begin{bmatrix} J_1 & J_2 & J_3 \end{bmatrix}'$ where J_i is the number of transmissions of message i. Note that we can write

$$\mathbf{K} = \mathbf{A}\mathbf{J} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{J}. \tag{1}$$

We also observe that since each transmission is an independent Bernoulli trial with success probability p, the components of $\mathbf{J}$ are iid geometric (p) random variables. Thus $E[J_i] = 1/p$ and $\text{Var}[J_i] = (1-p)/p^2$. Thus $\mathbf{J}$ has expected value

$$E[\mathbf{J}] = \boldsymbol{\mu}_J = \begin{bmatrix} E[J_1] & E[J_2] & E[J_3] \end{bmatrix}' = \begin{bmatrix} 1/p & 1/p & 1/p \end{bmatrix}'. \tag{2}$$

Since the components of $\mathbf{J}$ are independent, it has the diagonal covariance matrix

$$\mathbf{C}_J = \begin{bmatrix} \text{Var}[J_1] & 0 & 0 \\ 0 & \text{Var}[J_2] & 0 \\ 0 & 0 & \text{Var}[J_3] \end{bmatrix} = \frac{1-p}{p^2}\mathbf{I}. \tag{3}$$

Given these properties of $\mathbf{J}$, finding the same properties of $\mathbf{K} = \mathbf{AJ}$ is simple.

(a) The expected value of $\mathbf{K}$ is

$$\text{E}[\mathbf{K}] = \mathbf{A}\boldsymbol{\mu}_J = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/p \\ 1/p \\ 1/p \end{bmatrix} = \begin{bmatrix} 1/p \\ 2/p \\ 3/p \end{bmatrix}. \tag{4}$$

(b) From Theorem 8.8, the covariance matrix of $\mathbf{K}$ is

$$\begin{aligned} \mathbf{C}_K &= \mathbf{A}\mathbf{C}_J\mathbf{A}' \\ &= \frac{1-p}{p^2}\mathbf{A}\mathbf{I}\mathbf{A}' \\ &= \frac{1-p}{p^2} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1-p}{p^2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}. \end{aligned} \tag{5}$$

(c) Given the expected value vector $\boldsymbol{\mu}_K$ and the covariance matrix $\mathbf{C}_K$, we can use Theorem 8.7 to find the correlation matrix

$$\mathbf{R}_K = \mathbf{C}_K + \boldsymbol{\mu}_K \boldsymbol{\mu}_K'$$

$$= \frac{1-p}{p^2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1/p \\ 2/p \\ 3/p \end{bmatrix} \begin{bmatrix} 1/p & 2/p & 3/p \end{bmatrix} \tag{6}$$

$$= \frac{1-p}{p^2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} + \frac{1}{p^2} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \tag{7}$$

$$= \frac{1}{p^2} \begin{bmatrix} 2-p & 3-p & 4-p \\ 3-p & 6-2p & 8-2p \\ 4-p & 8-2p & 12-3p \end{bmatrix}. \tag{8}$$

Problem 8.4.9 Solution

In Example 5.23, we found the marginal PDF of Y_3 is

$$f_{Y_3}(y_3) = \begin{cases} 2(1-y_3) & 0 \le y_3 \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We also need to find the marginal PDFs of Y_1, Y_2, and Y_4. In Equation (5.78) of Example 5.23, we found the marginal PDF

$$f_{Y_1,Y_4}(y_1, y_4) = \begin{cases} 4(1-y_1)y_4 & 0 \le y_1 \le 1, 0 \le y_4 \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

We can use this result to show that

$$f_{Y_1}(y_1) = \int_0^1 f_{Y_1,Y_4}(y_1, y_4)\, dy_4 = 2(1-y_1), \qquad 0 \le y_1 \le 1, \tag{3}$$

$$f_{Y_4}(y_4) = \int_0^1 f_{Y_1,Y_4}(y_1, y_4)\, dy_1 = 2y_4, \qquad 0 \le y_4 \le 1. \tag{4}$$

The full expressions for the marginal PDFs are

$$f_{Y_1}(y_1) = \begin{cases} 2(1-y_1) & 0 \le y_1 \le 1, \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

$$f_{Y_4}(y_4) = \begin{cases} 2y_4 & 0 \le y_4 \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Similarly, we found in Equation (5.80) of Example 5.23 the marginal PDF

$$f_{Y_2,Y_3}(y_2, y_3) = \begin{cases} 4y_2(1-y_3) & 0 \le y_2 \le 1, 0 \le y_3 \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

This implies that for $0 \le y_2 \le 1$,

$$f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f_{Y_2,Y_3}(y_2, y_3)\ dy_3 = \int_0^1 4y_2(1-y_3)\,dy_3 = 2y_2 \tag{8}$$

It follows that the marginal PDF of Y_2 is

$$f_{Y_2}(y_2) = \begin{cases} 2y_2 & 0 \le y_2 \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Equations (1), (5), (6), and (9) imply

$$\mathrm{E}[Y_1] = \mathrm{E}[Y_3] = \int_0^1 2y(1-y)\,dy = 1/3, \tag{10}$$

$$\mathrm{E}[Y_2] = \mathrm{E}[Y_4] = \int_0^1 2y^2\,dy = 2/3. \tag{11}$$

Thus $\mathbf{Y}$ has expected value $\mathrm{E}[\mathbf{Y}] = \begin{bmatrix}1/3 & 2/3 & 1/3 & 2/3\end{bmatrix}'$. The second part of the problem is to find the correlation matrix $\mathbf{R_Y}$. In fact, we need to find $R_{\mathbf{Y}}(i,j) = \mathrm{E}[Y_iY_j]$ for each i,j pair. We will see that these are seriously tedious calculations. For $i = j$, the second moments are

$$\mathrm{E}\left[Y_1^2\right] = \mathrm{E}\left[Y_3^2\right] = \int_0^1 2y^2(1-y)\,dy = 1/6, \tag{12}$$

$$\mathrm{E}\left[Y_2^2\right] = \mathrm{E}\left[Y_4^2\right] = \int_0^1 2y^3\,dy = 1/2. \tag{13}$$

In terms of the correlation matrix,

$$R_{\mathbf{Y}}(1,1) = R_{\mathbf{Y}}(3,3) = 1/6, \qquad R_{\mathbf{Y}}(2,2) = R_{\mathbf{Y}}(4,4) = 1/2. \tag{14}$$

To find the off-diagonal terms $R_{\mathbf{Y}}(i,j) = \mathrm{E}[Y_iY_j]$, we need to find the marginal PDFs $f_{Y_i,Y_j}(y_i,y_j)$. Example 5.23 showed that

$$f_{Y_1,Y_4}(y_1,y_4) = \begin{cases} 4(1-y_1)y_4 & 0 \le y_1 \le 1, 0 \le y_4 \le 1, \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

$$f_{Y_2,Y_3}(y_2,y_3) = \begin{cases} 4y_2(1-y_3) & 0 \le y_2 \le 1, 0 \le y_3 \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Inspection will show that Y_1 and Y_4 are independent since $f_{Y_1,Y_4}(y_1,y_4) = f_{Y_1}(y_1)f_{Y_4}(y_4)$. Similarly, Y_2 and Y_3 are independent since $f_{Y_2,Y_3}(y_2,y_3) = f_{Y_2}(y_2)f_{Y_3}(y_3)$. This implies

$$R_{\mathbf{Y}}(1,4) = \mathrm{E}[Y_1Y_4] = \mathrm{E}[Y_1]\,\mathrm{E}[Y_4] = 2/9, \tag{17}$$

$$R_{\mathbf{Y}}(2,3) = \mathrm{E}[Y_2Y_3] = \mathrm{E}[Y_2]\,\mathrm{E}[Y_3] = 2/9. \tag{18}$$

We also need to calculate the marginal PDFs

$$f_{Y_1,Y_2}(y_1,y_2)\,, \; f_{Y_3,Y_4}(y_3,y_4)\,, \; f_{Y_1,Y_3}(y_1,y_3)\,, \text{ and } f_{Y_2,Y_4}(y_2,y_4)\,.$$

To start, for $0 \le y_1 \le y_2 \le 1$,

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{Y_1,Y_2,Y_3,Y_4}(y_1,y_2,y_3,y_4)\;dy_3\,dy_4 \\ &= \int_0^1\int_0^{y_4} 4\,dy_3\,dy_4 = \int_0^1 4y_4\,dy_4 = 2. \end{aligned} \tag{19}$$

Similarly, for $0 \le y_3 \le y_4 \le 1$,

$$\begin{aligned} f_{Y_3,Y_4}(y_3,y_4) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{Y_1,Y_2,Y_3,Y_4}(y_1,y_2,y_3,y_4)\; dy_1\, dy_2 \\ &= \int_0^1 \int_0^{y_2} 4\, dy_1\, dy_2 = \int_0^1 4y_2\, dy_2 = 2. \end{aligned} \tag{20}$$

In fact, these PDFs are the same in that

$$f_{Y_1,Y_2}(x,y) = f_{Y_3,Y_4}(x,y) = \begin{cases} 2 & 0 \le x \le y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

This implies $R_{\mathbf{Y}}(1,2) = R_{\mathbf{Y}}(3,4) = \mathrm{E}[Y_3Y_4]$ and that

$$\mathrm{E}\left[Y_3Y_4\right] = \int_0^1 \int_0^y 2xy\, dx\, dy = \int_0^1 \left(yx^2\Big|_0^y \right)\, dy = \int_0^1 y^3\, dy = \frac{1}{4}. \tag{22}$$

Continuing in the same way, we see for $0 \le y_1 \le 1$ and $0 \le y_3 \le 1$ that

$$\begin{aligned} f_{Y_1,Y_3}(y_1,y_3) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{Y_1,Y_2,Y_3,Y_4}(y_1,y_2,y_3,y_4)\; dy_2\, dy_4 \\ &= 4\left(\int_{y_1}^1 dy_2\right)\left(\int_{y_3}^1 dy_4\right) \\ &= 4(1-y_1)(1-y_3). \end{aligned} \tag{23}$$

We observe that Y_1 and Y_3 are independent since $f_{Y_1,Y_3}(y_1,y_3) = f_{Y_1}(y_1)f_{Y_3}(y_3)$. It follows that

$$R_{\mathbf{Y}}(1,3) = \mathrm{E}\left[Y_1Y_3\right] = \mathrm{E}\left[Y_1\right]\mathrm{E}\left[Y_3\right] = 1/9. \tag{24}$$

Finally, we need to calculate

$$\begin{aligned} f_{Y_2,Y_4}(y_2,y_4) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{Y_1,Y_2,Y_3,Y_4}(y_1,y_2,y_3,y_4)\; dy_1\, dy_3 \\ &= 4\left(\int_0^{y_2} dy_1\right)\left(\int_0^{y_4} dy_3\right) \\ &= 4y_2y_4. \end{aligned} \tag{25}$$

We observe that Y_2 and Y_4 are independent since $f_{Y_2,Y_4}(y_2, y_4) = f_{Y_2}(y_2) f_{Y_4}(y_4)$. It follows that $R_{\mathbf{Y}}(2,4) = \mathrm{E}[Y_2 Y_4] = \mathrm{E}[Y_2]\,\mathrm{E}[Y_4] = 4/9$. The above results give $R_{\mathbf{Y}}(i,j)$ for $i \le j$. Since $\mathbf{R_Y}$ is a symmetric matrix,

$$\mathbf{R_Y} = \begin{bmatrix} 1/6 & 1/4 & 1/9 & 2/9 \\ 1/4 & 1/2 & 2/9 & 4/9 \\ 1/9 & 2/9 & 1/6 & 1/4 \\ 2/9 & 4/9 & 1/4 & 1/2 \end{bmatrix}. \tag{26}$$

Since $\boldsymbol{\mu}_{\mathbf{X}} = \begin{bmatrix} 1/3 & 2/3 & 1/3 & 2/3 \end{bmatrix}'$, the covariance matrix is

$$\begin{aligned} \mathbf{C_Y} &= \mathbf{R_Y} - \boldsymbol{\mu}_{\mathbf{X}} \boldsymbol{\mu}'_{\mathbf{X}} \\ &= \begin{bmatrix} 1/6 & 1/4 & 1/9 & 2/9 \\ 1/4 & 1/2 & 2/9 & 4/9 \\ 2/9 & 4/9 & 1/4 & 1/2 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 1/3 & 2/3 \end{bmatrix} \\ &= \begin{bmatrix} 1/18 & 1/36 & 0 & 0 \\ 1/36 & 1/18 & 0 & 0 \\ 0 & 0 & 1/18 & 1/36 \\ 0 & 0 & 1/36 & 1/18 \end{bmatrix}. \end{aligned} \tag{27}$$

The off-diagonal zero blocks are a consequence of $\begin{bmatrix} Y_1 & Y_2 \end{bmatrix}'$ being independent of $\begin{bmatrix} Y_3 & Y_4 \end{bmatrix}'$. Along the diagonal, the two identical sub-blocks occur because $f_{Y_1,Y_2}(x,y) = f_{Y_3,Y_4}(x,y)$. In short, the matrix structure is the result of $\begin{bmatrix} Y_1 & Y_2 \end{bmatrix}'$ and $\begin{bmatrix} Y_3 & Y_4 \end{bmatrix}'$ being iid random vectors.

Problem 8.4.11 Solution

The 2-dimensional random vector $\mathbf{Y}$ has PDF

$$f_{\mathbf{Y}}(\mathbf{y}) = \begin{cases} 2 & \mathbf{y} \ge \mathbf{0}, \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{y} \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Rewritten in terms of the variables y_1 and y_2,

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 2 & y_1 \ge 0, y_2 \ge 0, y_1 + y_2 \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

In this problem, the PDF is simple enough that we can compute $\mathrm{E}[Y_i^n]$ for arbitrary integers $n \geq 0$.

$$\mathrm{E}\left[Y_1^n\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_1^n f_{Y_1,Y_2}(y_1, y_2)\ dy_1\, dy_2 = \int_0^1 \int_0^{1-y_2} 2y_1^n\, dy_1\, dy_2. \tag{3}$$

A little calculus yields

$$\begin{aligned}\mathrm{E}\left[Y_1^n\right] &= \int_0^1 \left(\frac{2}{n+1} y_1^{n+1} \Big|_0^{1-y_2} \right) dy_2 \\ &= \frac{2}{n+1} \int_0^1 (1-y_2)^{n+1}\, dy_2 = \frac{2}{(n+1)(n+2)}.\end{aligned} \tag{4}$$

Symmetry of the joint PDF $f_{Y_{1,2}}(y_{1,2})$ implies that $\mathrm{E}[Y_2^n] = \mathrm{E}[Y_1^n]$. Thus, $\mathrm{E}[Y_1] = \mathrm{E}[Y_2] = 1/3$ and

$$\mathrm{E}\left[\mathbf{Y}\right] = \boldsymbol{\mu}_{\mathbf{Y}} = \begin{bmatrix}1/3 & 1/3\end{bmatrix}'. \tag{5}$$

In addition,

$$R_{\mathbf{Y}}(1,1) = \mathrm{E}\left[Y_1^2\right] = 1/6, \qquad R_{\mathbf{Y}}(2,2) = \mathrm{E}\left[Y_2^2\right] = 1/6. \tag{6}$$

To complete the correlation matrix, we find

$$\begin{aligned}R_{\mathbf{Y}}(1,2) &= \mathrm{E}\left[Y_1 Y_2\right] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_1 y_2 f_{Y_1,Y_2}(y_1, y_2)\ dy_1\, dy_2 \\ &= \int_0^1 \int_0^{1-y_2} 2y_1 y_2\, dy_1\, dy_2.\end{aligned} \tag{7}$$

Following through on the calculus, we obtain

$$\begin{aligned}R_{\mathbf{Y}}(1,2) &= \int_0^1 \left(y_1^2 \big|_0^{1-y-2} \right) y_2\, dy_2 \\ &= \int_0^1 y_2 (1-y_2)^2\, dy_2 \\ &= \frac{1}{2} y_2^2 - \frac{2}{3} y_2^3 + \frac{1}{4} y_2^4 \Big|_0^1 = \frac{1}{12}.\end{aligned} \tag{8}$$

Thus we have found that

$$\mathbf{R_Y} = \begin{bmatrix} \mathrm{E}[Y_1^2] & \mathrm{E}[Y_1Y_2] \\ \mathrm{E}[Y_2Y_1] & \mathrm{E}[Y_2^2] \end{bmatrix} = \begin{bmatrix} 1/6 & 1/12 \\ 1/12 & 1/6 \end{bmatrix}. \tag{9}$$

Lastly, $\mathbf{Y}$ has covariance matrix

$$\mathbf{C_Y} = \mathbf{R_Y} - \boldsymbol{\mu}_\mathbf{Y}\boldsymbol{\mu}_\mathbf{Y}' = \begin{bmatrix} 1/6 & 1/12 \\ 1/12 & 1/6 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 \end{bmatrix}$$
$$= \begin{bmatrix} 1/9 & -1/36 \\ -1/36 & 1/9 \end{bmatrix}. \tag{10}$$

Problem 8.5.1 Solution

(a) From Theorem 8.7, the correlation matrix of $\mathbf{X}$ is

$$\mathbf{R}_X = \mathbf{C}_X + \boldsymbol{\mu}_X\boldsymbol{\mu}_X'$$
$$= \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix} \begin{bmatrix} 4 & 8 & 6 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix} + \begin{bmatrix} 16 & 32 & 24 \\ 32 & 64 & 48 \\ 24 & 48 & 36 \end{bmatrix} = \begin{bmatrix} 20 & 30 & 25 \\ 30 & 68 & 46 \\ 25 & 46 & 40 \end{bmatrix}. \tag{1}$$

(b) Let $\mathbf{Y} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}'$. Since $\mathbf{Y}$ is a subset of the components of $\mathbf{X}$, it is a Gaussian random vector with expected velue vector

$$\boldsymbol{\mu}_Y = \begin{bmatrix} \mathrm{E}[X_1] & \mathrm{E}[X_2] \end{bmatrix}' = \begin{bmatrix} 4 & 8 \end{bmatrix}'. \tag{2}$$

and covariance matrix

$$\mathbf{C}_Y = \begin{bmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}[X_1, X_2] \\ \mathbf{C}_{X_1}X_2 & \mathrm{Var}[X_2] \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}. \tag{3}$$

We note that $\det(\mathbf{C}_Y) = 12$ and that

$$\mathbf{C}_Y^{-1} = \frac{1}{12}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}. \tag{4}$$

This implies that

$$\begin{aligned}
(\mathbf{y} - \boldsymbol{\mu}_Y)'\mathbf{C}_Y^{-1}(\mathbf{y} - \boldsymbol{\mu}_Y) &= \begin{bmatrix} y_1 - 4 & y_2 - 8 \end{bmatrix} \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \begin{bmatrix} y_1 - 4 \\ y_2 - 8 \end{bmatrix} \\
&= \begin{bmatrix} y_1 - 4 & y_2 - 8 \end{bmatrix} \begin{bmatrix} y_1/3 + y_2/6 - 8/3 \\ y_1/6 + y_2/3 - 10/3 \end{bmatrix} \\
&= \frac{y_1^2}{3} + \frac{y_1 y_2}{3} - \frac{16 y_1}{3} - \frac{20 y_2}{3} + \frac{y_2^2}{3} + \frac{112}{3}. \qquad (5)
\end{aligned}$$

The PDF of $\mathbf{Y}$ is

$$\begin{aligned}
f_{\mathbf{Y}}(\mathbf{y}) &= \frac{1}{2\pi\sqrt{12}} e^{-(\mathbf{y}-\boldsymbol{\mu}_Y)'\mathbf{C}_Y^{-1}(\mathbf{y}-\boldsymbol{\mu}_Y)/2} \\
&= \frac{1}{\sqrt{48\pi^2}} e^{-(y_1^2 + y_1 y_2 - 16 y_1 - 20 y_2 + y_2^2 + 112)/6} \qquad (6)
\end{aligned}$$

Since $\mathbf{Y} = \begin{bmatrix} X_1, X_2 \end{bmatrix}'$, the PDF of X_1 and X_2 is simply

$$\begin{aligned}
f_{X_1,X_2}(x_1, x_2) &= f_{Y_1,Y_2}(x_1, x_2) \\
&= \frac{1}{\sqrt{48\pi^2}} e^{-(x_1^2 + x_1 x_2 - 16 x_1 - 20 x_2 + x_2^2 + 112)/6}. \qquad (7)
\end{aligned}$$

(c) We can observe directly from $\boldsymbol{\mu}_X$ and $\mathbf{C}_X$ that X_1 is a Gaussian $(4, 2)$ random variable. Thus,

$$\mathrm{P}[X_1 > 8] = \mathrm{P}\left[\frac{X_1 - 4}{2} > \frac{8 - 4}{2}\right] = Q(2) = 0.0228. \tag{8}$$

Problem 8.5.3 Solution

We are given that $\mathbf{X}$ is a Gaussian random vector with

$$\boldsymbol{\mu}_{\mathbf{X}} = \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix}, \qquad \mathbf{C}_{\mathbf{X}} = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix}. \tag{1}$$

We are also given that $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 1/2 & 2/3 \\ 1 & -1/2 & 2/3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}. \tag{2}$$

Since the two rows of $\mathbf{A}$ are linearly independent row vectors, $\mathbf{A}$ has rank 2. By Theorem 8.11, $\mathbf{Y}$ is a Gaussian random vector. Given these facts, the various parts of this problem are just straightforward calculations using Theorem 8.11.

(a) The expected value of $\mathbf{Y}$ is

$$\begin{aligned} \boldsymbol{\mu}_{\mathbf{Y}} &= \mathbf{A}\boldsymbol{\mu}_{\mathbf{X}} + \mathbf{b} \\ &= \begin{bmatrix} 1 & 1/2 & 2/3 \\ 1 & -1/2 & 2/3 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix} + \begin{bmatrix} -4 \\ -4 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}. \end{aligned} \tag{3}$$

(b) The covariance matrix of $\mathbf{Y}$ is

$$\begin{aligned} \mathbf{C}_{\mathbf{Y}} &= \mathbf{A}\mathbf{C}_{\mathbf{X}}\mathbf{A} \\ &= \begin{bmatrix} 1 & 1/2 & 2/3 \\ 1 & -1/2 & 2/3 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1/2 & -1/2 \\ 2/3 & 2/3 \end{bmatrix} \\ &= \frac{1}{9} \begin{bmatrix} 43 & 55 \\ 55 & 103 \end{bmatrix}. \end{aligned} \tag{4}$$

(c) $\mathbf{Y}$ has correlation matrix

$$\begin{aligned}\mathbf{R_Y} &= \mathbf{C_Y} + \boldsymbol{\mu}_\mathbf{Y}\boldsymbol{\mu}'_\mathbf{Y} \\ &= \frac{1}{9}\begin{bmatrix}43 & 55\\ 55 & 103\end{bmatrix} + \begin{bmatrix}8\\0\end{bmatrix}\begin{bmatrix}8 & 0\end{bmatrix} \\ &= \frac{1}{9}\begin{bmatrix}619 & 55\\ 55 & 103\end{bmatrix}. \end{aligned} \tag{5}$$

(d) From $\boldsymbol{\mu}_\mathbf{Y}$, we see that $\mathrm{E}[Y_2] = 0$. From the covariance matrix $\mathbf{C_Y}$, we learn that Y_2 has variance $\sigma_2^2 = C_\mathbf{Y}(2,2) = 103/9$. Since Y_2 is a Gaussian random variable,

$$\begin{aligned}\mathrm{P}\left[-1 \leq Y_2 \leq 1\right] &= \mathrm{P}\left[-\frac{1}{\sigma_2} \leq \frac{Y_2}{\sigma_2} \leq \frac{1}{\sigma_2}\right] \\ &= \Phi\left(\frac{1}{\sigma_2}\right) - \Phi\left(\frac{-1}{\sigma_2}\right) \\ &= 2\Phi\left(\frac{1}{\sigma_2}\right) - 1 \\ &= 2\Phi\left(\frac{3}{\sqrt{103}}\right) - 1 = 0.2325. \end{aligned} \tag{6}$$

Problem 8.5.5 Solution

(a) $\mathbf{C}$ must be symmetric since

$$\alpha = \beta = \mathrm{E}\left[X_1 X_2\right]. \tag{1}$$

In addition, α must be chosen so that $\mathbf{C}$ is positive semi-definite. Since the characteristic equation is

$$\begin{aligned}\det\left(\mathbf{C} - \lambda\mathbf{I}\right) &= (1-\lambda)(4-\lambda) - \alpha^2 \\ &= \lambda^2 - 5\lambda + 4 - \alpha^2 = 0, \end{aligned} \tag{2}$$

the eigenvalues of $\mathbf{C}$ are

$$\lambda_{1,2} = \frac{5 \pm \sqrt{25 - 4(4 - \alpha^2)}}{2}. \tag{3}$$

The eigenvalues are non-negative as long as $\alpha^2 \leq 4$, or $|\alpha| \leq 2$. Another way to reach this conclusion is through the requirement that $|\rho_{X_1 X_2}| \leq 1$.

(b) It remains true that $\alpha = \beta$ and $\mathbf{C}$ must be positive semi-definite. For $\mathbf{X}$ to be a Gaussian vector, $\mathbf{C}$ also must be positive definite. For the eigenvalues of $\mathbf{C}$ to be strictly positive, we must have $|\alpha| < 2$.

(c) Since $\mathbf{X}$ is a Gaussian vector, W is a Gaussian random variable. Thus, we need only calculate

$$\mathrm{E}[W] = 2\,\mathrm{E}[X_1] - \mathrm{E}[X_2] = 0, \tag{4}$$

and

$$\begin{aligned} \mathrm{Var}[W] = \mathrm{E}\left[W^2\right] &= \mathrm{E}\left[4X_1^2 - 4X_1X_2 + X_2^2\right] \\ &= 4\,\mathrm{Var}[X_1] - 4\,\mathrm{Cov}\left[X_1, X_2\right] + \mathrm{Var}[X_2] \\ &= 4 - 4\alpha + 4 = 4(2 - \alpha). \end{aligned} \tag{5}$$

The PDF of W is

$$f_W(w) = \frac{1}{\sqrt{8(2-\alpha)\pi}} e^{-w^2/8(2-\alpha)}. \tag{6}$$

Problem 8.5.7 Solution

(a) Since $\mathbf{X}$ is Gaussian, W is also Gaussian. Thus we need only compute the expected value

$$\mathrm{E}[W] = \mathrm{E}[X_1] + 2\,\mathrm{E}[X_2] = 0$$

and variance

$$\begin{aligned}\operatorname{Var}[W] = \mathrm{E}\left[W^2\right] &= \mathrm{E}\left[(X_1 + 2X_2)^2\right] \\ &= \mathrm{E}\left[X_1^2 + 4X_1X_2 + 4X_2^2\right] \\ &= C_{11} + 4C_{12} + 4C_{22} = 10. \end{aligned} \tag{1}$$

Thus W has the Gaussian $(0, \sqrt{10})$ PDF

$$f_W(w) = \frac{1}{\sqrt{20\pi}} e^{-w^2/20}.$$

(b) We first calculate

$$\mathrm{E}[V] = 0, \qquad \operatorname{Var}[V] = 4\operatorname{Var}[X_1] = 8, \tag{2}$$

$$\mathrm{E}[W] = 0, \qquad \operatorname{Var}[W] = 10, \tag{3}$$

and that V and W have correlation coefficient

$$\begin{aligned}\rho_{VW} &= \frac{\mathrm{E}[VW]}{\sqrt{\operatorname{Var}[V]\operatorname{Var}[W]}} \\ &= \frac{\mathrm{E}[2X_1(X_1 + 2X_2)]}{\sqrt{80}} \\ &= \frac{2C_{11} + 4C_{12}}{\sqrt{80}} = \frac{8}{\sqrt{80}} = \frac{2}{\sqrt{5}}. \end{aligned} \tag{4}$$

Now we recall that the conditional PDF $f_{V|W}(v|w)$ is Gaussian with conditional expected value

$$\begin{aligned}\mathrm{E}[V|W = w] &= \mathrm{E}[V] + \rho_{VW}\frac{\sigma_V}{\sigma_W}(w - \mathrm{E}[W]) \\ &= \frac{2}{\sqrt{5}}\frac{\sqrt{8}}{\sqrt{10}} w = 4w/5 \end{aligned} \tag{5}$$

and conditional variance

$$\operatorname{Var}[V|W] = \operatorname{Var}[V](1 - \rho_{VW}^2) = \frac{8}{5}. \tag{6}$$

It follows that

$$f_{V|W}(v|w) = \frac{1}{\sqrt{2\pi \operatorname{Var}[V|W]}} e^{-(v-\mathrm{E}[V|W])^2/2\operatorname{Var}[V|W]}$$
$$= \sqrt{\frac{5}{16\pi}} e^{-5(v-4w/5)^2/16}. \tag{7}$$

Problem 8.5.9 Solution

(a) First $b = c$ since a covariance matrix is always symmetric. Second, $a = \operatorname{Var}[X_1]$ and $b = \operatorname{Var}[X_2]$. Hence we must have $a > 0$ and $d > 0$. Third, $\mathbf{C}$ must be positive definite, i.e. the eigenvalues of $\mathbf{C}$ must be positive. This can be tackled directly from first principles by solving for the eigenvalues using $\det(()\mathbf{C} - \lambda\mathbf{I}) = 0$. If you do this, you will find, after some algebra that the eigenvalues are

$$\lambda = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4b^2}}{2}. \tag{1}$$

The requirement $\lambda > 0$ holds iff $b^2 < ad$. As it happens, this is precisely the same conditon as requiring the correlation coefficient to have magnitude less than 1:

$$|\rho_{X_1X_2}| = \left|\frac{b}{\sqrt{ad}}\right| < 1. \tag{2}$$

To summarize, there are four requirements:

$$a > 0, \qquad d > 0, \qquad b = c, \qquad b^2 < ad. \tag{3}$$

(b) This is easy: for Gaussian random variables, zero covariance implies X_1 and X_2 are independent. Hence the answer is $b = 0$.

(c) X_1 and X_2 are identical if they have the same variance: $a = d$.

Problem 8.5.11 Solution

(a) From Theorem 8.8, $\mathbf{Y}$ has covariance matrix

$$\begin{aligned}\mathbf{C_Y} &= \mathbf{QC_XQ}' \\ &= \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\sigma_1^2 & 0 \\ 0 & \sigma_2^2\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix} \\ &= \begin{bmatrix}\sigma_1^2\cos^2\theta + \sigma_2^2\sin^2\theta & (\sigma_1^2-\sigma_2^2)\sin\theta\cos\theta \\ (\sigma_1^2-\sigma_2^2)\sin\theta\cos\theta & \sigma_1^2\sin^2\theta + \sigma_2^2\cos^2\theta\end{bmatrix}.\end{aligned} \tag{1}$$

We conclude that Y_1 and Y_2 have covariance

$$\text{Cov}[Y_1, Y_2] = C_{\mathbf{Y}}(1,2) = (\sigma_1^2 - \sigma_2^2)\sin\theta\cos\theta. \tag{2}$$

Since Y_1 and Y_2 are jointly Gaussian, they are independent if and only if $\text{Cov}[Y_1, Y_2] = 0$. Thus, Y_1 and Y_2 are independent for all θ if and only if $\sigma_1^2 = \sigma_2^2$. In this case, when the joint PDF $f_{\mathbf{X}}(\mathbf{x})$ is symmetric in x_1 and x_2. In terms of polar coordinates, the PDF $f_{\mathbf{X}}(\mathbf{x}) = f_{X_1,X_2}(x_1, x_2)$ depends on $r = \sqrt{x_1^2 + x_2^2}$ but for a given r, is constant for all $\phi = \tan^{-1}(x_2/x_1)$. The transformation of $\mathbf{X}$ to $\mathbf{Y}$ is just a rotation of the coordinate system by θ preserves this circular symmetry.

(b) If $\sigma_2^2 > \sigma_1^2$, then Y_1 and Y_2 are independent if and only if $\sin\theta\cos\theta = 0$. This occurs in the following cases:

- $\theta = 0$: $Y_1 = X_1$ and $Y_2 = X_2$
- $\theta = \pi/2$: $Y_1 = -X_2$ and $Y_2 = -X_1$
- $\theta = \pi$: $Y_1 = -X_1$ and $Y_2 = -X_2$
- $\theta = -\pi/2$: $Y_1 = X_2$ and $Y_2 = X_1$

In all four cases, Y_1 and Y_2 are just relabeled versions, possibly with sign changes, of X_1 and X_2. In these cases, Y_1 and Y_2 are independent because X_1 and X_2 are independent. For other values of θ, each Y_i is a linear combination of both X_1 and X_2. This mixing results in correlation between Y_1 and Y_2.

Problem 8.5.13 Solution

As given in the problem statement, we define the m-dimensional vector $\mathbf{X}$, the n-dimensional vector $\mathbf{Y}$ and $\mathbf{W} = \begin{bmatrix} \mathbf{X}' \\ \mathbf{Y}' \end{bmatrix}'$. Note that $\mathbf{W}$ has expected value

$$\boldsymbol{\mu}_{\mathbf{W}} = \mathrm{E}[\mathbf{W}] = \mathrm{E}\left[\begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix}\right] = \begin{bmatrix} \mathrm{E}[\mathbf{X}] \\ \mathrm{E}[\mathbf{Y}] \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu}_{\mathbf{X}} \\ \boldsymbol{\mu}_{\mathbf{Y}} \end{bmatrix}. \tag{1}$$

The covariance matrix of $\mathbf{W}$ is

$$\begin{aligned}
\mathbf{C}_{\mathbf{W}} &= \mathrm{E}[(\mathbf{W} - \boldsymbol{\mu}_{\mathbf{W}})(\mathbf{W} - \boldsymbol{\mu}_{\mathbf{W}})'] \\
&= \mathrm{E}\left[\begin{bmatrix} \mathbf{X} - \boldsymbol{\mu}_{\mathbf{X}} \\ \mathbf{Y} - \boldsymbol{\mu}_{\mathbf{Y}} \end{bmatrix} \begin{bmatrix} (\mathbf{X} - \boldsymbol{\mu}_{\mathbf{X}})' & (\mathbf{Y} - \boldsymbol{\mu}_{\mathbf{Y}})' \end{bmatrix}\right] \\
&= \begin{bmatrix} \mathrm{E}[(\mathbf{X} - \boldsymbol{\mu}_{\mathbf{X}})(\mathbf{X} - \boldsymbol{\mu}_{\mathbf{X}})'] & \mathrm{E}[(\mathbf{X} - \boldsymbol{\mu}_{\mathbf{X}})(\mathbf{Y} - \boldsymbol{\mu}_{\mathbf{Y}})'] \\ \mathrm{E}[(\mathbf{Y} - \boldsymbol{\mu}_{\mathbf{Y}})(\mathbf{X} - \boldsymbol{\mu}_{\mathbf{X}})'] & \mathrm{E}[(\mathbf{Y} - \boldsymbol{\mu}_{\mathbf{Y}})(\mathbf{Y} - \boldsymbol{\mu}_{\mathbf{Y}})'] \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{C}_{\mathbf{X}} & \mathbf{C}_{\mathbf{XY}} \\ \mathbf{C}_{\mathbf{YX}} & \mathbf{C}_{\mathbf{Y}} \end{bmatrix}.
\end{aligned} \tag{2}$$

The assumption that $\mathbf{X}$ and $\mathbf{Y}$ are independent implies that

$$\mathbf{C}_{\mathbf{XY}} = \mathrm{E}[(\mathbf{X} - \boldsymbol{\mu}_{\mathbf{X}})(\mathbf{Y}' - \boldsymbol{\mu}'_{\mathbf{Y}})] = (\mathrm{E}[(\mathbf{X} - \boldsymbol{\mu}_{\mathbf{X}})]\,\mathrm{E}[(\mathbf{Y}' - \boldsymbol{\mu}'_{\mathbf{Y}})] = \mathbf{0}. \tag{3}$$

This also implies $\mathbf{C}_{\mathbf{YX}} = \mathbf{C}'_{\mathbf{XY}} = \mathbf{0}'$. Thus

$$\mathbf{C}_{\mathbf{W}} = \begin{bmatrix} \mathbf{C}_{\mathbf{X}} & \mathbf{0} \\ \mathbf{0}' & \mathbf{C}_{\mathbf{Y}} \end{bmatrix}. \tag{4}$$

Problem 8.6.1 Solution

We can use Theorem 8.11 since the scalar Y is also a 1-dimensional vector. To do so, we write

$$Y = \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix} \mathbf{X} = \mathbf{A}\mathbf{X}. \tag{1}$$

By Theorem 8.11, Y is a Gaussian vector with expected value

$$\begin{aligned} \mathrm{E}[Y] = \mathbf{A}\boldsymbol{\mu}_X &= (\mathrm{E}[X_1] + \mathrm{E}[X_2] + \mathrm{E}[X_3])/3 \\ &= (4+8+6)/3 = 6. \end{aligned} \tag{2}$$

and covariance matrix

$$\begin{aligned} \mathbf{C}_Y = \mathrm{Var}[Y] &= \mathbf{A}\mathbf{C}_X\mathbf{A}' \\ &= \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \frac{2}{3}. \end{aligned} \tag{3}$$

Thus Y is a Gaussian $(6, \sqrt{2/3})$ random variable, implying

$$\begin{aligned} \mathrm{P}[Y > 4] &= \mathrm{P}\left[\frac{Y-6}{\sqrt{2/3}} > \frac{4-6}{\sqrt{2/3}}\right] \\ &= 1 - \Phi(-\sqrt{6}) = \Phi(\sqrt{6}) = 0.9928. \end{aligned} \tag{4}$$

Problem 8.6.3 Solution

Under the model of Quiz 8.6, the temperature on day i and on day j have covariance

$$\mathrm{Cov}[T_i, T_j] = C_T[i-j] = \frac{36}{1+|i-j|}. \tag{1}$$

From this model, the vector $\mathbf{T} = \begin{bmatrix} T_1 & \cdots & T_{31} \end{bmatrix}'$ has covariance matrix

$$\mathbf{C_T} = \begin{bmatrix} C_T[0] & C_T[1] & \cdots & C_T[30] \\ C_T[1] & C_T[0] & \ddots & \vdots \\ \vdots & \ddots & \ddots & C_T[1] \\ C_T[30] & \cdots & C_T[1] & C_T[0] \end{bmatrix}. \tag{2}$$

If you have read the solution to Quiz 8.6, you know that $\mathbf{C_T}$ is a symmetric Toeplitz matrix and that MATLAB has a `toeplitz` function to generate

Toeplitz matrices. Using the toeplitz function to generate the covariance matrix, it is easy to use gaussvector to generate samples of the random vector **T**.Here is the code for estimating P[A] using m samples.

```
function p=julytemp583(m);
c=36./(1+(0:30));
CT=toeplitz(c);
mu=80*ones(31,1);
T=gaussvector(mu,CT,m);
Y=sum(T)/31;
Tmin=min(T);
p=sum((Tmin>=72) & (Y <= 82))/m;
```

```
julytemp583(100000)
ans =
    0.0684
>> julytemp583(100000)
ans =
    0.0706
>> julytemp583(100000)
ans =
    0.0714
>> julytemp583(100000)
ans =
    0.0701
```

We see from repeated experiments with $m = 100{,}000$ trials that P[A] $\approx$ 0.07.

Problem 8.6.5 Solution

When we built poissonrv.m, we went to some trouble to be able to generate m iid samples at once. In this problem, each Poisson random variable that we generate has an expected value that is different from that of any other Poisson random variables. Thus, we must generate the daily jackpots sequentially. Here is a simple program for this purpose.

```
function jackpot=lottery1(jstart,M,D)
%Usage: function j=lottery1(jstart,M,D)
%Perform M trials of the D day lottery
%of Problem 5.5.5 and initial jackpot jstart
jackpot=zeros(M,1);
for m=1:M,
    disp('trm)
jackpot(m)=jstart;
for d=1:D,
    jackpot(m)=jackpot(m)+(0.5*poissonrv(jackpot(m),1));
end
end
```

The main problem with `lottery1` is that it will run *very* slowly. Each call to `poissonrv` generates an entire Poisson PMF $P_X(x)$ for $x = 0, 1, \ldots, x_{\max}$ where $x_{\max} \geq 2 \cdot 10^6$. This is slow in several ways. First, we repeating the calculation of $\sum_{j=1}^{x_{\max}} \log j$ with each call to `poissonrv`. Second, each call to `poissonrv` asks for a Poisson sample value with expected value $\alpha > 1 \cdot 10^6$. In these cases, for small values of x, $P_X(x) = \alpha^x e^{-\alpha x}/x!$ is so small that it is less than the smallest nonzero number that MATLAB can store!

To speed up the simulation, we have written a program `bigpoissonrv` which generates Poisson (α) samples for large α. The program makes an approximation that for a Poisson (α) random variable X, $P_X(x) \approx 0$ for $|x - \alpha| > 6\sqrt{\alpha}$. Since X has standard deviation $\sqrt{\alpha}$, we are assuming that X cannot be more than six standard deviations away from its mean value. The error in this approximation is very small. In fact, for a Poisson (a) random variable, the program `poissonsigma(a,k)` calculates the error $\mathrm{P}[|X - a| > k\sqrt{a}]$. Here is `poissonsigma.m` and some simple calculations:

```
function err=poissonsigma(a,k);
xmin=max(0,floor(a-k*sqrt(a)));
xmax=a+ceil(k*sqrt(a));
sx=xmin:xmax;
logfacts =cumsum([0,log(1:xmax)]);
  %logfacts includes 0 in case xmin=0
  %Now we extract needed values:
logfacts=logfacts(sx+1);
  %pmf(i,:) is a Poisson a(i) PMF
  %    from xmin to xmax
pmf=exp(-a+ (log(a)*sx)-(logfacts));
err=1-sum(pmf);
```

```
>> poissonsigma(1,6)
ans =
  1.0249e-005
>> poissonsigma(10,6)
ans =
  2.5100e-007
>> poissonsigma(100,6)
ans =
  1.2620e-008
>> poissonsigma(1000,6)
ans =
  2.6777e-009
>> poissonsigma(10000,6)
ans =
  1.8081e-009
>> poissonsigma(100000,6)
ans =
 -1.6383e-010
```

The error reported by `poissonsigma(a,k)` should always be positive. In fact, we observe negative errors for very large `a`.For large α and x, numerical calculation of $P_X(x) = \alpha^x e^{-\alpha}/x!$ is tricky because we are taking ratios of very large numbers. In fact, for $\alpha = x =$ 1,000,000, MATLAB calculation of α^x and $x!$ will report infinity while $e^{-\alpha}$ will evaluate as zero. Our method of calculating the Poisson (α) PMF is to use the fact that $\ln x! = \sum_{j=1}^{x} \ln j$ to calculate

$$\exp\left(\ln P_X(x)\right) = \exp\left(x \ln \alpha - \alpha - \sum_{j=1}^{x} \ln j\right). \qquad (1)$$

This method works reasonably well except that the calculation of the logarithm has finite precision. The consequence is that the calculated sum over the PMF can vary from 1 by a very small amount, on the order of 10^{-7} in our experiments. In our problem, the error is inconsequential, however, one should keep in mind that this may not be the case in other other experiments using large Poisson random variables. In any case, we can conclude that within the accuracy of MATLAB's simulated experiments, the approximations to be used by `bigpoissonrv` are not significant.

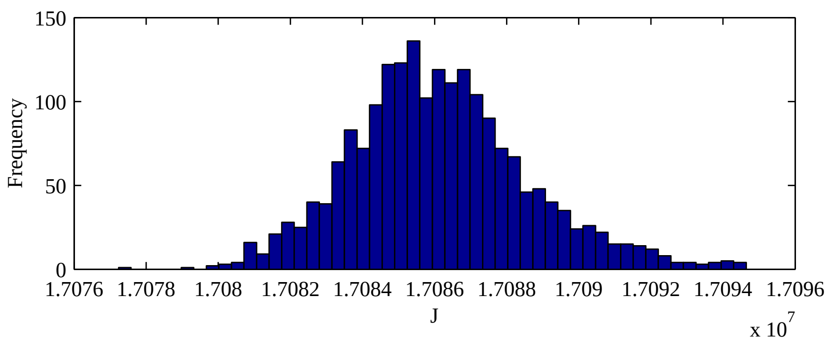
The other feature of `bigpoissonrv` is that for a vector `alpha` corresponding to expected values $\begin{bmatrix}\alpha_1 & \cdots & \alpha_m\end{bmatrix}'$, `bigpoissonrv` returns a vector `X` such that `X(i)` is a Poisson `alpha(i)` sample. The work of calculating the sum of logarithms is done only once for all calculated samples. The result is a significant savings in cpu time as long as the values of `alpha` are reasonably close to each other.

```
function x=bigpoissonrv(alpha)
%for vector alpha, returns a vector x such that
% x(i) is a Poisson (alpha(i)) rv
%set up Poisson CDF from xmin to xmax for each alpha(i)
alpha=alpha(:);
amin=min(alpha(:));
amax=max(alpha(:));
%Assume Poisson PMF is negligible +-6 sigma from the average
xmin=max(0,floor(amin-6*sqrt(amax)));
xmax=amax+ceil(6*sqrt(amax));%set max range
sx=xmin:xmax;
%Now we include the basic code of poissonpmf (but starting at xmin)
logfacts =cumsum([0,log(1:xmax)]); %include 0 in case xmin=0
logfacts=logfacts(sx+1); %extract needed values
%pmf(i,:) is a Poisson alpha(i) PMF from xmin to xmax
pmf=exp(-alpha*ones(size(sx))+ ...
        (log(alpha)*sx)-(ones(size(alpha))*logfacts));
cdf=cumsum(pmf,2); %each row is a cdf
x=(xmin-1)+sum((rand(size(alpha))*ones(size(sx)))<=cdf,2);
```

Finally, given `bigpoissonrv`, we can write a short program `lottery` that simulates trials of the jackpot experiment. Ideally, we would like to use `lottery` to perform $m = 1{,}000$ trials in a single pass. In general, MATLAB is more efficient when calculations are executed in parallel using vectors. However, in `bigpoissonrv`, the matrix `pmf` will have m rows and at least $12\sqrt{\alpha} = 12{,}000$ columns. For m more than several hundred, MATLAB running on my laptop reported an "Out of Memory" error. Thus, we wrote the program lottery to perform M trials at once and to repeat that N times. The output is an $M \times N$ matrix where each i, j entry is a sample jackpot after seven days.

```
function jackpot=lottery(jstart,M,N,D)
%Usage: function j=lottery(jstart,M,N,D)
%Perform M trials of the D day lottery
%of Problem 5.5.5 and initial jackpot jstart
jackpot=zeros(M,N);
for n=1:N,
jackpot(:,n)=jstart*ones(M,1);
for d=1:D,
    disp(d);
    jackpot(:,n)=jackpot(:,n)+(0.5*bigpoissonrv(jackpot(:,n)));
end
end
```

Executing `J=lottery(1e6,200,10,7)` generates a matrix `J` of 2,000 sample jackpots. The command `hist(J(:),50)` generates a histogram of the values with 50 bins. An example is shown here:



Problem Solutions – Chapter 9

Problem 9.1.1 Solution

Let $Y = X_1 - X_2$.

(a) Since $Y = X_1 + (-X_2)$, Theorem 9.1 says that the expected value of the difference is

$$\mathrm{E}[Y] = \mathrm{E}[X_1] + \mathrm{E}[-X_2] = \mathrm{E}[X] - \mathrm{E}[X] = 0. \tag{1}$$

(b) By Theorem 9.2, the variance of the difference is

$$\mathrm{Var}[Y] = \mathrm{Var}[X_1] + \mathrm{Var}[-X_2] = 2\,\mathrm{Var}[X]. \tag{2}$$

Problem 9.1.3 Solution

(a) The PMF of N_1, the number of phone calls needed to obtain the correct answer, can be determined by observing that if the correct answer is given on the nth call, then the previous $n-1$ calls must have given wrong answers so that

$$P_{N_1}(n) = \begin{cases} (3/4)^{n-1}(1/4) & n = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(b) N_1 is a geometric random variable with parameter $p = 1/4$. In Theorem 3.5, the mean of a geometric random variable is found to be $1/p$. For our case, $\mathrm{E}[N_1] = 4$.

(c) Using the same logic as in part (a) we recognize that in order for n to be the fourth correct answer, that the previous $n-1$ calls must have contained exactly 3 correct answers and that the fourth correct answer arrived on the n-th call. This is described by a Pascal random variable.

$$P_{N_4}(n_4) = \begin{cases} \binom{n-1}{3}(3/4)^{n-4}(1/4)^4 & n = 4, 5, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(d) Using the hint given in the problem statement we can find the mean of N_4 by summing up the means of the 4 identically distributed geometric random variables each with mean 4. This gives $\mathrm{E}[N_4] = 4\,\mathrm{E}[N_1] = 16$.

Problem 9.1.5 Solution

We can solve this problem using Theorem 9.2 which says that

$$\mathrm{Var}[W] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}\,[X, Y]\,. \tag{1}$$

The first two moments of X are

$$\mathrm{E}\,[X] = \int_0^1 \int_0^{1-x} 2x\,dy\,dx = \int_0^1 2x(1-x)\,dx = 1/3, \tag{2}$$

$$\mathrm{E}\left[X^2\right] = \int_0^1 \int_0^{1-x} 2x^2\,dy\,dx = \int_0^1 2x^2(1-x)\,dx = 1/6. \tag{3}$$

Thus the variance of X is $\mathrm{Var}[X] = \mathrm{E}[X^2] - (\mathrm{E}[X])^2 = 1/18$. By symmetry, it should be apparent that $\mathrm{E}[Y] = \mathrm{E}[X] = 1/3$ and $\mathrm{Var}[Y] = \mathrm{Var}[X] = 1/18$. To find the covariance, we first find the correlation

$$\mathrm{E}\,[XY] = \int_0^1 \int_0^{1-x} 2xy\,dy\,dx = \int_0^1 x(1-x)^2\,dx = 1/12. \tag{4}$$

The covariance is

$$\mathrm{Cov}\,[X, Y] = \mathrm{E}\,[XY] - \mathrm{E}\,[X]\,\mathrm{E}\,[Y] = 1/12 - (1/3)^2 = -1/36. \tag{5}$$

Finally, the variance of the sum $W = X + Y$ is

$$\begin{aligned} \mathrm{Var}[W] &= \mathrm{Var}[X] + \mathrm{Var}[Y] - 2\,\mathrm{Cov}\,[X, Y] \\ &= 2/18 - 2/36 = 1/18. \end{aligned} \tag{6}$$

For this specific problem, it's arguable whether it would easier to find $\mathrm{Var}[W]$ by first deriving the CDF and PDF of W. In particular, for $0 \le w \le 1$,

$$\begin{aligned} F_W(w) &= \mathrm{P}\,[X + Y \le w] \\ &= \int_0^w \int_0^{w-x} 2\,dy\,dx \\ &= \int_0^w 2(w-x)\,dx = w^2. \end{aligned} \tag{7}$$

Hence, by taking the derivative of the CDF, the PDF of W is

$$f_W(w) = \begin{cases} 2w & 0 \le w \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

From the PDF, the first and second moments of W are

$$\mathrm{E}[W] = \int_0^1 2w^2\,dw = 2/3, \qquad \mathrm{E}\left[W^2\right] = \int_0^1 2w^3\,dw = 1/2. \tag{9}$$

The variance of W is $\mathrm{Var}[W] = \mathrm{E}[W^2] - (\mathrm{E}[W])^2 = 1/18$. Not surprisingly, we get the same answer both ways.

Problem 9.2.1 Solution

For a constant $a > 0$, a zero mean Laplace random variable X has PDF

$$f_X(x) = \frac{a}{2} e^{-a|x|} \qquad -\infty < x < \infty \tag{1}$$

The moment generating function of X is

$$\begin{aligned} \phi_X(s) = \mathrm{E}\left[e^{sX}\right] &= \frac{a}{2}\int_{-\infty}^{0} e^{sx}e^{ax}\,dx + \frac{a}{2}\int_0^\infty e^{sx}e^{-ax}\,dx \\ &= \frac{a}{2}\frac{e^{(s+a)x}}{s+a}\bigg|_{-\infty}^{0} + \frac{a}{2}\frac{e^{(s-a)x}}{s-a}\bigg|_0^\infty \\ &= \frac{a}{2}\left(\frac{1}{s+a} - \frac{1}{s-a}\right) \\ &= \frac{a^2}{a^2 - s^2}. \end{aligned} \tag{2}$$

Problem 9.2.3 Solution

We find the MGF by calculating $\mathrm{E}[e^{sX}]$ from the PDF $f_X(x)$.

$$\phi_X(s) = \mathrm{E}\left[e^{sX}\right] = \int_a^b e^{sX}\frac{1}{b-a}\,dx = \frac{e^{bs} - e^{as}}{s(b-a)}. \tag{1}$$

Now to find the first moment, we evaluate the derivative of $\phi_X(s)$ at $s = 0$.

$$\mathrm{E}[X] = \left.\frac{d\phi_X(s)}{ds}\right|_{s=0} = \left.\frac{s\left[be^{bs} - ae^{as}\right] - \left[e^{bs} - e^{as}\right]}{(b-a)s^2}\right|_{s=0}. \tag{2}$$

Direct evaluation of the above expression at $s = 0$ yields $0/0$ so we must apply l'Hôpital's rule and differentiate the numerator and denominator.

$$\begin{aligned}\mathrm{E}[X] &= \lim_{s\to 0} \frac{be^{bs} - ae^{as} + s\left[b^2e^{bs} - a^2e^{as}\right] - \left[be^{bs} - ae^{as}\right]}{2(b-a)s} \\ &= \lim_{s\to 0} \frac{b^2e^{bs} - a^2e^{as}}{2(b-a)} = \frac{b+a}{2}.\end{aligned} \tag{3}$$

To find the second moment of X, we first find that the second derivative of $\phi_X(s)$ is

$$\frac{d^2\phi_X(s)}{ds^2} = \frac{s^2\left[b^2e^{bs} - a^2e^{as}\right] - 2s\left[be^{bs} - ae^{as}\right] + 2\left[be^{bs} - ae^{as}\right]}{(b-a)s^3}. \tag{4}$$

Substituting $s = 0$ will yield $0/0$ so once again we apply l'Hôpital's rule and differentiate the numerator and denominator.

$$\begin{aligned}\mathrm{E}\left[X^2\right] = \lim_{s\to 0}\frac{d^2\phi_X(s)}{ds^2} &= \lim_{s\to 0}\frac{s^2\left[b^3e^{bs} - a^3e^{as}\right]}{3(b-a)s^2} \\ &= \frac{b^3 - a^3}{3(b-a)} = (b^2 + ab + a^2)/3.\end{aligned} \tag{5}$$

In this case, it is probably simpler to find these moments without using the MGF.

Problem 9.2.5 Solution

The PMF of K is

$$P_K(k) = \begin{cases} 1/n & k = 1, 2, \ldots, n, \\ 0 & \text{otherwise.}\end{cases} \tag{1}$$

The corresponding MGF of K is

$$\begin{aligned}\phi_K(s) = \mathrm{E}\left[e^{sK}\right] &= \frac{1}{n}\left(e^s + e^2s + \cdots + e^{ns}\right) \\ &= \frac{e^s}{n}\left(1 + e^s + e^{2s} + \cdots + e^{(n-1)s}\right) \\ &= \frac{e^s(e^{ns}-1)}{n(e^s-1)}. \end{aligned} \tag{2}$$

We can evaluate the moments of K by taking derivatives of the MGF. Some algebra will show that

$$\frac{d\phi_K(s)}{ds} = \frac{ne^{(n+2)s} - (n+1)e^{(n+1)s} + e^s}{n(e^s-1)^2}. \tag{3}$$

Evaluating $d\phi_K(s)/ds$ at $s = 0$ yields $0/0$. Hence, we apply l'Hôpital's rule twice (by twice differentiating the numerator and twice differentiating the denominator) when we write

$$\begin{aligned}\left.\frac{d\phi_K(s)}{ds}\right|_{s=0} &= \lim_{s\to 0} \frac{n(n+2)e^{(n+2)s} - (n+1)^2 e^{(n+1)s} + e^s}{2n(e^s-1)} \\ &= \lim_{s\to 0} \frac{n(n+2)^2 e^{(n+2)s} - (n+1)^3 e^{(n+1)s} + e^s}{2ne^s} \\ &= (n+1)/2. \end{aligned} \tag{4}$$

A significant amount of algebra will show that the second derivative of the MGF is

$$\begin{aligned}&\frac{d^2\phi_K(s)}{ds^2} \\ &\quad = \frac{n^2 e^{(n+3)s} - (2n^2+2n-1)e^{(n+2)s} + (n+1)^2 e^{(n+1)s} - e^{2s} - e^s}{n(e^s-1)^3}. \end{aligned} \tag{5}$$

Evaluating $d^2\phi_K(s)/ds^2$ at $s = 0$ yields $0/0$.Because $(e^s-1)^3$ appears in the denominator, we need to use l'Hôpital's rule three times to obtain our answer.

$$\begin{aligned}
&\left.\frac{d^2\phi_K(s)}{ds^2}\right|_{s=0} \\
&= \lim_{s\to 0} \frac{n^2(n+3)^3e^{(n+3)s} - (2n^2+2n-1)(n+2)^3e^{(n+2)s} + (n+1)^5 - 8e^{2s} - e^s}{6ne^s} \\
&= \frac{n^2(n+3)^3 - (2n^2+2n-1)(n+2)^3 + (n+1)^5 - 9}{6n} \\
&= (2n+1)(n+1)/6. \qquad (6)
\end{aligned}$$

We can use these results to derive two well known results. We observe that we can directly use the PMF $P_K(k)$ to calculate the moments

$$\mathrm{E}[K] = \frac{1}{n}\sum_{k=1}^{n} k, \qquad \mathrm{E}\left[K^2\right] = \frac{1}{n}\sum_{k=1}^{n} k^2. \qquad (7)$$

Using the answers we found for $\mathrm{E}[K]$ and $\mathrm{E}[K^2]$, we have the formulas

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \qquad (8)$$

Problem 9.3.1 Solution

N is a binomial $(n = 100, p = 0.4)$ random variable. M is a binomial $(n = 50, p = 0.4)$ random variable. Thus N is the sum of 100 independent Bernoulli $(p = 0.4)$ and M is the sum of 50 independent Bernoulli $(p = 0.4)$ random variables. Since M and N are independent, $L = M + N$ is the sum of 150 independent Bernoulli $(p = 0.4)$ random variables. Hence L is a binomial $n = 150, p = 0.4)$ and has PMF

$$P_L(l) = \binom{150}{l}(0.4)^l(0.6)^{150-l}. \qquad (1)$$

Problem 9.3.3 Solution

In the iid random sequence $K_1, K_2, \ldots$, each K_i has PMF

$$P_K(k) = \begin{cases} 1-p & k=0, \\ p & k=1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) The MGF of K is $\phi_K(s) = \text{E}[e^{sK}] = 1 - p + pe^s$.

(b) By Theorem 9.6, $M = K_1 + K_2 + \ldots + K_n$ has MGF

$$\phi_M(s) = [\phi_K(s)]^n = [1 - p + pe^s]^n. \tag{2}$$

(c) Although we could just use the fact that the expectation of the sum equals the sum of the expectations, the problem asks us to find the moments using $\phi_M(s)$. In this case,

$$\begin{aligned} \text{E}[M] &= \left.\frac{d\phi_M(s)}{ds}\right|_{s=0} \\ &= n(1-p+pe^s)^{n-1}pe^s\Big|_{s=0} = np. \end{aligned} \tag{3}$$

The second moment of M can be found via

$$\begin{aligned} \text{E}\left[M^2\right] &= \left.\frac{d\phi_M(s)}{ds}\right|_{s=0} \\ &= np\left((n-1)(1-p+pe^s)pe^{2s} + (1-p+pe^s)^{n-1}e^s\right)\Big|_{s=0} \\ &= np[(n-1)p+1]. \end{aligned} \tag{4}$$

The variance of M is

$$\text{Var}[M] = \text{E}\left[M^2\right] - (\text{E}[M])^2 = np(1-p) = n\,\text{Var}[K]. \tag{5}$$

Problem 9.3.5 Solution

K_i has PMF

$$P_{K_i}(k) = \begin{cases} 2^k e^{-2}/k! & k = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Let $R_i = K_1 + K_2 + \ldots + K_i$

(a) From Table 9.1, we find that the Poisson ($\alpha = 2$) random variable K has MGF $\phi_K(s) = e^{2(e^s - 1)}$.

(b) The MGF of R_i is the product of the MGFs of the K_i's.

$$\phi_{R_i}(s) = \prod_{n=1}^{i} \phi_K(s) = e^{2i(e^s - 1)}. \quad (2)$$

(c) Since the MGF of R_i is of the same form as that of the Poisson with parameter, $\alpha = 2i$. Therefore we can conclude that R_i is in fact a Poisson random variable with parameter $\alpha = 2i$. That is,

$$P_{R_i}(r) = \begin{cases} (2i)^r e^{-2i}/r! & r = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

(d) Because R_i is a Poisson random variable with parameter $\alpha = 2i$, the mean and variance of R_i are then both $2i$.

Problem 9.3.7 Solution

By Theorem 9.6, we know that $\phi_M(s) = [\phi_K(s)]^n$.

(a) The first derivative of $\phi_M(s)$ is

$$\frac{d\phi_M(s)}{ds} = n\left[\phi_K(s)\right]^{n-1} \frac{d\phi_K(s)}{ds}. \quad (1)$$

We can evaluate $d\phi_M(s)/ds$ at $s = 0$ to find E[M].

$$\begin{aligned} \mathrm{E}[M] &= \left.\frac{d\phi_M(s)}{ds}\right|_{s=0} \\ &= n\left[\phi_K(s)\right]^{n-1} \left.\frac{d\phi_K(s)}{ds}\right|_{s=0} = n\,\mathrm{E}[K]. \end{aligned} \tag{2}$$

(b) The second derivative of $\phi_M(s)$ is

$$\begin{aligned} \frac{d^2\phi_M(s)}{ds^2} &= n(n-1)\left[\phi_K(s)\right]^{n-2} \left(\frac{d\phi_K(s)}{ds}\right)^2 \\ &\quad + n\left[\phi_K(s)\right]^{n-1} \frac{d^2\phi_K(s)}{ds^2}. \end{aligned} \tag{3}$$

Evaluating the second derivative at $s = 0$ yields

$$\begin{aligned} \mathrm{E}\left[M^2\right] &= \left.\frac{d^2\phi_M(s)}{ds^2}\right|_{s=0} \\ &= n(n-1)\left(\mathrm{E}[K]\right)^2 + n\,\mathrm{E}\left[K^2\right]. \end{aligned} \tag{4}$$

Problem 9.4.1 Solution

(a) From Table 9.1, we see that the exponential random variable X has MGF

$$\phi_X(s) = \frac{\lambda}{\lambda - s}. \tag{1}$$

(b) Note that K is a geometric random variable identical to the geometric random variable X in Table 9.1 with parameter $p = 1 - q$. From Table 9.1, we know that random variable K has MGF

$$\phi_K(s) = \frac{(1-q)e^s}{1 - qe^s}. \tag{2}$$

Since K is independent of each X_i, $V = X_1 + \cdots + X_K$ is a random sum of random variables. From Theorem 9.10,

$$\begin{aligned}\phi_V(s) &= \phi_K(\ln \phi_X(s)) \\ &= \frac{(1-q)\frac{\lambda}{\lambda-s}}{1-q\frac{\lambda}{\lambda-s}} = \frac{(1-q)\lambda}{(1-q)\lambda - s}.\end{aligned} \tag{3}$$

We see that the MGF of V is that of an exponential random variable with parameter $(1-q)\lambda$. The PDF of V is

$$f_V(v) = \begin{cases} (1-q)\lambda e^{-(1-q)\lambda v} & v \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Problem 9.4.3 Solution

In this problem, $Y = X_1 + \cdots + X_N$ is not a straightforward random sum of random variables because N and the X_i's are dependent. In particular, given $N = n$, then we know that there were exactly 100 heads in N flips. Hence, given N, $X_1 + \cdots + X_N = 100$ *no matter what is the actual value of* N. Hence $Y = 100$ every time and the PMF of Y is

$$P_Y(y) = \begin{cases} 1 & y = 100, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Problem 9.4.5 Solution

Since each ticket is equally likely to have one of $\binom{46}{6}$ combinations, the probability a ticket is a winner is

$$q = \frac{1}{\binom{46}{6}}. \tag{1}$$

Let $X_i = 1$ if the ith ticket sold is a winner; otherwise $X_i = 0$. Since the number K of tickets sold has a Poisson PMF with $E[K] = r$, the number of winning tickets is the random sum

$$V = X_1 + \cdots + X_K. \tag{2}$$

From Appendix A,

$$\phi_X(s) = (1-q) + qe^s, \qquad \phi_K(s) = e^{r[e^s-1]}. \tag{3}$$

By Theorem 9.10,

$$\phi_V(s) = \phi_K(\ln \phi_X(s)) = e^{r[\phi_X(s)-1]}. = e^{rq(e^s-1)} \tag{4}$$

Hence, we see that V has the MGF of a Poisson random variable with mean $\mathrm{E}[V] = rq$. The PMF of V is

$$P_V(v) = \begin{cases} (rq)^v e^{-rq}/v! & v = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Problem 9.4.7 Solution

The way to solve for the mean and variance of U is to use conditional expectations. Given $K = k$, $U = X_1 + \cdots + X_k$ and

$$\begin{aligned} \mathrm{E}[U|K=k] &= \mathrm{E}[X_1 + \cdots + X_k | X_1 + \cdots + X_n = k] \\ &= \sum_{i=1}^{k} \mathrm{E}[X_i | X_1 + \cdots + X_n = k]. \end{aligned} \tag{1}$$

Since X_i is a Bernoulli random variable,

$$\begin{aligned} \mathrm{E}[X_i | X_1 + \cdots + X_n = k] &= \mathrm{P}\left[X_i = 1 | \sum_{j=1}^{n} X_j = k\right] \\ &= \frac{\mathrm{P}\left[X_i = 1, \sum_{j \neq i} X_j = k-1\right]}{\mathrm{P}\left[\sum_{j=1}^{n} X_j = k\right]}. \end{aligned} \tag{2}$$

Note that $\sum_{j=1}^{n} X_j$ is just a binomial random variable for n trials while $\sum_{j \neq i} X_j$ is a binomial random variable for $n-1$ trials. In addition, X_i

and $\sum_{j\neq i} X_j$ are independent random variables. This implies

$$\mathrm{E}\left[X_i|X_1+\cdots+X_n=k\right] = \frac{\mathrm{P}\left[X_i=1\right]\mathrm{P}\left[\sum_{j\neq i} X_j = k-1\right]}{\mathrm{P}\left[\sum_{j=1}^{n} X_j = k\right]}$$

$$= \frac{p\binom{n-1}{k-1}p^{k-1}(1-p)^{n-1-(k-1)}}{\binom{n}{k}p^k(1-p)^{n-k}} = \frac{k}{n}. \tag{3}$$

A second way is to argue that symmetry implies

$$\mathrm{E}\left[X_i|X_1+\cdots+X_n=k\right] = \gamma, \tag{4}$$

the same for each i. In this case,

$$n\gamma = \sum_{i=1}^{n} \mathrm{E}\left[X_i|X_1+\cdots+X_n=k\right]$$

$$= \mathrm{E}\left[X_1+\cdots+X_n|X_1+\cdots+X_n=k\right] = k. \tag{5}$$

Thus $\gamma = k/n$. At any rate, the conditional mean of U is

$$\mathrm{E}\left[U|K=k\right] = \sum_{i=1}^{k} \mathrm{E}\left[X_i|X_1+\cdots+X_n=k\right]$$

$$= \sum_{i=1}^{k} \frac{k}{n} = \frac{k^2}{n}. \tag{6}$$

This says that the random variable $\mathrm{E}[U|K] = K^2/n$. Using iterated expectations, we have

$$\mathrm{E}\left[U\right] = \mathrm{E}\left[\mathrm{E}\left[U|K\right]\right] = \mathrm{E}\left[K^2/n\right]. \tag{7}$$

Since K is a binomial random variable, we know that $\mathrm{E}[K] = np$ and $\mathrm{Var}[K] = np(1-p)$. Thus,

$$\mathrm{E}\left[U\right] = \frac{\mathrm{E}\left[K^2\right]}{n} = \frac{\mathrm{Var}[K] + (\mathrm{E}\left[K\right])^2}{n} = p(1-p) + np^2. \tag{8}$$

On the other hand, V is just and ordinary random sum of independent random variables and the mean of $\mathrm{E}[V] = \mathrm{E}[X]\,\mathrm{E}[M] = np^2$.

Problem 9.5.1 Solution

We know that the waiting time, W is uniformly distributed on [0,10] and therefore has the following PDF.

$$f_W(w) = \begin{cases} 1/10 & 0 \leq w \leq 10, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We also know that the total time is 3 milliseconds plus the waiting time, that is $X = W + 3$.

(a) The expected value of X is $E[X] = E[W+3] = E[W] + 3 = 5 + 3 = 8$.

(b) The variance of X is $\text{Var}[X] = \text{Var}[W+3] = \text{Var}[W] = 25/3$.

(c) The expected value of A is $E[A] = 12\,E[X] = 96$.

(d) The standard deviation of A is $\sigma_A = \sqrt{\text{Var}[A]} = \sqrt{12(25/3)} = 10$.

(e) $P[A > 116] = 1 - \Phi(\frac{116-96}{10}) = 1 - \Phi(2) = 0.02275$.

(f) $P[A < 86] = \Phi(\frac{86-96}{10}) = \Phi(-1) = 1 - \Phi(1) = 0.1587$.

Problem 9.5.3 Solution

(a) Let $X_1, \ldots, X_{120}$ denote the set of call durations (measured in minutes) during the month. From the problem statement, each $X - I$ is an exponential (λ) random variable with $E[X_i] = 1/\lambda = 2.5$ min and $\text{Var}[X_i] = 1/\lambda^2 = 6.25$ min^2. The total number of minutes used during the month is $Y = X_1 + \cdots + X_{120}$. By Theorem 9.1 and Theorem 9.3,

$$\begin{aligned} E[Y] &= 120\,E[X_i] = 300 \\ \text{Var}[Y] &= 120\,\text{Var}[X_i] = 750. \end{aligned} \tag{1}$$

The subscriber's bill is $30 + 0.4(y - 300)^+$ where $x^+ = x$ if $x \geq 0$ or $x^+ = 0$ if $x < 0$. the subscribers bill is exactly \$36 if $Y = 315$. The probability the subscribers bill exceeds \$36 equals

$$\begin{aligned} P[Y > 315] &= P\left[\frac{Y - 300}{\sigma_Y} > \frac{315 - 300}{\sigma_Y}\right] \\ &= Q\left(\frac{15}{\sqrt{750}}\right) = 0.2919. \end{aligned} \tag{2}$$

(b) If the actual call duration is X_i, the subscriber is billed for $M_i = \lceil X_i \rceil$ minutes. Because each X_i is an exponential (λ) random variable, Theorem 4.9 says that M_i is a geometric (p) random variable with $p = 1 - e^{-\lambda} = 0.3297$. Since M_i is geometric,

$$E[M_i] = \frac{1}{p} = 3.033, \qquad \text{Var}[M_i] = \frac{1-p}{p^2} = 6.167. \tag{3}$$

The number of billed minutes in the month is $B = M_1 + \cdots + M_{120}$. Since $M_1, \ldots, M_{120}$ are iid random variables,

$$E[B] = 120\, E[M_i] = 364.0, \quad \text{Var}[B] = 120\, \text{Var}[M_i] = 740.08. \tag{4}$$

Similar to part (a), the subscriber is billed \$36 if $B = 315$ minutes. The probability the subscriber is billed more than \$36 is

$$\begin{aligned} P[B > 315] &= P\left[\frac{B - 364}{\sqrt{740.08}} > \frac{315 - 365}{\sqrt{740.08}}\right] \\ &= Q(-1.8) = \Phi(1.8) = 0.964. \end{aligned} \tag{5}$$

Problem 9.5.5 Solution

(a) Since the number of requests N has expected value $\mathrm{E}[N] = 300$ and variance $\mathrm{Var}[N] = 300$, we need C to satisfy

$$\begin{aligned} \mathrm{P}\left[N > C\right] &= \mathrm{P}\left[\frac{N-300}{\sqrt{300}} > \frac{C-300}{\sqrt{300}}\right] \\ &= 1 - \Phi\left(\frac{C-300}{\sqrt{300}}\right) = 0.05. \end{aligned} \tag{1}$$

From Table 4.2, we note that $\Phi(1.65) = 0.9505$. Thus,

$$C = 300 + 1.65\sqrt{300} = 328.6. \tag{2}$$

(b) For $C = 328.6$, the exact probability of overload is

$$\begin{aligned} \mathrm{P}\left[N > C\right] &= 1 - \mathrm{P}\left[N \leq 328\right] \\ &= 1 - \texttt{poissoncdf(300,328)} = 0.0516, \end{aligned} \tag{3}$$

which shows the central limit theorem approximation is reasonable.

(c) This part of the problem could be stated more carefully. Re-examining Definition 3.9 for the Poisson random variable and the accompanying discussion in Chapter 3, we observe that the webserver has an arrival rate of $\lambda = 300$ hits/min, or equivalently $\lambda = 5$ hits/sec. Thus in a one second interval, the number of requests N' is a Poisson ($\alpha = 5$) random variable.

However, since the server "capacity" in a one second interval is not precisely defined, we will make the somewhat arbitrary definition that the server capacity is $C' = 328.6/60 = 5.477$ packets/sec. With this somewhat arbitrary definition, the probability of overload in a one second interval is

$$\mathrm{P}\left[N' > C'\right] = 1 - \mathrm{P}\left[N' \leq 5.477\right] = 1 - \mathrm{P}\left[N' \leq 5\right]. \tag{4}$$

Because the number of arrivals in the interval is small, it would be a mistake to use the Central Limit Theorem to estimate this overload probability. However, the direct calculation of the overload probability is not hard. For $E[N'] = \alpha = 5$,

$$1 - P[N' \leq 5] = 1 - \sum_{n=0}^{5} P_N(n)$$
$$= 1 - e^{-\alpha} \sum_{n=0}^{5} \frac{\alpha^n}{n!} = 0.3840. \tag{5}$$

(d) Here we find the smallest C such that $P[N' \leq C] \geq 0.95$. From the previous step, we know that $C > 5$. Since N' is a Poisson ($\alpha = 5$) random variable, we need to find the smallest C such that

$$P[N \leq C] = \sum_{n=0}^{C} \alpha^n e^{-\alpha}/n! \geq 0.95. \tag{6}$$

Some experiments with `poissoncdf(alpha,c)` will show that

$$P[N \leq 8] = 0.9319, \qquad P[N \leq 9] = 0.9682. \tag{7}$$

Hence $C = 9$.

(e) If we use the Central Limit theorem to estimate the overload probability in a one second interval, we would use the facts that $E[N'] = 5$ and $\text{Var}[N'] = 5$ to estimate the the overload probability as

$$1 - P[N' \leq 5] = 1 - \Phi\left(\frac{5-5}{\sqrt{5}}\right) = 0.5, \tag{8}$$

which overestimates the overload probability by roughly 30 percent. We recall from Chapter 3 that a Poisson random is the limiting case of the (n,p) binomial random variable when n is large and $np = \alpha$.In general, for fixed p, the Poisson and binomial PMFs become closer as n increases. Since large n is also the case for which the central limit theorem applies, it is not surprising that the the CLT approximation for the Poisson (α) CDF is better when $\alpha = np$ is large.

Comment: Perhaps a more interesting question is why the overload probability in a one-second interval is so much higher than that in a one-minute interval? To answer this, consider a T-second interval in which the number of requests N_T is a Poisson (λT) random variable while the server capacity is cT hits. In the earlier problem parts, $c = 5.477$ hits/sec. We make the assumption that the server system is reasonably well-engineered in that $c > \lambda$. (In fact, to assume otherwise means that the backlog of requests will grow without bound.) Further, assuming T is fairly large, we use the CLT to estimate the probability of overload in a T-second interval as

$$P[N_T \geq cT] = P\left[\frac{N_T - \lambda T}{\sqrt{\lambda T}} \geq \frac{cT - \lambda T}{\sqrt{\lambda T}}\right] = Q\left(k\sqrt{T}\right), \tag{9}$$

where $k = (c - \lambda)/\sqrt{\lambda}$. As long as $c > \lambda$, the overload probability decreases with increasing T. In fact, the overload probability goes rapidly to zero as T becomes large. The reason is that the gap $cT - \lambda T$ between server capacity cT and the expected number of requests λT grows linearly in T while the standard deviation of the number of requests grows proportional to $\sqrt{T}$.

However, one should add that the definition of a T-second overload is somewhat arbitrary. In fact, one can argue that as T becomes large, the requirement for no overloads simply becomes less stringent. Using more advanced techniques found in the Markov Chains Supplement, a system such as this webserver can be evaluated in terms of the average backlog of requests and the average delay in serving in serving a request. These statistics won't depend on a particular time period T and perhaps better describe the system performance.

Problem 9.5.7 Solution

Random variable K_n has a binomial distribution for n trials and success probability $P[V] = 3/4$.

(a) The expected number of video packets out of 48 packets is

$$E[K_{48}] = 48\, P[V] = 36. \tag{1}$$

(b) The variance of K_{48} is

$$\mathrm{Var}[K_{48}] = 48\,\mathrm{P}[V](1-\mathrm{P}[V]) = 48(3/4)(1/4) = 9 \tag{2}$$

Thus K_{48} has standard deviation $\sigma_{K_{48}} = 3$.

(c) Using the ordinary central limit theorem and Table 4.2 yields

$$\begin{aligned}\mathrm{P}[30 \le K_{48} \le 42] &\approx \Phi\left(\frac{42-36}{3}\right) - \Phi\left(\frac{30-36}{3}\right)\\ &= \Phi(2) - \Phi(-2)\end{aligned} \tag{3}$$

Recalling that $\Phi(-x) = 1 - \Phi(x)$, we have

$$\mathrm{P}[30 \le K_{48} \le 42] \approx 2\Phi(2) - 1 = 0.9545. \tag{4}$$

(d) Since K_{48} is a discrete random variable, we can use the De Moivre-Laplace approximation to estimate

$$\begin{aligned}\mathrm{P}[30 \le K_{48} \le 42] &\approx \Phi\left(\frac{42+0.5-36}{3}\right) - \Phi\left(\frac{30-0.5-36}{3}\right)\\ &= 2\Phi(2.16666) - 1 = 0.9687.\end{aligned} \tag{5}$$

Problem 9.5.9 Solution

By symmetry, $\mathrm{E}[X] = 0$. Since X is a continuous $(a=-1, b=1)$ uniform random variable, its variance is $\mathrm{Var}[X] = (b-a)^2/12 = 1/3$. Working with the moments of X, we can write

$$\begin{aligned}\mathrm{E}[Y] &= \mathrm{E}\left[20 + 15X^2\right]\\ &= 20 + 15\,\mathrm{E}\left[X^2\right]\\ &= 20 + 15\,\mathrm{Var}[X^2] = 25,\end{aligned} \tag{1}$$

where we recall that $\mathrm{E}[X] = 0$ implies $\mathrm{E}[X^2] = \mathrm{Var}[X]$. Next we observe that

$$\begin{aligned}\mathrm{Var}[Y] &= \mathrm{Var}[20 + 15X^2]\\ &= \mathrm{Var}[15X^2]\\ &= 225\,\mathrm{Var}[X^2] = 225\left(\mathrm{E}\left[(X^2)^2\right] - (\mathrm{E}\left[X^2\right])^2\right)\end{aligned} \tag{2}$$

Since $\mathrm{E}[X^2] = \mathrm{Var}[X] = 1/3$, and since $\mathrm{E}[(X^2)^2] = \mathrm{E}[X^4]$, it follows that

$$\begin{aligned}\mathrm{Var}[Y] &= 225\left(\mathrm{E}\left[X^4\right] - (\mathrm{Var}[X])^2\right)\\ &= 225\left(\int_{-1}^{1} \frac{1}{2}x^4\,dx - \left(\frac{1}{3}\right)^2\right) = 225\left(\frac{1}{5} - \frac{1}{9}\right) = 20. \qquad (3)\end{aligned}$$

To use the central limit theorem, we approximate the CDF of W by a Gaussian CDF with expected value $\mathrm{E}[W]$ and variance $\mathrm{Var}[W]$. Since the expectation of the sum equals the sum of the expectations,

$$\begin{aligned}\mathrm{E}[W] &= \mathrm{E}\left[\frac{1}{100}\sum_{i=1}^{100} Y_i\right]\\ &= \frac{1}{100}\sum_{i=1}^{100}\mathrm{E}[Y_i] = \mathrm{E}[Y] = 25. \qquad (4)\end{aligned}$$

Since the independence of the Y_i follows from the independence of the X_i, we can write

$$\mathrm{Var}[W] = \frac{1}{100^2}\mathrm{Var}\left[\sum_{i=1}^{100} Y_i\right] = \frac{1}{100^2}\sum_{i=1}^{100}\mathrm{Var}[Y_i] = \frac{\mathrm{Var}[Y]}{100} = 0.2. \qquad (5)$$

By a CLT approximation,

$$\begin{aligned}\mathrm{P}[W \leq 25.4] &= \mathrm{P}\left[\frac{W-25}{\sqrt{0.2}} \leq \frac{25.4-25}{\sqrt{0.2}}\right]\\ &\approx \Phi\left(\frac{0.4}{\sqrt{0.2}}\right) = \Phi(2/\sqrt{5}) = 0.8145. \qquad (6)\end{aligned}$$

Problem 9.5.11 Solution

(a) On quiz i, your score X_i is the sum of $n = 10$ independent Bernoulli trials and so X_i is a binomial $(n = 10, p = 0.8)$ random variable, which has PMF

$$P_{X_i}(x) = \binom{10}{x}(0.8)^x(0.2)^{10-x}. \qquad (1)$$

(b) First we note that $\mathrm{E}[X_i] = np = 8$ and that

$$\mathrm{Var}[X_i] = np(1-p) = 10(0.8)(0.2) = 1.6. \tag{2}$$

Since X is a scaled sum of 100 Bernoulli trials, it is appropriate to use a central limit theorem approximation. All we need to do is calculate the expected value and variance of X:

$$\mu_X = \mathrm{E}\left[X\right] = 0.01 \sum_{i=1}^{10} \mathrm{E}\left[X_i\right] = 0.8, \tag{3}$$

$$\begin{aligned} \sigma_X^2 = \mathrm{Var}[X] &= (0.01)^2 \,\mathrm{Var}\left[\sum_{i=1}^{10} X_i\right] \\ &= 10^{-4} \sum_{i=1}^{10} \mathrm{Var}[X_i] = 16 \times 10^{-4}. \end{aligned} \tag{4}$$

To use the central limit theorem, we write

$$\begin{aligned} \mathrm{P}\left[A\right] = \mathrm{P}\left[X \geq 0.9\right] &= \mathrm{P}\left[\frac{X - \mu_X}{\sigma_X} \geq \frac{0.9 - \mu_X}{\sigma_X}\right] \\ &\approx \mathrm{P}\left[Z \geq \frac{0.9 - 0.8}{0.04}\right] \\ &= \mathrm{P}\left[Z \geq 2.5\right] = Q(2.5). \end{aligned} \tag{5}$$

A nicer way to do this same calculation is to observe that

$$\mathrm{P}\left[A\right] = \mathrm{P}\left[X \geq 0.9\right] = \mathrm{P}\left[\sum_{i=1}^{10} X_i \geq 90\right]. \tag{6}$$

Now we define $W = \sum_{i=1}^{10} X_i$ and use the central limit theorem on W. In this case,

$$\mathrm{E}\left[W\right] = 10\,\mathrm{E}\left[X_i\right] = 80, \qquad \mathrm{Var}[W] = 10\,\mathrm{Var}[X_i] = 16. \tag{7}$$

Our central limit theorem approximation can now be written as

$$\begin{aligned} P[A] = P[W \geq 90] &= P\left[\frac{W-80}{\sqrt{16}} \geq \frac{90-80}{\sqrt{16}}\right] \\ &\approx P[Z \geq 2.5] = Q(2.5). \end{aligned} \tag{8}$$

We will see that this second approach is more useful in the next problem.

(c) With n attendance quizzes,

$$\begin{aligned} P[A] &= P[X' \geq 0.9] \\ &= P\left[10n + \sum_{i=1}^{10} X_i \geq 9n + 90\right] = P[W \geq 90 - n], \end{aligned} \tag{9}$$

where $W = \sum_{i=1}^{10} X_i$ is the same as in the previous part. Thus

$$\begin{aligned} P[A] &= P\left[\frac{W - E[W]}{\sqrt{\text{Var}[W]}} \geq \frac{90 - n - E[W]}{\sqrt{\text{Var}[W]}}\right] \\ &= Q\left(\frac{10-n}{4}\right) = Q(2.5 - 0.25n). \end{aligned} \tag{10}$$

(d) Without the scoring change on quiz 1, your grade will be based on

$$X = \frac{8 + \sum_{i=2}^{10} X_i}{100} = \frac{8 + Y}{100}. \tag{11}$$

With the corrected scoring, your grade will be based on

$$X' = \frac{9 + \sum_{i=2}^{10} X_i}{100} = \frac{9 + Y}{100} = 0.01 + X. \tag{12}$$

The only time this change will matter is when X is on the borderline between two grades. Specifically, your grade will change if $X \in \{0.59, 0.69, 0.79, 0.89\}$. Equivalently,

$$\begin{aligned} P[U^c] &= P[Y = 51] + P[Y = 61] + P[Y = 71] + P[Y = 81] \\ &= P_Y(51) + P_Y(61) + P_Y(71) + P_Y(81). \end{aligned} \tag{13}$$

If you're curious, we note that since Y is binomial with $\mathrm{E}[Y] = 72$, the dominant term in the above sum is $P_Y(71)$ and that

$$\mathrm{P}\left[U^c\right] \approx \binom{90}{71}(0.8)^{71}(0.2)^{19} \approx 0.099. \tag{14}$$

This near 10 percent probability is fairly high because the student is a borderline B/C student. That is, the point matters if you are a borderline student. Of course, in real life, you don't know if you're a borderline student.

Problem 9.6.1 Solution

Note that W_n is a binomial $(10^n, 0.5)$ random variable. We need to calculate

$$\begin{aligned}\mathrm{P}\left[B_n\right] &= \mathrm{P}\left[0.499 \times 10^n \leq W_n \leq 0.501 \times 10^n\right] \\ &= \mathrm{P}\left[W_n \leq 0.501 \times 10^n\right] - \mathrm{P}\left[W_n < 0.499 \times 10^n\right].\end{aligned} \tag{1}$$

A complication is that the event $W_n < w$ is not the same as $W_n \leq w$ when w is an integer. In this case, we observe that

$$\mathrm{P}\left[W_n < w\right] = \mathrm{P}\left[W_n \leq \lceil w \rceil - 1\right] = F_{W_n}\left(\lceil w \rceil - 1\right). \tag{2}$$

Thus

$$\mathrm{P}\left[B_n\right] = F_{W_n}(0.501 \times 10^n) - F_{W_n}\left(\left\lceil 0.499 \times 10^9 \right\rceil - 1\right). \tag{3}$$

For $n = 1, \ldots, N$, we can calculate $\mathrm{P}[B_n]$ in this MATLAB program:

```
function pb=binomialcdftest(N);
pb=zeros(1,N);
for n=1:N,
    w=[0.499 0.501]*10^n;
    w(1)=ceil(w(1))-1;
    pb(n)=diff(binomialcdf(10^n,0.5,w));
end
```

Unfortunately, on this user's machine (a Windows XP laptop), the program fails for $N = 4$. The problem, as noted earlier is that `binomialcdf.m` uses `binomialpmf.m`, which fails for a binomial $(10000, p)$ random variable. Of course, your mileage may vary. A slightly better solution is to use the `bignomialcdf.m` function, which is identical to `binomialcdf.m` except it calls `bignomialpmf.m` rather than `binomialpmf.m`. This enables calculations for larger values of n, although at some cost in numerical accuracy. Here is the code:

```
function pb=bignomialcdftest(N);
pb=zeros(1,N);
for n=1:N,
    w=[0.499 0.501]*10^n;
    w(1)=ceil(w(1))-1;
    pb(n)=diff(bignomialcdf(10^n,0.5,w));
end
```

For comparison, here are the outputs of the two programs:

```
>> binomialcdftest(4)
ans =
  0.2461  0.0796  0.0756     NaN
>> bignomialcdftest(6)
ans =
  0.2461  0.0796  0.0756  0.1663  0.4750  0.9546
```

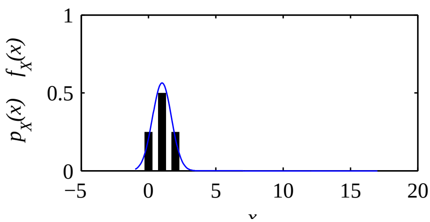
The result 0.9546 for $n = 6$ corresponds to the exact probability in Example 9.14 which used the CLT to estimate the probability as 0.9544. Unfortunately for this user, `bignomialcdftest(7)` failed.

Problem 9.6.3 Solution

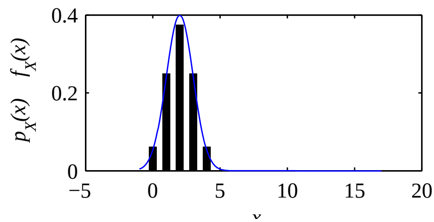
In this problem, we re-create the plots of Figure 9.3 except we use the binomial PMF and corresponding Gaussian PDF. Here is a MATLAB program that compares the binomial (n, p) PMF and the Gaussian PDF with the same expected value and variance.

```
function y=binomcltpmf(n,p)
x=-1:17;
xx=-1:0.05:17;
y=binomialpmf(n,p,x);
std=sqrt(n*p*(1-p));
clt=gausspdf(n*p,std,xx);
hold off;
pmfplot(x,y,'\it x','\it p_X(x)    f_X(x)');
hold on; plot(xx,clt); hold off;
```

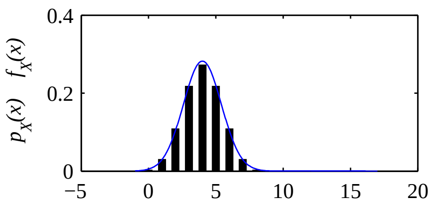
Here are the output plots for $p = 1/2$ and $n = 2, 4, 8, 16$.



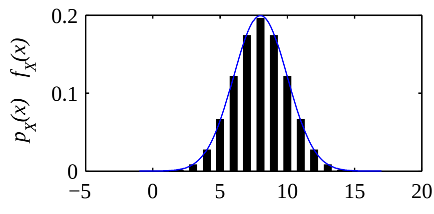
binomcltpmf(2,0.5)



binomcltpmf(4,0.5)



binomcltpmf(8,0.5)



binomcltpmf(16,0.5)

To see why the values of the PDF and PMF are roughly the same, consider the Gaussian random variable Y. For small Δ,

$$f_Y(x)\,\Delta \approx \frac{F_Y(x+\Delta/2) - F_Y(x-\Delta/2)}{\Delta}. \tag{1}$$

For $\Delta = 1$, we obtain

$$f_Y(x) \approx F_Y(x+1/2) - F_Y(x-1/2)\,. \tag{2}$$

Since the Gaussian CDF is approximately the same as the CDF of the binomial (n, p) random variable X, we observe for an integer x that

$$f_Y(x) \approx F_X(x + 1/2) - F_X(x - 1/2) = P_X(x) . \tag{3}$$

Although the equivalence in heights of the PMF and PDF is only an approximation, it can be useful for checking the correctness of a result.

Problem 9.6.5 Solution

In Example 10.4, the height of a storm surge X is a Gaussian $(5.5, 1)$ random variable. Here were are asked to estimate

$$\mathrm{P}[X > 7] = \mathrm{P}[X - 5.5 > 1.5] = 1 - \Phi(1.5). \tag{1}$$

using the `uniform12.m` function defined in Example 9.18.

The exact correct value is $1 - \Phi(1.5) = 0.0668$. You may wonder why this problem asks you to estimate $1 - \Phi(1.5)$ when we can calculate it exactly. The goal of this exercise is really to understand the limitations of using a sum of 12 uniform random variables as an approximation to a Gaussian.

Unfortunately, in the function `uniform12.m`, the vector `T=(-3:3)` is hard-coded, making it hard to directly reuse the function to solve our problem. So instead, let's redefine a new `unif12sum.m` function that accepts the number of trials `m` and the threshold value `T` as arguments:

```
function FT = unif12sum(m,T)
%Using m samples of a sum of 12 uniform random variables,
%FT is an estimate of P(X<T) for a Gaussian (0,1) rv X
x=sum(rand(12,m))-6;
FT=(count(x,T)/m)';
end
```

Before looking at some experimental runs, we note that `unif12sum` is making two different approximations. First, samples consisting of the sum of 12 uniform random variables are being used as an approximation for a Gaussian $(0, 1)$ random variable X. Second, we are using the relative frequency of samples below the threshold T as an approximation or estimate of $\mathrm{P}[X < T]$.

(a) Here are some sample runs for $m = 1000$ sample values:

```
>> m=1000;t=1.5;
>> 1-[unif12sum(m,t) unif12sum(m,t) unif12sum(m,t)]
ans =
    0.0640    0.0620    0.0610
>> 1-[unif12sum(m,t) unif12sum(m,t) unif12sum(m,t)]
ans =
    0.0810    0.0670    0.0690
```

We see that six trials yields six close but different estimates.

(b) Here are some sample runs for $m = 10{,}000$ sample values:

```
>> m=10000;t=1.5;
>> 1-[unif12sum(m,t) unif12sum(m,t) unif12sum(m,t)]
ans =
    0.0667    0.0709    0.0697
>> 1-[unif12sum(m,t) unif12sum(m,t) unif12sum(m,t)]
ans =
    0.0686    0.0672    0.0708
```

Casual inspection gives the impression that 10,000 samples provide better estimates than 1000 samples. Although the small number of tests here is definitely not sufficient to make such an assertion, we will see in Chapter 10 that this is indeed true.

Problem 9.6.7 Solution

The function `sumfinitepmf` generalizes the method of Example 9.17.

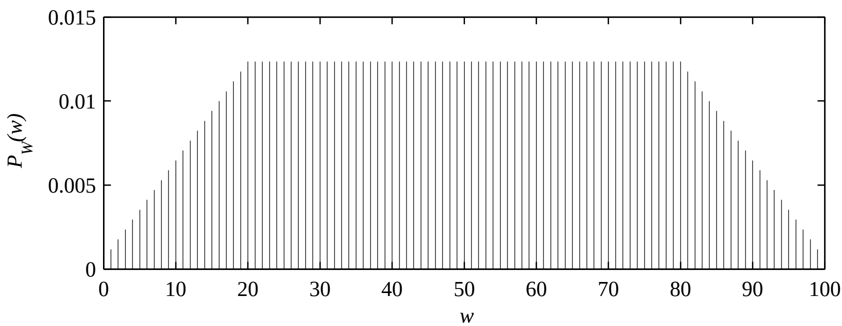
```
function [pw,sw]=sumfinitepmf(px,sx,py,sy);
[SX,SY]=ndgrid(sx,sy);
[PX,PY]=ndgrid(px,py);
SW=SX+SY;PW=PX.*PY;
sw=unique(SW);
pw=finitepmf(SW,PW,sw);
```

The only difference is that the PMFs px and py and ranges sx and sy are not hard coded, but instead are function inputs.

As an example, suppose X is a discrete uniform $(0, 20)$ random variable and Y is an independent discrete uniform $(0, 80)$ random variable. The following program sum2unif will generate and plot the PMF of $W = X + Y$.

```
%sum2unif.m
sx=0:20;px=ones(1,21)/21;
sy=0:80;py=ones(1,81)/81;
[pw,sw]=sumfinitepmf(px,sx,py,sy);
h=pmfplot(sw,pw,'\it w','\it P_W(w)');
set(h,'LineWidth',0.25);
```

Here is the graph generated by sum2unif.



Problem Solutions – Chapter 10

Problem 10.1.1 Solution

Recall that $X_1, X_2 \ldots X_n$ are independent exponential random variables with mean value $\mu_X = 5$ so that for $x \geq 0$, $F_X(x) = 1 - e^{-x/5}$.

(a) Using Theorem 10.1, $\sigma^2_{M_n(x)} = \sigma^2_X/n$. Realizing that $\sigma^2_X = 25$, we obtain

$$\text{Var}[M_9(X)] = \frac{\sigma^2_X}{9} = \frac{25}{9}. \tag{1}$$

(b)

$$\begin{aligned} P[X_1 \geq 7] &= 1 - P[X_1 \leq 7] \\ &= 1 - F_X(7) \\ &= 1 - (1 - e^{-7/5}) = e^{-7/5} \approx 0.247. \end{aligned} \tag{2}$$

(c) First we express $P[M_9(X) > 7]$ in terms of $X_1, \ldots, X_9$.

$$\begin{aligned} P[M_9(X) > 7] &= 1 - P[M_9(X) \leq 7] \\ &= 1 - P[(X_1 + \ldots + X_9) \leq 63]. \end{aligned} \tag{3}$$

Now the probability that $M_9(X) > 7$ can be approximated using the Central Limit Theorem (CLT).

$$\begin{aligned} P[M_9(X) > 7] &= 1 - P[(X_1 + \ldots + X_9) \leq 63] \\ &\approx 1 - \Phi\left(\frac{63 - 9\mu_X}{\sqrt{9}\sigma_X}\right) \\ &= 1 - \Phi(6/5). \end{aligned} \tag{4}$$

Consulting with Table 4.2 yields $P[M_9(X) > 7] \approx 0.1151$.

Problem 10.1.3 Solution

This problem is in the wrong section since the *standard error* isn't defined until Section 10.4. However is we peek ahead to this section, the problem isn't very hard. Given the sample mean estimate $M_n(X)$, the standard error is defined as the standard deviation $e_n = \sqrt{\text{Var}[M_n(X)]}$. In our problem, we use samples X_i to generate $Y_i = X_i^2$. For the sample mean $M_n(Y)$, we need to find the standard error

$$e_n = \sqrt{\text{Var}[M_n(Y)]} = \sqrt{\frac{\text{Var}[Y]}{n}}. \tag{1}$$

Since X is a uniform $(0, 1)$ random variable,

$$\text{E}[Y] = \text{E}\left[X^2\right] = \int_0^1 x^2\,dx = 1/3, \tag{2}$$

$$\text{E}\left[Y^2\right] = \text{E}\left[X^4\right] = \int_0^1 x^4\,dx = 1/5. \tag{3}$$

Thus $\text{Var}[Y] = 1/5-(1/3)^2 = 4/45$ and the sample mean $M_n(Y)$ has standard error

$$e_n = \sqrt{\frac{4}{45n}}. \tag{4}$$

Problem 10.2.1 Solution

If the average weight of a Maine black bear is 500 pounds with standard deviation equal to 100 pounds, we can use the Chebyshev inequality to upper bound the probability that a randomly chosen bear will be more then 200 pounds away from the average.

$$\text{P}\left[|W - \text{E}[W]| \geq 200\right] \leq \frac{\text{Var}[W]}{200^2} \leq \frac{100^2}{200^2} = 0.25. \tag{1}$$

Problem 10.2.3 Solution

The arrival time of the third elevator is $W = X_1 + X_2 + X_3$. Since each X_i is uniform $(0, 30)$, $\mathrm{E}[X_i] = 15$ and $\mathrm{Var}[X_i] = (30-0)^2/12 = 75$. Thus $\mathrm{E}[W] = 3\,\mathrm{E}[X_i] = 45$, and $\mathrm{Var}[W] = 3\,\mathrm{Var}[X_i] = 225$.

(a) By the Markov inequality,

$$\mathrm{P}[W > 75] \le \frac{\mathrm{E}[W]}{75} = \frac{45}{75} = \frac{3}{5}. \tag{1}$$

(b) By the Chebyshev inequality,

$$\begin{aligned} \mathrm{P}[W > 75] &= \mathrm{P}[W - \mathrm{E}[W] > 30] \\ &\le \mathrm{P}[|W - \mathrm{E}[W]| > 30] \\ &\le \frac{\mathrm{Var}[W]}{30^2} = \frac{1}{4}. \end{aligned} \tag{2}$$

Problem 10.2.5 Solution

On each roll of the dice, a success, namely snake eyes, occurs with probability $p = 1/36$. The number of trials, R, needed for three successes is a Pascal $(k = 3, p)$ random variable with

$$\mathrm{E}[R] = \frac{3}{p} = 108, \qquad \mathrm{Var}[R] = \frac{3(1-p)}{p^2} = 3780. \tag{1}$$

(a) By the Markov inequality,

$$\mathrm{P}[R \ge 250] \le \frac{\mathrm{E}[R]}{250} = \frac{54}{125} = 0.432. \tag{2}$$

(b) By the Chebyshev inequality,

$$\begin{aligned} \mathrm{P}[R \ge 250] = \mathrm{P}[R - 108 \ge 142] &= \mathrm{P}[|R - 108| \ge 142] \\ &\le \frac{\mathrm{Var}[R]}{(142)^2} = 0.1875. \end{aligned} \tag{3}$$

(c) The exact value is $P[R \geq 250] = 1 - \sum_{r=3}^{249} P_R(r)$. Since there is no way around summing the Pascal PMF to find the CDF, this is what `pascalcdf` does.

```
>> 1-pascalcdf(3,1/36,249)
ans =
    0.0299
```

Thus the Markov and Chebyshev inequalities are valid bounds but not good estimates of $P[R \geq 250]$.

Problem 10.2.7 Solution

For an $N[\mu, \sigma^2]$ random variable X, we can write

$$P[X \geq c] = P[(X-\mu)/\sigma \geq (c-\mu)/\sigma] = P[Z \geq (c-\mu)/\sigma]. \tag{1}$$

Since Z is $N[0,1]$, we can apply the result of Problem 10.2.6 with c replaced by $(c-\mu)/\sigma$. This yields

$$P[X \geq c] = P[Z \geq (c-\mu)/\sigma] \leq e^{-(c-\mu)^2/2\sigma^2} \tag{2}$$

Problem 10.2.9 Solution

This problem is solved completely in the solution to Quiz 10.2! We repeat that solution here. Since $W = X_1 + X_2 + X_3$ is an Erlang $(n = 3, \lambda = 1/2)$ random variable, Theorem 4.11 says that for any $w > 0$, the CDF of W satisfies

$$F_W(w) = 1 - \sum_{k=0}^{2} \frac{(\lambda w)^k e^{-\lambda w}}{k!} \tag{1}$$

Equivalently, for $\lambda = 1/2$ and $w = 20$,

$$\begin{aligned} P[W > 20] &= 1 - F_W(20) \\ &= e^{-10}\left(1 + \frac{10}{1!} + \frac{10^2}{2!}\right) = 61e^{-10} = 0.0028. \end{aligned} \tag{2}$$

Problem 10.3.1 Solution

$X_1, X_2, \ldots$ are iid random variables each with mean 75 and standard deviation 15.

(a) We would like to find the value of n such that

$$P[74 \leq M_n(X) \leq 76] = 0.99. \tag{1}$$

When we know only the mean and variance of X_i, our only real tool is the Chebyshev inequality which says that

$$\begin{aligned} P[74 \leq M_n(X) \leq 76] &= 1 - P[|M_n(X) - E[X]| \geq 1] \\ &\geq 1 - \frac{\text{Var}[X]}{n} = 1 - \frac{225}{n} \geq 0.99. \end{aligned} \tag{2}$$

This yields $n \geq 22{,}500$.

(b) If each X_i is a Gaussian, the sample mean, $M_n(X)$ will also be Gaussian with mean and variance

$$E[M_{n'}(X)] = E[X] = 75, \tag{3}$$

$$\text{Var}[M_{n'}(X)] = \text{Var}[X]/n' = 225/n' \tag{4}$$

In this case,

$$\begin{aligned} P[74 \leq M_{n'}(X) \leq 76] &= \Phi\left(\frac{76-\mu}{\sigma}\right) - \Phi\left(\frac{74-\mu}{\sigma}\right) \\ &= \Phi(\sqrt{n'}/15) - \Phi(-\sqrt{n'}/15) \\ &= 2\Phi(\sqrt{n'}/15) - 1 = 0.99. \end{aligned} \tag{5}$$

Thus, $n' = 1{,}521$.

Since even under the Gaussian assumption, the number of samples n' is so large that even if the X_i are not Gaussian, the sample mean may be approximated by a Gaussian. Hence, about 1500 samples probably is about right. However, in the absence of any information about the PDF of X_i beyond the mean and variance, we cannot make any guarantees stronger than that given by the Chebyshev inequality.

Problem 10.3.3 Solution

(a) As $n \to \infty$, Y_{2n} is a sum of a large number of iid random variables, so we can use the central limit theorem. Since $E[Y_{2n}] = n$ and $\text{Var}[Y_{2n}] = 2np(1-p) = n/2$,

$$\begin{aligned} P\left[|Y_{2n} - n| \leq \sqrt{n/2}\right] &= P\left[Y_{2n} - E\left[Y_{2n}\right] \leq \sqrt{n/2}\right] \\ &= P\left[\frac{|Y_{2n} - E\left[Y_{2n}\right]|}{\sqrt{\text{Var}[Y_{2n}]}} \leq 1\right] \\ &= P\left[-1 \leq Z_n \leq 1\right]. \end{aligned} \tag{1}$$

By the central limit theorem, $Z_n = (Y_{2n} - E[Y_{2n}])/\sqrt{\text{Var}[Y_{2n}]}$ is converging to a Gaussian $(0, 1)$ random variable Z. Thus

$$\begin{aligned} \lim_{n\to\infty} P\left[|Y_{2n} - n| \leq \sqrt{n/2}\right] &= P\left[-1 \leq Z \leq 1\right] \\ &= \Phi(1) - \Phi(-1) \\ &= 2\Phi(1) - 1 = 0.68. \end{aligned} \tag{2}$$

(b) Note that $Y_{2n}/(2n)$ is a sample mean for $2n$ samples of X_n. Since $E[X_n] = 1/2$, the weak law says that given any $\epsilon > 0$,

$$\lim_{n\to\infty} P\left[\left|\frac{Y_{2n}}{2n} - \frac{1}{2}\right| > \epsilon\right] = 0. \tag{3}$$

An equivalent statement is

$$\lim_{n\to\infty} P\left[|Y_{2n} - n| > 2n\epsilon\right] = 0. \tag{4}$$

Problem 10.3.5 Solution

Given $N_0, N_1, \ldots$, there are jN_j chips used in cookies with j chips. The total number of chips used is $\sum_{k=0}^{\infty} kN_k$. You are equally likely to be any one of the chips in a cookie, so the probability you landed in a cookie with j chips is

$$P[J = j] = \frac{jN_j}{\sum_{k=0}^{\infty} kN_k} = \frac{j\frac{N_j}{n}}{\sum_{k=0}^{\infty} k\frac{N_k}{n}}. \tag{1}$$

First, we note that $P[J = 0] = 0$ since a chip cannot land in a cookie that has zero chips. Second, we note that N_j/n is the relative frequency of cookies with j chips out of all cookies. By comparison, $P_K(j)$ is the probability a cookie has j chips. As $n \to \infty$, the law of large numbers implies $N_j/n \to P_K(j)$. It follows for $j \geq 1$ that as $n \to \infty$,

$$P_J(j) \to \frac{jP_K(j)}{\sum_{k=0}^{\infty} kP_K(k)} = \frac{j(10)^j e^{-10}/j!}{E[K]} = \frac{(10)^{j-1}e^{-10}}{(j-1)!}. \tag{2}$$

Problem 10.3.7 Solution

(a) We start by observing that $E[R_1] = E[X_1] = q$. Next, we write

$$R_n = \frac{(n-1)R_{n-1}}{n} + \frac{X_n}{n}. \tag{1}$$

If follows that

$$\begin{aligned} E[R_n|R_{n-1} = r] &= \frac{(n-1)E[R_{n-1}|R_{n-1} = r]}{n} + \frac{E[X_n|R_{n-1} = r]}{n} \\ &= \frac{(n-1)r}{n} + \frac{r}{n} = r. \end{aligned} \tag{2}$$

Thus $E[R_n|R_{n-1}] = R_{n-1}$ and by iterated expectation, $E[R_n] = E[R_{n-1}]$. By induction, it follows that $E[R_n] = E[R_1] = q$.

(b) From the start, $R_1 = X_1$ is Gaussian. Given $R_1 = r$, $R_2 = R_1/2 + X_2/2$ where X_2 is conditionally Gaussian given R_1. Since R_1 is Gaussian, it follows that R_1 and X_2 are jointly Gaussian. It follows that R_2 is also Gaussian since it is a linear combination of jointly Gaussian random variables. Similarly, X_n is conditionally Gaussian given R_{n-1} and thus X_n and R_{n-1} are jointly Gaussian. Thus R_n which is a linear combination of X_n and R_{n-1} is Gaussian. Since $\mathrm{E}[R_n] = q$, we can define $\sigma_n^2 = \mathrm{Var}[R_n]$ and write the PDF of R_n as

$$f_{R_n}(r) = \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-(r-q)^2/2\sigma_n^2}. \tag{3}$$

The parameter σ_n^2 still needs to be determined.

(c) Following the hint, given $R_{n-1} = r$, $R_n = (n-1)r/n + X_n/n$. It follows that

$$\begin{aligned}
\mathrm{E}\left[R_n^2 | R_{n-1} = r\right] &= \mathrm{E}\left[\left(\frac{(n-1)r}{n} + \frac{X_n}{n}\right)^2 | R_{n-1} = r\right] \\
&= \mathrm{E}\left[\frac{(n-1)^2 r^2}{n^2} + 2r\frac{(n-1)X_n}{n^2} + \frac{X_n^2}{n^2} | R_{n-1} = r\right] \\
&= \frac{(n-1)^2 r^2}{n^2} + \frac{2(n-1)r^2}{n^2} + \frac{\mathrm{E}\left[X_n^2 | R_{n-1} = r\right]}{n^2}.
\end{aligned} \tag{4}$$

Given $R_{n-1} = r$, X_n is Gaussian $(r, 1)$. Since $\mathrm{Var}[X_n | R_{n-1} = r] = 1$,

$$\begin{aligned}
\mathrm{E}\left[X_n^2 | R_{n-1} = r\right] &= \mathrm{Var}[X_n | R_{n-1} = r] + \left(\mathrm{E}\left[X_n | R_{n-1} = r\right]\right)^2 \\
&= 1 + r^2.
\end{aligned} \tag{5}$$

This implies

$$\begin{aligned}
\mathrm{E}\left[R_n^2 | R_{n-1} = r\right] &= \frac{(n-1)^2 r^2}{n^2} + \frac{2(n-1)r^2}{n^2} + \frac{1 + r^2}{n^2} \\
&= r^2 + \frac{1}{n^2},
\end{aligned} \tag{6}$$

and thus

$$\mathrm{E}\left[R_n^2|R_{n-1}\right] = R_{n-1}^2 + \frac{1}{n^2}. \tag{7}$$

By the iterated expectation,

$$\mathrm{E}\left[R_n^2\right] = \mathrm{E}\left[R_{n-1}^2\right] + \frac{1}{n^2}. \tag{8}$$

Since $\mathrm{E}[R_1^2] = \mathrm{E}[X_1^2] = 1 + q^2$, it follows that

$$\mathrm{E}\left[R_n^2\right] = q^2 + \sum_{j=1}^{n} \frac{1}{j^2}. \tag{9}$$

Hence

$$\mathrm{Var}[R_n] = \mathrm{E}\left[R_n^2\right] - (\mathrm{E}\left[R_n\right])^2 = \sum_{j=1}^{n} \frac{1}{j^2}. \tag{10}$$

Note that $\mathrm{Var}[R_n]$ is an increasing sequence and that $\lim_{n\to\infty} \mathrm{Var}[R_n] \approx 1.645$.

(d) When the prior ratings have no influence, the review scores X_n are iid and by the law of large numbers, R_n will converge to q, the "true quality" of the movie. In our system, the possibility of early misjudgments will lead to randomness in the final rating. The eventual rating $R_n = \lim_{n\to\infty} R_n$ is a random variable with $\mathrm{E}[R] = q$ and $\mathrm{Var}[R] \approx 1.645$. This may or may not be representative of how bad movies can occasionally get high ratings.

Problem 10.4.1 Solution

For an an arbitrary Gaussian (μ, σ) random variable Y,

$$\begin{aligned} \mathrm{P}\left[\mu - \sigma \le Y \le \mu + \sigma\right] &= \mathrm{P}\left[-\sigma \le Y - \mu \le \sigma\right] \\ &= \mathrm{P}\left[-1 \le \frac{Y - \mu}{\sigma} \le 1\right] \\ &= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.6827. \end{aligned} \tag{1}$$

Note that Y can be any Gaussian random variable, including, for example, $M_n(X)$ when X is Gaussian. When X is not Gaussian, the same claim holds to the extent that the central limit theorem promises that $M_n(X)$ is nearly Gaussian for large n.

Problem 10.4.3 Solution

This problem is really very simple. If we let $Y = X_1X_2$ and for the ith trial, let $Y_i = X_1(i)X_2(i)$, then $\hat{R}_n = M_n(Y)$, the sample mean of random variable Y. By Theorem 10.9, $M_n(Y)$ is unbiased. Since $\text{Var}[Y] = \text{Var}[X_1X_2] < \infty$, Theorem 10.11 tells us that $M_n(Y)$ is a consistent sequence.

Problem 10.4.5 Solution

Note that we can write Y_k as

$$\begin{aligned} Y_k &= \left(\frac{X_{2k-1} - X_{2k}}{2}\right)^2 + \left(\frac{X_{2k} - X_{2k-1}}{2}\right)^2 \\ &= \frac{(X_{2k} - X_{2k-1})^2}{2}. \end{aligned} \tag{1}$$

Hence,

$$\begin{aligned} \text{E}\left[Y_k\right] &= \frac{1}{2}\,\text{E}\left[X_{2k}^2 - 2X_{2k}X_{2k-1} + X_{2k-1}^2\right] \\ &= \text{E}\left[X^2\right] - (\text{E}\left[X\right])^2 = \text{Var}[X]. \end{aligned} \tag{2}$$

Next we observe that $Y_1, Y_2, \ldots$ is an iid random sequence. If this independence is not obvious, consider that Y_1 is a function of X_1 and X_2, Y_2 is a function of X_3 and X_4, and so on. Since $X_1, X_2, \ldots$ is an idd sequence, $Y_1, Y_2, \ldots$ is an iid sequence. Hence, $\text{E}[M_n(Y)] = \text{E}[Y] = \text{Var}[X]$, implying $M_n(Y)$ is an unbiased estimator of $\text{Var}[X]$. We can use Theorem 10.9 to prove that $M_n(Y)$ is consistent if we show that $\text{Var}[Y]$ is finite.Since $\text{Var}[Y] \leq \text{E}[Y^2]$, it is sufficient to prove that $\text{E}[Y^2] < \infty$. Note that

$$Y_k^2 = \frac{X_{2k}^4 - 4X_{2k}^3X_{2k-1} + 6X_{2k}^2X_{2k-1}^2 - 4X_{2k}X_{2k-1}^3 + X_{2k-1}^4}{4}. \tag{3}$$

Taking expectations yields

$$\mathrm{E}\left[Y_k^2\right] = \frac{1}{2}\mathrm{E}\left[X^4\right] - 2\,\mathrm{E}\left[X^3\right]\mathrm{E}\left[X\right] + \frac{3}{2}\left(\mathrm{E}\left[X^2\right]\right)^2. \tag{4}$$

Hence, if the first four moments of X are finite, then $\mathrm{Var}[Y] \leq \mathrm{E}[Y^2] < \infty$. By Theorem 10.9, the sequence $M_n(Y)$ is consistent.

Problem 10.5.1 Solution

X has the Bernoulli (0.9) PMF

$$P_X(x) = \begin{cases} 0.1 & x = 0, \\ 0.9 & x = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) $\mathrm{E}[X]$ is in fact the same as $P_X(1)$ because X is Bernoulli.

(b) We can use the Chebyshev inequality to find

$$\begin{aligned} \mathrm{P}\left[|M_{90}(X) - P_X(1)| \geq .05\right] &= \mathrm{P}\left[|M_{90}(X) - \mathrm{E}\left[X\right]| \geq .05\right] \\ &\leq \alpha. \end{aligned} \tag{2}$$

In particular, the Chebyshev inequality states that

$$\alpha = \frac{\sigma_X^2}{90(.05)^2} = \frac{.09}{90(.05)^2} = 0.4. \tag{3}$$

(c) Now we wish to find the value of n such that

$$\mathrm{P}\left[|M_n(X) - P_X(1)| \geq .03\right] \leq 0.1. \tag{4}$$

From the Chebyshev inequality, we write

$$0.1 = \frac{\sigma_X^2}{n(.03)^2}. \tag{5}$$

Since $\sigma_X^2 = 0.09$, solving for n yields $n = 100$.

Problem 10.5.3 Solution

First we observe that the interval estimate can be expressed as

$$\left|\hat{P}_n(A) - P[A]\right| < 0.05. \tag{1}$$

Since $\hat{P}_n(A) = M_n(X_A)$ and $E[M_n(X_A)] = P[A]$, we can use Theorem 10.5(b) to write

$$P\left[\left|\hat{P}_n(A) - P[A]\right| < 0.05\right] \geq 1 - \frac{\text{Var}[X_A]}{n(0.05)^2}. \tag{2}$$

Note that $\text{Var}[X_A] = P[A](1 - P[A]) \leq 0.25$. Thus for confidence coefficient 0.9, we require that

$$1 - \frac{\text{Var}[X_A]}{n(0.05)^2} \geq 1 - \frac{0.25}{n(0.05)^2} \geq 0.9. \tag{3}$$

This implies $n \geq 1{,}000$ samples are needed.

Problem 10.6.1 Solution

In this problem, we have to keep straight that the Poisson expected value $\alpha = 1$ is a different α than the confidence coefficient $1 - \alpha$. That said, we will try avoid using α for the confidence coefficient. Using X to denote the Poisson ($\alpha = 1$) random variable, the trace of the sample mean is the sequence $M_1(X), M_2(X), \ldots$ The confidence interval estimate of α has the form

$$M_n(X) - c \leq \alpha \leq M_n(X) + c. \tag{1}$$

The confidence coefficient of the estimate based on n samples is

$$\begin{aligned} P[M_n(X) - c \leq \alpha \leq M_n(X) + c] &= P[\alpha - c \leq M_n(X) \leq \alpha + c] \\ &= P[-c \leq M_n(X) - \alpha \leq c]. \end{aligned} \tag{2}$$

Since $\text{Var}[M_n(X)] = \text{Var}[X]/n = 1/n$, the 0.9 confidence interval shrinks with increasing n. In particular, $c = c_n$ will be a decreasing sequence. Using a Central Limit Theorem approximation, a 0.9 confidence implies

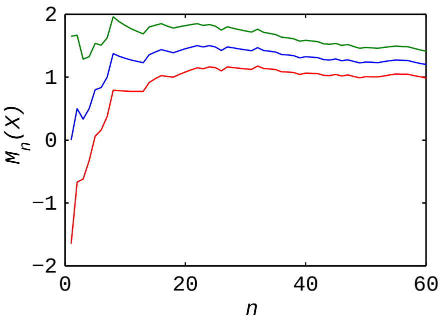
$$\begin{aligned} 0.9 &= \mathrm{P}\left[\frac{-c_n}{\sqrt{1/n}} \leq \frac{M_n(X)-\alpha}{\sqrt{1/n}} \leq \frac{c_n}{\sqrt{1/n}}\right] \\ &= \Phi(c_n\sqrt{n}) - \Phi(-c_n\sqrt{n}) = 2\Phi(c_n\sqrt{n}) - 1. \end{aligned} \tag{3}$$

Equivalently, $\Phi(c_n\sqrt{n}) = 0.95$ or $c_n = 1.65/\sqrt{n}$.

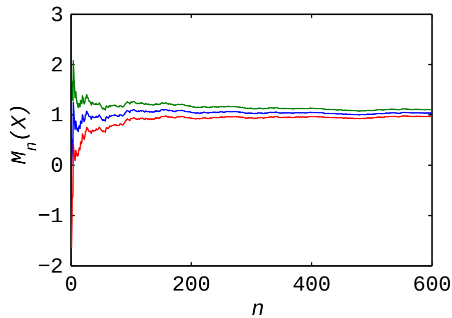
Thus, as a function of the number of samples n, we plot three functions: the sample mean $M_n(X)$, and the upper limit $M_n(X) + 1.65/\sqrt{n}$ and lower limit $M_n(X) - 1.65/\sqrt{n}$ of the 0.9 confidence interval. We use the MATLAB function `poissonmeanseq(n)` to generate these sequences for n sample values.

```
function M=poissonmeanseq(n);
x=poissonrv(1,n);
nn=(1:n)';
M=cumsum(x)./nn;
r=(1.65)./sqrt(nn);
plot(nn,M,nn,M+r,nn,M-r);
```

Here are two output graphs:



`poissonmeanseq(60)`



`poissonmeanseq(600)`

Problem 10.6.3 Solution

First, we need to determine whether the relative performance of the two estimators depends on the actual value of λ. To address this, we observe that if Y is an exponential (1) random variable, then Theorem 6.3 tells us that $X = Y/\lambda$ is an exponential (λ) random variable. Thus if $Y_1, Y_2, \ldots$ are iid samples of Y, then $Y_1/\lambda, Y_2/\lambda, \ldots$ are iid samples of X. Moreover, the sample mean of X is

$$M_n(X) = \frac{1}{n\lambda} \sum_{i=1}^{n} Y_i = \frac{1}{\lambda} M_n(Y). \tag{1}$$

Similarly, the sample variance of X satisfies

$$\begin{aligned} V_n'(X) &= \frac{1}{n-1} \sum_{i=1}^{n} (X_i - M_n(X))^2 \\ &= \frac{1}{n-1} \sum_{i=1}^{n} \left(\frac{Y_i}{\lambda} - \frac{1}{\lambda} M_n(Y) \right)^2 = \frac{V_n'(Y)}{\lambda^2}. \end{aligned} \tag{2}$$

We can conclude that

$$\hat{\lambda} = \frac{\lambda}{M_n(Y)}, \qquad \tilde{\lambda} = \frac{\lambda}{\sqrt{V_n'(Y)}}. \tag{3}$$

For $\lambda \neq 1$, the estimators $\hat{\lambda}$ and $\tilde{\lambda}$ are just scaled versions of the estimators for the case $\lambda = 1$. Hence it is sufficient to consider only the $\lambda = 1$ case. The function `z=lamest(n,m)` returns the estimation errors for m trials of each estimator where each trial uses n iid exponential (1) samples.

```
function z=lamest(n,m);
x=exponentialrv(1,n*m);
x=reshape(x,n,m);
mx=sum(x)/n;
MX=ones(n,1)*mx;
vx=sum((x-MX).^2)/(n-1);
z=[(1./mx); (1./sqrt(vx))]-1;
```

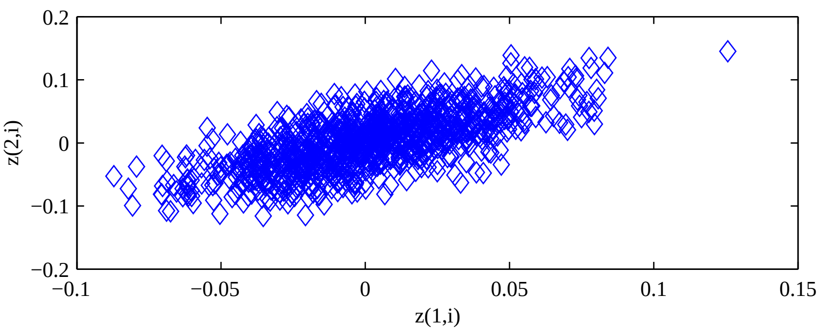
In `lamest.m`, each column of matrix `x` represents one trial. Note that `mx` is a row vector such that `mx(i)` is the sample mean for trial i. The matrix `MX` has `mx(i)` for every element in column i. Thus `vx` is a row vector such that `vx(i)` is the sample variance for trial i.

Finally, `z` is a $2 \times m$ matrix such that column i of `z` records the estimation errors for trial i. If $\hat{\lambda}_i$ and $\tilde{\lambda}_i$ are the estimates for for trial i, then `z(1,i)` is the error $\hat{Z}_i = \hat{\lambda}_i - 1$ while `z(2,i)` is the error $\tilde{Z}_i = \tilde{\lambda}_i - 1$.

Now that we can simulate the errors generated by each estimator, we need to determine which estimator is better. We start by using the commands

```
z=lamest(1000,1000);
plot(z(1,:),z(2,:),'bd')
```

to perform 1,000 trials, each using 1,000 samples. The `plot` command generates a scatter plot of the error pairs $(\hat{Z}_i, \tilde{Z}_i)$ for each trial. Here is an example of the resulting scatter plot:



In the scatter plot, each diamond marks an independent pair $(\hat{Z}, \tilde{Z})$ where $\hat{Z}$ is plotted on the x-axis and $\tilde{Z}$ is plotted on the y-axis. (Although it is outside the scope of this solution, it is interesting to note that the errors $\hat{Z}$ and $\tilde{Z}$ appear to be positively correlated.) From the plot, it may not be obvious that one estimator is better than the other. However, by reading the axis ticks carefully, one can observe that it appears that typical values for $\hat{Z}$ are in the interval $(-0.05, 0.05)$ while typical values for $\tilde{Z}$ are in the interval $(-0.1, 0.1)$. This suggests that $\hat{Z}$ may be superior.To verify this observation, we calculate the sample mean for each squared errors

$$M_m(\hat{Z}^2) = \frac{1}{m}\sum_{i=1}^{m} \hat{Z}_i^2, \qquad M_m(\tilde{Z}^2) = \frac{1}{m}\sum_{i=1}^{m} \tilde{Z}_i^2. \tag{4}$$

From our MATLAB experiment with $m = 1{,}000$ trials, we calculate

```
>> sum(z.^2,2)/1000
ans =
    0.0010
    0.0021
```

That is, $M_{1,000}(\hat{Z}^2) = 0.0010$ and $M_{1,000}(\tilde{Z}^2) = 0.0021$. In fact, one can show (with a lot of work) for large m that

$$M_m(\hat{Z}^2) \approx 1/m, \qquad\qquad M_m(\tilde{Z}^2) = 2/m, \tag{5}$$

and that

$$\lim_{m\to\infty} \frac{M_m(\tilde{Z}^2)}{M_m(\hat{Z}^2)} = 2. \tag{6}$$

In short, the mean squared error of the $\tilde{\lambda}$ estimator is twice that of the $\hat{\lambda}$ estimator.

Problem 10.6.5 Solution

The difficulty in this problem is that although $\mathrm{E}[X]$ exists, EX^2 and higher order moments are infinite. Thus $\mathrm{Var}[X]$ is also infinite. It also follows for any finite n that the sample mean $M_n(X)$ has infinite variance. In this case, we *cannot* apply the Chebyshev inequality to the sample mean to show the convergence in probability of $M_n(X)$ to $\mathrm{E}[X]$.

If $\lim_{n\to\infty} \mathrm{P}[|M_n(X) - \mathrm{E}[X]| \geq \epsilon] = p$, then there are two distinct possibilities:

- $p > 0$, or
- $p = 0$ but the Chebyshev inequality isn't a sufficient powerful technique to verify this fact.

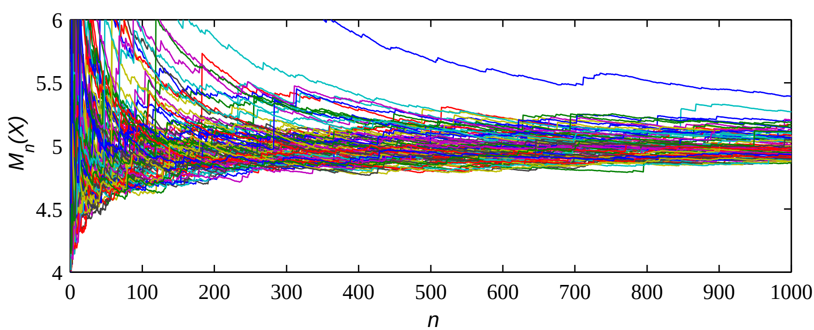
To resolve whether $p = 0$ (and the sample mean converges to the expected value) one can spend time trying to prove either $p = 0$ or $p > 0$. At this point, we try some simulation experiments to see if the experimental evidence points one way or the other.

As requested by the problem, the MATLAB function `samplemeantest(n,a)` simulates one hundred traces of the sample mean when $E[X] = a$. Each trace is a length n sequence $M_1(X), M_2(X), \ldots, M_n(X)$.

```
function mx=samplemeantest(n,a);
u=rand(n,100);
x=a-2+(1./sqrt(1-u));
d=((1:n)')*ones(1,100);
mx=cumsum(x)./d;
plot(mx);
xlabel('\it n'); ylabel('\it M_n(X)');
axis([0 n a-1 a+1]);
```

The $n \times 100$ matrix `x` consists of iid samples of X. Taking cumulative sums along each column of x, and dividing row i by i, each column of `mx` is a length n sample mean trace. we then plot the traces.

The following graph was generated by `samplemeantest(1000,5)`:



Frankly, it is difficult to draw strong conclusions from the graph. If the sample sequences $M_n(X)$ are converging to $E[X]$, the convergence is fairly slow. Even after averaging 1,000 samples, typical values for the sample mean appear to range from $a - 0.5$ to $a + 0.5$. There may also be outlier sequences which are still off the charts since we truncated the y-axis range. On the other hand, the sample mean sequences do not appear to be diverging (which is also possible since $\text{Var}[X] = \infty$.) Note the above graph was generated using 10^5 sample values. Repeating the experiment with more samples, say `samplemeantest(10000,5)`, will yield a similarly inconclusive result. Even if your version of MATLAB can support the generation of 100 times as many samples, you won't know for sure whether the sample mean sequence *always* converges. On the other hand, the experiment is probably enough that if you pursue the analysis, you should start by trying to prove that $p = 0$.

Problem Solutions – Chapter 11

Problem 11.1.1 Solution

Assuming the coin is fair, we must choose a rejection region R such that $\alpha = \mathrm{P}[R] = 0.05$. We can choose a rejection region $R = \{L > r\}$. What remains is to choose r so that $\mathrm{P}[R] = 0.05$. Note that $L > l$ if we first observe l tails in a row. Under the hypothesis that the coin is fair, l tails in a row occurs with probability

$$\mathrm{P}[L > l] = (1/2)^l. \tag{1}$$

Thus, we need

$$\mathrm{P}[R] = \mathrm{P}[L > r] = 2^{-r} = 0.05. \tag{2}$$

Thus, $r = -\log_2(0.05) = \log_2(20) = 4.32$. In this case, we reject the hypothesis that the coin is fair if $L \geq 5$. The significance level of the test is $\alpha = \mathrm{P}[L > 4] = 2^{-4} = 0.0625$ which close to but not exactly 0.05.

The shortcoming of this test is that we always accept the hypothesis that the coin is fair whenever heads occurs on the first, second, third or fourth flip. If the coin was biased such that the probability of heads was much higher than 1/2, say 0.8 or 0.9, we would hardly ever reject the hypothesis that the coin is fair. In that sense, our test cannot identify that kind of biased coin.

Problem 11.1.3 Solution

We reject the null hypothesis when the call rate M is too high. Note that

$$\mathrm{E}[M] = \mathrm{E}[N_i] = 2.5, \qquad \mathrm{Var}[M] = \frac{\mathrm{Var}[N_i]}{T} = \frac{2.5}{T}. \tag{1}$$

For large T, we use a central limit theorem approximation to calculate the rejection probability

$$\begin{aligned} P[R] &= P[M \geq m_0] \\ &= P\left[\frac{M - 2.5}{\sigma_M} \geq \frac{m_0 - 2.5}{\sigma_M}\right] \\ &= 1 - \Phi\left(\frac{m_0 - 2.5}{\sqrt{\frac{2.5}{T}}}\right) = 0.05. \end{aligned} \tag{2}$$

It follows that

$$\frac{m_0 - 2.5}{\sqrt{2.5/T}} = 1.65 \implies m_0 = 2.5 + \frac{1.65\sqrt{2.5}}{\sqrt{T}} = 2.5 + \frac{2.6}{\sqrt{T}}. \tag{3}$$

That is, we reject the null hypothesis whenever

$$M \geq 2.5 + \frac{2.6}{\sqrt{T}}. \tag{4}$$

As T gets larger, smaller deviations of M above the expected value $E[M] = 2.5$ are sufficient to reject the null hypothesis.

Problem 11.1.5 Solution

In order to test just a small number of pacemakers, we test n pacemakers and we reject the null hypothesis if *any* pacemaker fails the test. Moreover, we choose the smallest n such that we meet the required significance level of the test.

The number of pacemakers that fail the test is X, a binomial $(n, q_0 = 10^{-4})$ random variable. The significance level of the test is

$$\alpha = P[X > 0] = 1 - P[X = 0] = 1 - (1 - q_0)^n. \tag{1}$$

For a significance level $\alpha = 0.01$, we have that

$$n = \frac{\ln(1 - \alpha)}{\ln(1 - q_0)} = 100.5. \tag{2}$$

Comment: For $\alpha = 0.01$, keep in mind that there is a one percent probability that a normal factory will fail the test. That is, a test failure is quite unlikely if the factory is operating normally.

Problem 11.1.7 Solution

A reasonable test would reject the null hypothesis that the plant is operating normally if one or more of the chips fail the one-day test. Exactly how many shold be tested and how many failures N are needed to reject the null hypothesis would depend on the significance level of the test.

(a) The lifetime of a chip is X, an exponential (λ) random variable with $\lambda = (T/200)^2$. The probability p that a chip passes the one-day test is

$$p = \mathrm{P}[X \geq 1/365] = e^{-\lambda/365}. \tag{1}$$

For an m chip test, the significance level of the test is

$$\begin{aligned} \alpha = \mathrm{P}[N > 0] &= 1 - \mathrm{P}[N = 0] \\ &= 1 - p^m = 1 - e^{-m\lambda/365}. \end{aligned} \tag{2}$$

(b) At $T = 100°$, $\lambda = 1/4$ and we obtain a significance level of $\alpha = 0.01$ if

$$m = -\frac{365 \ln(0.99)}{\lambda} = \frac{3.67}{\lambda} = 14.74. \tag{3}$$

In fact, at $m = 15$ chips, the significance level is $\alpha = 0.0102$.

(c) Raising T raises the failure rate $\lambda = (T/200)^2$ and thus lowers $m = 3.67/\lambda$. In essence, raising the temperature makes a “tougher” test and thus requires fewer chips to be tested for the same significance level.

Problem 11.1.9 Solution

Since the null hypothesis H_0 asserts that the two exams have the same mean and variance, we reject H_0 if the difference in sample means is large. That is, $R = \{|D| \geq d_0\}$.

Under H_0, the two sample means satisfy

$$\begin{aligned} \mathrm{E}[M_A] &= \mathrm{E}[M_B] = \mu, \\ \mathrm{Var}[M_A] &= \mathrm{Var}[M_B] = \frac{\sigma^2}{n} = \frac{100}{n}. \end{aligned} \tag{1}$$

Since n is large, it is reasonable to use the Central Limit Theorem to approximate M_A and M_B as Gaussian random variables. Since M_A and M_B are independent, D is also Gaussian with

$$\begin{aligned} \mathrm{E}[D] &= \mathrm{E}[M_A] - \mathrm{E}[M_B] = 0, \\ \mathrm{Var}[D] &= \mathrm{Var}[M_A] + \mathrm{Var}[M_B] = \frac{200}{n}. \end{aligned} \tag{2}$$

Under the Gaussian assumption, we can calculate the significance level of the test as

$$\alpha = \mathrm{P}[|D| \geq d_0] = 2\left(1 - \Phi(d_0/\sigma_D)\right). \tag{3}$$

For $\alpha = 0.05$, $\Phi(d_0/\sigma_D) = 0.975$, or $d_0 = 1.96\sigma_D = 1.96\sqrt{200/n}$. If $n = 100$ students take each exam, then $d_0 = 2.77$ and we reject the null hypothesis that the exams are the same if the sample means differ by more than 2.77 points.

Problem 11.2.1 Solution

For the MAP test, we must choose acceptance regions A_0 and A_1 for the two hypotheses H_0 and H_1. From Theorem 11.2, the MAP rule is

$$n \in A_0 \text{ if } \frac{P_{N|H_0}(n)}{P_{N|H_1}(n)} \geq \frac{\mathrm{P}[H_1]}{\mathrm{P}[H_0]}; \qquad n \in A_1 \text{ otherwise.} \tag{1}$$

Since $P_{N|H_i}(n) = \lambda_i^n e^{-\lambda_i}/n!$, the MAP rule becomes

$$n \in A_0 \text{ if } \left(\frac{\lambda_0}{\lambda_1}\right)^n e^{-(\lambda_0-\lambda_1)} \geq \frac{\mathrm{P}[H_1]}{\mathrm{P}[H_0]}; \qquad n \in A_1 \text{ otherwise.} \tag{2}$$

By taking logarithms and assuming $\lambda_1 > \lambda_0$ yields the final form of the MAP rule

$$n \in A_0 \text{ if } n \leq n^* = \frac{\lambda_1 - \lambda_0 + \ln(\mathrm{P}[H_0]/\mathrm{P}[H_1])}{\ln(\lambda_1/\lambda_0)}; \qquad n \in A_1 \text{ otherwise.} \tag{3}$$

From the MAP rule, we can get the ML rule by setting the a priori probabilities to be equal. This yields the ML rule

$$n \in A_0 \text{ if } n \leq n^* = \frac{\lambda_1 - \lambda_0}{\ln(\lambda_1/\lambda_0)}; \qquad n \in A_1 \text{ otherwise.} \tag{4}$$

Problem 11.2.3 Solution

By Theorem 11.5, the decision rule is

$$n \in A_0 \text{ if } L(n) = \frac{P_{N|H_0}(n)}{P_{N|H_1}(n)} \geq \gamma; \qquad n \in A_1 \text{ otherwise,} \tag{1}$$

where γ is the largest possible value such that $\sum_{L(n)<\gamma} P_{N|H_0}(n) \leq \alpha$.

Given H_0, N is Poisson ($a_0 = 1{,}000$) while given H_1, N is Poisson ($a_1 = 1{,}300$). We can solve for the acceptance set A_0 by observing that $n \in A_0$ if

$$\frac{P_{N|H_0}(n)}{P_{N|H_1}(n)} = \frac{a_0^n e^{-a_0}/n!}{a_1^n e^{-a_1}/n!} \geq \gamma. \tag{2}$$

Cancelling common factors and taking the logarithm, we find that $n \in A_0$ if

$$n \ln \frac{a_0}{a_1} \geq (a_0 - a_1) + \ln \gamma. \tag{3}$$

Since $\ln(a_0/a_1) < 0$, dividing through reverses the inequality and shows that

$$n \in A_0 \text{ if } n \leq n^* = \frac{(a_0 - a_1) + \ln \gamma}{\ln(a_0/a_1)} = \frac{(a_1 - a_0) - \ln \gamma}{\ln(a_1/a_0)}; \qquad n \in A_1 \text{ otherwise.}$$

However, we still need to determine the constant γ. In fact, it is easier to work with the threshold n^* directly. Note that $L(n) < \gamma$ if and only if $n > n^*$. Thus we choose the smallest n^* such that

$$\mathrm{P}\left[N > n^* | H_0\right] = \sum_{n > n^*} P_{N|H_0}(n)\, \alpha \leq 10^{-6}. \tag{4}$$

To find n^* a reasonable approach would be to use Central Limit Theorem approximation since given H_0, N is a Poisson (1,000) random variable, which has the same PDF as the sum on 1,000 independent Poisson (1) random variables. Given H_0, N has expected value a_0 and variance a_0. From the CLT,

$$\begin{aligned} \mathrm{P}\left[N > n^* | H_0\right] &= \mathrm{P}\left[\frac{N - a_0}{\sqrt{a_0}} > \frac{n^* - a_0}{\sqrt{a_0}} | H_0\right] \\ &\approx Q\left(\frac{n^* - a_0}{\sqrt{a_0}}\right) \leq 10^{-6}. \end{aligned} \tag{5}$$

From Table 4.3, $Q(4.75) = 1.02 \times 10^{-6}$ and $Q(4.76) < 10^{-6}$, implying

$$n^* = a_0 + 4.76\sqrt{a_0} = 1150.5. \tag{6}$$

On the other hand, perhaps the CLT should be used with some caution since $\alpha = 10^{-6}$ implies we are using the CLT approximation far from the center of the distribution. In fact, we can check out answer using the `poissoncdf` functions:

```
>> nstar=[1150 1151 1152 1153 1154 1155];
>> (1.0-poissoncdf(1000,nstar))'
ans =
 1.0e-005 *
   0.1644   0.1420   0.1225   0.1056   0.0910   0.0783
>>
```

Thus we see that n^*1154. Using this threshold, the miss probability is

$$\begin{aligned} \mathrm{P}\left[N \leq n^* | H_1\right] &= \mathrm{P}\left[N \leq 1154 | H_1\right] \\ &= \texttt{poissoncdf(1300,1154)} = 1.98 \times 10^{-5}. \end{aligned} \tag{7}$$

Keep in mind that this is the smallest possible P_{MISS} subject to the constraint that $P_{\mathrm{FA}} \leq 10^{-6}$.

Problem 11.2.5 Solution

Given H_0, X is Gaussian $(0,1)$. Given H_1, X is Gaussian $(v,1)$. By Theorem 11.4, the Neyman-Pearson test is

$$x \in A_0 \text{ if } L(x) = \frac{f_{X|H_0}(x)}{f_{X|H_1}(x)} = \frac{e^{-x^2/2}}{e^{-(x-v)^2/2}} \geq \gamma; \qquad x \in A_1 \text{ otherwise.} \tag{1}$$

This rule simplifies to

$$x \in A_0 \text{ if } L(x) = e^{-[x^2-(x-v)^2]/2} = e^{-vx+v^2/2} \geq \gamma; \qquad x \in A_1 \text{ otherwise.} \tag{2}$$

Taking logarithms, the Neyman-Pearson rule becomes

$$x \in A_0 \text{ if } x \leq x_0 = \frac{v}{2} - \frac{1}{v}\ln\gamma; \qquad x \in A_1 \text{ otherwise.} \tag{3}$$

The choice of γ has a one-to-one correspondence with the choice of the threshold x_0. Moreoever $L(x) \geq \gamma$ if and only if $x \leq x_0$. In terms of x_0, the false alarm probability is

$$P_{\text{FA}} = \mathrm{P}\left[L(X) < \gamma|H_0\right] = \mathrm{P}\left[X \geq x_0|H_0\right] = Q(x_0). \tag{4}$$

Thus we choose x_0 such that $Q(x_0) = \alpha$.

Problem 11.2.7 Solution

Given H_0, $M_n(T)$ has expected value $\mathrm{E}[V]/n = 3/n$ and variance $\mathrm{Var}[V]/n = 9/n$. Given H_1, $M_n(T)$ has expected value $\mathrm{E}[D]/n = 6/n$ and variance $\mathrm{Var}[D]/n = 36/n$.

(a) Using a Central Limit Theorem approximation, the false alarm probability is

$$\begin{aligned} P_{\text{FA}} &= \mathrm{P}\left[M_n(T) > t_0|H_0\right] \\ &= \mathrm{P}\left[\frac{M_n(T)-3}{\sqrt{9/n}} > \frac{t_0-3}{\sqrt{9/n}}\right] = Q(\sqrt{n}[t_0/3-1]). \end{aligned} \tag{1}$$

(b) Again, using a CLT Approximation, the miss probability is

$$\begin{aligned} P_{\text{MISS}} &= \mathrm{P}\left[M_n(T) \le t_0 | H_1\right] \\ &= \mathrm{P}\left[\frac{M_n(T) - 6}{\sqrt{36/n}} \le \frac{t_0 - 6}{\sqrt{36/n}}\right] = \Phi(\sqrt{n}[t_0/6 - 1]). \end{aligned} \tag{2}$$

(c) From Theorem 11.6, the maximum likelihood decision rule is

$$t \in A_0 \text{ if } \frac{f_{M_n(T)|H_0}(t)}{f_{M_n(T)|H_1}(t)} \ge 1; \qquad t \in A_1 \text{ otherwise.} \tag{3}$$

We will see shortly that using a CLT approximation for the likelihood functions is something of a detour. Nevertheless, with a CLT approximation, the likelihood functions are

$$\begin{aligned} f_{M_n(T)|H_0}(t) &= \sqrt{\frac{n}{18\pi}} e^{-n(t-3)^2/18}, \\ f_{M_n(T)|H_1}(t) &= \sqrt{\frac{n}{72\pi}} e^{-n(t-6)^2/72}. \end{aligned} \tag{4}$$

From the CLT approximation, the ML decision rule is

$$t \in A_0 \text{ if } \sqrt{\frac{72}{18}} \frac{e^{-n(t-3)^2/18}}{e^{-n(t-6)^2/72}} \ge 1; \qquad t \in A_1 \text{ otherwise.} \tag{5}$$

This simplifies to

$$t \in A_0 \text{ if } 2e^{-n[4(t-3)^2-(t-6)^2]/72} \ge 1; \qquad t \in A_1 \text{ otherwise.} \tag{6}$$

After more algebra, this rule further simplifies to

$$t \in A_0 \text{ if } t^2 - 4t - \frac{24 \ln 2}{n} \le 0; \qquad t \in A_1 \text{ otherwise.} \tag{7}$$

Since the quadratic $t^2 - 4t - 24\ln(2)/n$ has two zeros, we use the quadratic formula to find the roots. One root corresponds to a negative value of t and can be discarded since $M_n(T) \geq 0$. Thus the ML rule (for $n = 9$) becomes

$$t \in A_0 \text{ if } t \leq t_{ML} = 2 + 2\sqrt{1 + 6\ln(2)/n} = 4.42; \qquad t \in A_1 \text{ otherwise.}$$

The negative root of the quadratic is the result of the Gaussian assumption which allows for a nonzero probability that $M_n(T)$ will be negative. In this case, hypothesis H_1 which has higher variance becomes more likely. However, since $M_n(T) \geq 0$, we can ignore this root since it is just an artifact of the CLT approximation.

In fact, the CLT approximation gives an incorrect answer. Note that $M_n(T) = Y_n/n$ where Y_n is a sum of iid exponential random variables. Under hypothesis H_0, Y_n is an Erlang $(n, \lambda_0 = 1/3)$ random variable. Under hypothesis H_1, Y_n is an Erlang $(n, \lambda_1 = 1/6)$ random variable. Since $M_n(T) = Y_n/n$ is a scaled version of Y_n, Theorem 6.3 tells us that given hypothesis H_i, $M_n(T)$ is an Erlang $(n, n\lambda_i)$ random variable. Thus $M_n(T)$ has likelihood functions

$$f_{M_n(T)|H_i}(t) = \begin{cases} \frac{(n\lambda_i)^n t^{n-1} e^{-n\lambda_i t}}{(n-1)!} & t \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Using the Erlang likelihood functions, the ML rule becomes

$$t \in \begin{cases} A_0 & \text{if } \frac{f_{M_n(T)|H_0}(t)}{f_{M_n(T)|H_1}(t)} = \left(\frac{\lambda_0}{\lambda_1}\right)^n e^{-n(\lambda_0 - \lambda_1)t} \geq 1; \\ A_1 & \text{otherwise.} \end{cases} \tag{9}$$

This rule simplifies to

$$t \in A_0 \text{ if } t \leq t_{\text{ML}}; \qquad t \in A_1 \text{ otherwise.} \tag{10}$$

where

$$t_{\text{ML}} = \frac{\ln(\lambda_0/\lambda_1)}{\lambda_0 - \lambda_1} = 6\ln 2 = 4.159. \tag{11}$$

Since $6 \ln 2 = 4.159$, this rule is not the same as the rule derived using a CLT approximation. Using the exact Erlang PDF, the ML rule does not depend on n. Moreoever, even if $n \to \infty$, the exact Erlang-derived rule and the CLT approximation rule remain different. In fact, the CLT-based rule is simply an approximation to the correct rule. This highlights that we should first check whether a CLT approximation is necessary before we use it.

(d) In this part, we will use the exact Erlang PDFs to find the MAP decision rule. From 11.2, the MAP rule is

$$t \in \begin{cases} A_0 & \text{if } \frac{f_{M_n(T)|H_0}(t)}{f_{M_n(T)|H_1}(t)} = \left(\frac{\lambda_0}{\lambda_1}\right)^n e^{-n(\lambda_0-\lambda_1)t} \geq \frac{\mathrm{P}[H_1]}{\mathrm{P}[H_0]}; \\ A_1 & \text{otherwise.} \end{cases} \tag{12}$$

Since $\mathrm{P}[H_0] = 0.8$ and $\mathrm{P}[H_1] = 0.2$, the MAP rule simplifies to

$$t \in \begin{cases} A_0 & \text{if } t \leq t_{\mathrm{MAP}} = \frac{\ln \frac{\lambda_0}{\lambda_1} - \frac{1}{n} \ln \frac{\mathrm{P}[H_1]}{\mathrm{P}[H_0]}}{\lambda_0 - \lambda_1} = 6\left[\ln 2 + \frac{\ln 4}{n}\right]; \\ A_1 & \text{otherwise.} \end{cases} \tag{13}$$

For $n = 9$, $t_{\mathrm{MAP}} = 5.083$.

(e) Although we have seen it is incorrect to use a CLT approximation to derive the decision rule, the CLT approximation used in parts (a) and (b) remains a good way to estimate the false alarm and miss probabilities. However, given H_i, $M_n(T)$ is an Erlang $(n, n\lambda_i)$ random variable. In particular, given H_0, $M_n(T)$ is an Erlang $(n, n/3)$ random variable while given H_1, $M_n(T)$ is an Erlang $(n, n/6)$. Thus we can also use `erlangcdf` for an exact calculation of the false alarm and miss probabilities. To summarize the results of parts (a) and (b), a threshold t_0

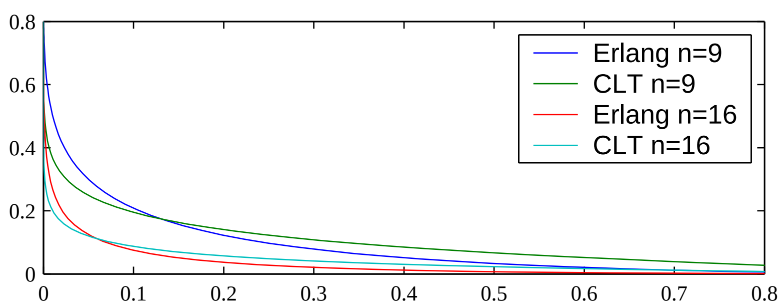
implies that

$$P_{\text{FA}} = \mathrm{P}\left[M_n(T) > t_0 | H_0\right]$$
$$= \texttt{1-erlangcdf(n,n/3,t0)} \approx Q(\sqrt{n}[t_0/3 - 1]), \tag{14}$$
$$P_{\text{MISS}} = \mathrm{P}\left[M_n(T) \le t_0 | H_1\right]$$
$$= \texttt{erlangcdf(n,n/6,t0)} \approx \Phi(\sqrt{n}[t_0/6 - 1]). \tag{15}$$

Here is a program that generates the receiver operating curve.

```
%voicedatroc.m
t0=1:0.1:8';
n=9;
PFA9=1.0-erlangcdf(n,n/3,t0);
PFA9clt=1-phi(sqrt(n)*((t0/3)-1));
PM9=erlangcdf(n,n/6,t0);
PM9clt=phi(sqrt(n)*((t0/6)-1));
n=16;
PFA16=1.0-erlangcdf(n,n/3,t0);
PFA16clt=1.0-phi(sqrt(n)*((t0/3)-1));
PM16=erlangcdf(n,n/6,t0);
PM16clt=phi(sqrt(n)*((t0/6)-1));
plot(PFA9,PM9,PFA9clt,PM9clt,PFA16,PM16,PFA16clt,PM16clt);
axis([0 0.8 0 0.8]);
legend('Erlang n=9','CLT n=9','Erlang n=16','CLT n=16');
```

Here are the resulting ROCs.



Both the true curve and CLT-based approximations are shown. The graph makes it clear that the CLT approximations are somewhat innaccurate. It is also apparent that the ROC for $n = 16$ is clearly better than for $n = 9$.

Problem 11.2.9 Solution

Given hypothesis H_0 that $X = 0$, $Y = W$ is an exponential ($\lambda = 1$) random variable. Given hypothesis H_1 that $X = 1$, $Y = V + W$ is an Erlang ($n = 2, \lambda = 1$) random variable. That is,

$$f_{Y|H_0}(y) = \begin{cases} e^{-y} & y \geq 0, \\ 0 & \text{otherwise,} \end{cases} \quad f_{Y|H_1}(y) = \begin{cases} ye^{-y} & y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The probability of a decoding error is minimized by the MAP rule. Since $P[H_0] = P[H_1] = 1/2$, the MAP rule is

$$y \in A_0 \text{ if } \frac{f_{Y|H_0}(y)}{f_{Y|H_1}(y)} = \frac{e^{-y}}{ye^{-y}} \geq \frac{P[H_1]}{P[H_0]} = 1; \qquad y \in A_1 \text{ otherwise.} \tag{2}$$

Thus the MAP rule simplifies to

$$y \in A_0 \text{ if } y \leq 1; \qquad y \in A_1 \text{ otherwise.} \tag{3}$$

The probability of error is

$$\begin{aligned} P_{\text{ERR}} &= P[Y > 1|H_0]\,P[H_0] + P[Y \leq 1|H_1]\,P[H_1] \\ &= \frac{1}{2}\int_1^\infty e^{-y}\,dy + \frac{1}{2}\int_0^1 ye^{-y}\,dy \\ &= \frac{e^{-1}}{2} + \frac{1 - 2e^{-1}}{2} = \frac{1 - e^{-1}}{2}. \end{aligned} \tag{4}$$

Problem 11.2.11 Solution

(a) Since $\mathbf{Y}$ is continuous, the conditioning event is $\mathbf{y} < \mathbf{Y} \leq \mathbf{y} + d\mathbf{y}$. We then write

$$\begin{aligned} \mathrm{P}[X = 1|\mathbf{y} < \mathbf{Y} \leq \mathbf{y} + d\mathbf{y}] \\ &= \frac{\mathrm{P}[\mathbf{y} < \mathbf{Y} \leq \mathbf{y} + d\mathbf{y}|X = 1]\,\mathrm{P}[X = 1]}{\mathrm{P}[\mathbf{y} < \mathbf{Y} \leq \mathbf{y} + d\mathbf{y}]} \\ &= \frac{\frac{1}{2} f_{\mathbf{Y}|X}(\mathbf{y}|1)\, d\mathbf{y}}{\frac{1}{2} f_{\mathbf{Y}|X}(\mathbf{y}|1)\ d\mathbf{y} + \frac{1}{2} f_{\mathbf{Y}|X}(\mathbf{y}|-1)\ d\mathbf{y}}. \end{aligned} \tag{1}$$

We conclude that

$$\mathrm{P}[X = 1|\mathbf{Y} = \mathbf{y}] = \frac{f_{\mathbf{Y}|X}(\mathbf{y}|1)}{f_{\mathbf{Y}|X}(\mathbf{y}|1) + f_{\mathbf{Y}|X}(\mathbf{y}|-1)} = \frac{1}{1 + \frac{f_{\mathbf{Y}|X}(\mathbf{y}|-1)}{f_{\mathbf{Y}|X}(\mathbf{y}|1)}}. \tag{2}$$

Given $X = x$, $Y_i = x + W_i$ and

$$f_{Y_i|X}(y_i|x) = \frac{1}{\sqrt{2\pi}} e^{(y_i - x)^2/2}. \tag{3}$$

Since the W_i are iid and independent of X, given $X = x$, the Y_i are conditionally iid. That is,

$$f_{\mathbf{Y}|X}(\mathbf{y}|x) = \prod_{i=1}^{n} f_{Y_i|X}(y_i|x) = \frac{1}{(2\pi)^{n/2}} e^{-\sum_{i=1}^{n}(y_i - x)^2/2}. \tag{4}$$

This implies

$$\begin{aligned} L(\mathbf{y}) = \frac{f_{\mathbf{Y}|X}(\mathbf{y}|-1)}{f_{\mathbf{Y}|X}(\mathbf{y}|1)} &= \frac{e^{-\sum_{i=1}^{n}(y_i+1)^2/2}}{e^{-\sum_{i=1}^{n}(y_i-1)^2/2}} \\ &= e^{-\sum_{i=1}^{n}[(y_i+1)^2-(y_i-1)^2]/2} = e^{-2\sum_{i=1}^{n} y_i} \end{aligned} \tag{5}$$

and that

$$\mathrm{P}[X = 1|\mathbf{Y} = \mathbf{y}] = \frac{1}{1 + L(\mathbf{y})} = \frac{1}{1 + e^{-2\sum_{i=1}^{n} y_i}}. \tag{6}$$

(b)

$$\begin{aligned} P_e = \mathrm{P}\left[X^* \neq X | X = 1\right] &= \mathrm{P}\left[X^* = -1 | X = 1\right] \\ &= \mathrm{P}\left[\frac{1}{1 + L(\mathbf{Y})} < \frac{1}{2} | X = 1\right] \\ &= \mathrm{P}\left[L(\mathbf{Y}) > 1 | X = 1\right] \\ &= \mathrm{P}\left[e^{-2\sum_{i=1}^{n} Y_i} > 1 | X = 1\right] \\ &= \mathrm{P}\left[\sum_{i=1}^{n} Y_i < 0 | X = 1\right]. \end{aligned} \tag{7}$$

Given $X = 1$, $Y_i = 1 + W_i$ and $\sum_{i=1}^{n} Y_i = n + W$ where $W = \sum_{i=1}^{n} W_n$ is a Gaussian $(0, \sqrt{n})$ random variable. Since, W is independent of X,

$$P_e = \mathrm{P}\left[n + W < 0 | X = 1\right] = \mathrm{P}\left[W < -n\right] = \mathrm{P}\left[W > n\right] = Q(\sqrt{n}). \tag{8}$$

(c) First we observe that we decide $X^* = 1$ on stage n iff

$$\begin{aligned} \hat{X}_n(\mathbf{Y}) > 1 - \epsilon &\Rightarrow 2\,\mathrm{P}\left[X = 1 | \mathbf{Y} = \mathbf{y}\right] - 1 < 1 - \epsilon \\ &\Rightarrow \mathrm{P}\left[X = 1 | \mathbf{Y} = \mathbf{y}\right] > 1 - \epsilon/2 = \epsilon_2. \end{aligned} \tag{9}$$

However, when we decide $X^* = 1$ given $\mathbf{Y} = \mathbf{y}$, the probability of a correct decision is $\mathrm{P}[X = 1 | \mathbf{Y} = \mathbf{y}]$. The probability of an error thus satisfies

$$P_e = 1 - \mathrm{P}\left[X = 1 | \mathbf{Y} = \mathbf{y}\right] < \epsilon/2. \tag{10}$$

This answer is simple if the logic occurs to you. In some ways, the following lower bound derivation is more straightforward. If $X = 1$, an error occurs after transmission $n = 1$ if $\hat{X}_1(\mathbf{y}) < -1 + \epsilon$. Thus

$$\begin{aligned} P_e &\geq \mathrm{P}\left[\hat{X}_1(\mathbf{y}) < -1 + \epsilon | X = 1\right] \\ &= \mathrm{P}\left[\frac{1 - L(\mathbf{y})}{1 + L(\mathbf{y})} < -1 + \epsilon | X = 1\right] \\ &= \mathrm{P}\left[L(\mathbf{y}) > (2/\epsilon) - 1 | X = 1\right]. \end{aligned} \tag{11}$$

For $n = 1$, $L(\mathbf{y}) = e^{-2Y_1} = e^{-2(X+W_1)}$. This implies

$$\begin{aligned} P_e &\geq \mathrm{P}\left[e^{-2(X+W_1)} > 2/\epsilon - 1 | X = 1\right] \\ &= \mathrm{P}\left[1 + W_1 < -\frac{1}{2}\ln\left(\frac{2}{\epsilon} - 1\right)\right] \\ &= \mathrm{P}\left[W_1 < -1 - \ln\sqrt{\frac{2}{\epsilon} - 1}\right] = Q\left(1 + \sqrt{\frac{2}{\epsilon} - 1}\right) = \epsilon_1. \end{aligned} \tag{12}$$

Problem 11.2.13 Solution

The key to this problem is to observe that

$$\mathrm{P}[A_0|H_0] = 1 - \mathrm{P}[A_1|H_0], \qquad \mathrm{P}[A_1|H_1] = 1 - \mathrm{P}[A_0|H_1]. \tag{1}$$

The total expected cost can be written as

$$\begin{aligned} \mathrm{E}[C'] = {} & \mathrm{P}[A_1|H_0]\,\mathrm{P}[H_0]\,C'_{10} + (1 - \mathrm{P}[A_1|H_0])\,\mathrm{P}[H_0]\,C'_{00} \\ & + \mathrm{P}[A_0|H_1]\,\mathrm{P}[H_1]\,C'_{01} + (1 - \mathrm{P}[A_0|H_1])\,\mathrm{P}[H_1]\,C'_{11}. \end{aligned} \tag{2}$$

Rearranging terms, we have

$$\begin{aligned} \mathrm{E}[C'] = {} & \mathrm{P}[A_1|H_0]\,\mathrm{P}[H_0]\,(C'_{10} - C'_{00}) + \mathrm{P}[A_0|H_1]\,\mathrm{P}[H_1]\,(C'_{01} - C'_{11}) \\ & + \mathrm{P}[H_0]\,C'_{00} + \mathrm{P}[H_1]\,C'_{11}. \end{aligned} \tag{3}$$

Since $\mathrm{P}[H_0]C'_{00} + \mathrm{P}[H_1]C'_{11}$ does not depend on the acceptance sets A_0 and A_1, the decision rule that minimizes $\mathrm{E}[C']$ is the same decision rule that minimizes

$$\mathrm{E}[C''] = \mathrm{P}[A_1|H_0]\,\mathrm{P}[H_0]\,(C'_{10} - C'_{00}) + \mathrm{P}[A_0|H_1]\,\mathrm{P}[H_1]\,(C'_{01} - C'_{11}). \tag{4}$$

The decision rule that minimizes $\mathrm{E}[C'']$ is the same as the minimum cost test in Theorem 11.3 with the costs C_{01} and C_{10} replaced by the differential costs $C'_{01} - C'_{11}$ and $C'_{10} - C'_{00}$.

Problem 11.3.1 Solution

Since the three hypotheses H_0, H_1, and H_2 are equally likely, the MAP and ML hypothesis tests are the same. From Theorem 11.8, the MAP rule is

$$x \in A_m \text{ if } f_{X|H_m}(x) \geq f_{X|H_j}(x) \text{ for all } j. \tag{1}$$

Since N is Gaussian with zero mean and variance σ_N^2, the conditional PDF of X given H_i is

$$f_{X|H_i}(x) = \frac{1}{\sqrt{2\pi\sigma_N^2}} e^{-(x-a(i-1))^2/2\sigma_N^2}. \tag{2}$$

Thus, the MAP rule is

$$x \in A_m \text{ if } (x - a(m-1))^2 \leq (x - a(j-1))^2 \text{ for all } j. \tag{3}$$

This implies that the rule for membership in A_0 is

$$x \in A_0 \text{ if } (x+a)^2 \leq x^2 \text{ and } (x+a)^2 \leq (x-a)^2. \tag{4}$$

This rule simplifies to

$$x \in A_0 \text{ if } x \leq -a/2. \tag{5}$$

Similar rules can be developed for A_1 and A_2. These are:

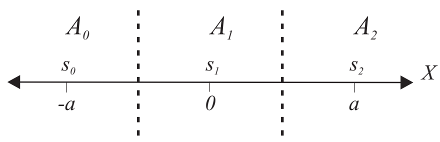
$$x \in A_1 \text{ if } -a/2 \leq x \leq a/2, \tag{6}$$

$$x \in A_2 \text{ if } x \geq a/2. \tag{7}$$

To summarize, the three acceptance regions are

$$A_0 = \{x|x \leq -a/2\}, A_1 = \{x| -a/2 < x \leq a/2\}, A_2 = \{x|x > a/2\}. \tag{8}$$

Graphically, the signal space is one dimensional and the acceptance regions are



Just as in the QPSK system of Example 11.13, the additive Gaussian noise dictates that the acceptance region A_i is the set of observations x that are closer to $s_i = (i-1)a$ than any other s_j.

Problem 11.3.3 Solution

Let H_i denote the hypothesis that symbol a_i was transmitted. Since the four hypotheses are equally likely, the ML tests will maximize the probability of a correct decision. Given H_i, N_1 and N_2 are independent and thus X_1 and X_2 are independent. This implies

$$\begin{aligned} f_{X_1,X_2|H_i}(x_1,x_2) &= f_{X_1|H_i}(x_1)\, f_{X_2|H_i}(x_2) \\ &= \frac{1}{2\pi\sigma^2} e^{-(x_1-s_{i1})^2/2\sigma^2} e^{-(x_2-s_{i2})^2/2\sigma^2} \\ &= \frac{1}{2\pi\sigma^2} e^{-[(x_1-s_{i1})^2+(x_2-s_{i2})^2]/2\sigma^2}. \end{aligned} \tag{1}$$

From Definition 11.2 the acceptance regions A_i for the ML multiple hypothesis test must satisfy

$$(x_1,x_2) \in A_i \text{ if } f_{X_1,X_2|H_i}(x_1,x_2) \geq f_{X_1,X_2|H_j}(x_1,x_2) \text{ for all } j. \tag{2}$$

Equivalently, the ML acceptance regions are

$$(x_1,x_2) \in A_i \text{ if } (x_1-s_{i1})^2 + (x_2-s_{i2})^2 \leq \min_j (x_1-s_{j1})^2 + (x_2-s_{j2})^2. \tag{3}$$

In terms of the vectors $\mathbf{x}$ and $\mathbf{s}_i$, the acceptance regions are defined by the rule

$$\mathbf{x} \in A_i \text{ if } \|\mathbf{x}-\mathbf{s}_i\|^2 \leq \|\mathbf{x}-\mathbf{s}_j\|^2. \tag{4}$$

Just as in the case of QPSK, the acceptance region A_i is the set of vectors $\mathbf{x}$ that are closest to $\mathbf{s}_i$.

Problem 11.3.5 Solution

(a) Hypothesis H_i is that $\mathbf{X} = \mathbf{s}_i + \mathbf{N}$, where $\mathbf{N}$ is a Gaussian random vector independent of which signal was transmitted. Thus, given H_i, $\mathbf{X}$ is a Gaussian $(\mathbf{s}_i, \sigma^2\mathbf{I})$ random vector. Since $\mathbf{X}$ is two-dimensional,

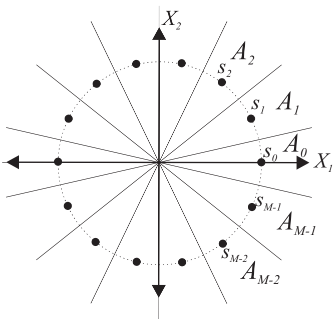
$$f_{\mathbf{X}|H_i}(\mathbf{x}) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}(\mathbf{x}-\mathbf{s}_i)'\sigma^2\mathbf{I}^{-1}(\mathbf{x}-\mathbf{s}_i)} = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\|\mathbf{x}-\mathbf{s}_i\|^2}. \tag{1}$$

Since the hypotheses H_i are equally likely, the MAP and ML rules are the same and achieve the minimum probability of error. In this case, from the vector version of Theorem 11.8, the MAP rule is

$$\mathbf{x} \in A_m \text{ if } f_{\mathbf{X}|H_m}(\mathbf{x}) \geq f_{\mathbf{X}|H_j}(\mathbf{x}) \text{ for all } j. \tag{2}$$

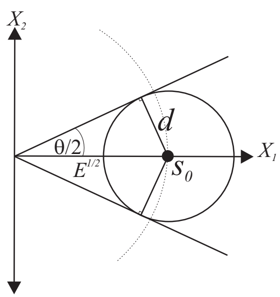
Using the conditional PDFs $f_{\mathbf{X}|H_i}(\mathbf{x})$, the MAP rule becomes

$$\mathbf{x} \in A_m \text{ if } \|\mathbf{x} - \mathbf{s}_m\|^2 \leq \|\mathbf{x} - \mathbf{s}_j\|^2 \text{ for all } j. \tag{3}$$



In terms of geometry, the interpretation is that all vectors $\mathbf{x}$ closer to $\mathbf{s}_m$ than to any other signal $\mathbf{s}_j$ are assigned to A_m. In this problem, the signal constellation (i.e., the set of vectors $\mathbf{s}_i$) is the set of vectors on the circle of radius E. The acceptance regions are the "pie slices" around each signal vector.

(b) Consider the following sketch to determine d.



Geometrically, the largest d such that $\|\mathbf{x} - \mathbf{s}_i\| \leq d$ defines the largest circle around $\mathbf{s}_i$ that can be inscribed into the pie slice A_i. By symmetry, this is the same for every A_i, hence we examine A_0. Each pie slice has angle $\theta = 2\pi/M$. Since the length of each signal vector is $\sqrt{E}$, the sketch shows that $\sin(\theta/2) = d/\sqrt{E}$. Thus $d = \sqrt{E}\sin(\pi/M)$.

(c) By symmetry, P_{ERR} is the same as the conditional probability of error $1 - \mathrm{P}[A_i|H_i]$, no matter which $\mathbf{s}_i$ is transmitted. Let B denote a circle of radius d at the origin and let B_i denote the circle of radius d around $\mathbf{s}_i$. Since $B_0 \subset A_0$,

$$\mathrm{P}\left[A_0|H_0\right] = \mathrm{P}\left[\mathbf{X} \in A_0|H_0\right] \geq \mathrm{P}\left[\mathbf{X} \in B_0|H_0\right] = \mathrm{P}\left[\mathbf{N} \in B\right]. \tag{4}$$

Since the components of $\mathbf{N}$ are iid Gaussian $(0, \sigma^2)$ random variables,

$$\begin{aligned} \mathrm{P}\left[\mathbf{N} \in B\right] &= \iint_B f_{N_1,N_2}(n_1, n_2)\, dn_1\, dn_2 \\ &= \frac{1}{2\pi\sigma^2} \iint_B e^{-(n_1^2+n_2^2)/2\sigma^2}\, dn_1\, dn_2. \end{aligned} \tag{5}$$

By changing to polar coordinates,

$$\begin{aligned} \mathrm{P}\left[\mathbf{N} \in B\right] &= \frac{1}{2\pi\sigma^2} \int_0^d \int_0^{2\pi} e^{-r^2/2\sigma^2} r\, d\theta\, dr \\ &= \frac{1}{\sigma^2} \int_0^d re^{-r^2/2\sigma^2} r\, dr \\ &= -e^{-r^2/2\sigma^2} \Big|_0^d = 1 - e^{-d^2/2\sigma^2} \\ &= 1 - e^{-E \sin^2(\pi/M)/2\sigma^2}. \end{aligned} \tag{6}$$

Thus

$$P_{\mathrm{ERR}} = 1 - \mathrm{P}\left[A_0|H_0\right] \leq 1 - \mathrm{P}\left[\mathbf{N} \in B\right] = e^{-E \sin^2(\pi/M)/2\sigma^2}. \tag{7}$$

Problem 11.3.7 Solution

Let $p_i = \mathrm{P}[H_i]$. From Theorem 11.8, the MAP multiple hypothesis test is

$$(x_1, x_2) \in A_i \text{ if } p_i f_{X_1,X_2|H_i}(x_1, x_2) \geq p_j f_{X_1,X_2|H_j}(x_1, x_2) \text{ for all } j. \tag{1}$$

From Example 11.13, the conditional PDF of X_1, X_2 given H_i is

$$f_{X_1,X_2|H_i}(x_1, x_2) = \frac{1}{2\pi\sigma^2} e^{-[(x_1-\sqrt{E}\cos\theta_i)^2+(x_2-\sqrt{E}\sin\theta_i)^2]/2\sigma^2}. \tag{2}$$

Using this conditional joint PDF, the MAP rule becomes

- $(x_1, x_2) \in A_i$ if for all j,

$$- \frac{(x_1 - \sqrt{E}\cos\theta_i)^2 + (x_2 - \sqrt{E}\sin\theta_i)^2}{2\sigma^2} + \frac{(x_1 - \sqrt{E}\cos\theta_j)^2 + (x_2 - \sqrt{E}\sin\theta_j)^2}{2\sigma^2} \geq \ln\frac{p_j}{p_i}. \tag{3}$$

Expanding the squares and using the identity $\cos^2\theta + \sin^2\theta = 1$ yields the simplified rule

- $(x_1, x_2) \in A_i$ if for all j,

$$x_1[\cos\theta_i - \cos\theta_j] + x_2[\sin\theta_i - \sin\theta_j] \geq \frac{\sigma^2}{\sqrt{E}}\ln\frac{p_j}{p_i}. \tag{4}$$

Note that the MAP rules define linear constraints in x_1 and x_2. Since $\theta_i = \pi/4 + i\pi/2$, we use the following table to enumerate the constraints:

	$\cos\theta_i$	$\sin\theta_i$
$i = 0$	$1/\sqrt{2}$	$1/\sqrt{2}$
$i = 1$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$i = 2$	$-1/\sqrt{2}$	$-1/\sqrt{2}$
$i = 3$	$1/\sqrt{2}$	$-1/\sqrt{2}$

(5)

To be explicit, to determine whether $(x_1, x_2) \in A_i$, we need to check the MAP rule for each $j \neq i$. Thus, each A_i is defined by three constraints. Using the above table, the acceptance regions are

- $(x_1, x_2) \in A_0$ if

$$x_1 \geq \frac{\sigma^2}{\sqrt{2E}}\ln\frac{p_1}{p_0}, \quad x_2 \geq \frac{\sigma^2}{\sqrt{2E}}\ln\frac{p_3}{p_0}, \quad x_1 + x_2 \geq \frac{\sigma^2}{\sqrt{2E}}\ln\frac{p_2}{p_0}. \tag{6}$$

- $(x_1, x_2) \in A_1$ if

$$x_1 \leq \frac{\sigma^2}{\sqrt{2E}} \ln \frac{p_1}{p_0}, \quad x_2 \geq \frac{\sigma^2}{\sqrt{2E}} \ln \frac{p_2}{p_1}, \quad -x_1 + x_2 \geq \frac{\sigma^2}{\sqrt{2E}} \ln \frac{p_3}{p_1}. \tag{7}$$

- $(x_1, x_2) \in A_2$ if

$$x_1 \leq \frac{\sigma^2}{\sqrt{2E}} \ln \frac{p_2}{p_3}, \quad x_2 \leq \frac{\sigma^2}{\sqrt{2E}} \ln \frac{p_2}{p_1}, \quad x_1 + x_2 \geq \frac{\sigma^2}{\sqrt{2E}} \ln \frac{p_2}{p_0}. \tag{8}$$

- $(x_1, x_2) \in A_3$ if

$$x_1 \geq \frac{\sigma^2}{\sqrt{2E}} \ln \frac{p_2}{p_3}, \quad x_2 \leq \frac{\sigma^2}{\sqrt{2E}} \ln \frac{p_3}{p_0}, \quad -x_1 + x_2 \geq \frac{\sigma^2}{\sqrt{2E}} \ln \frac{p_2}{p_3}. \tag{9}$$

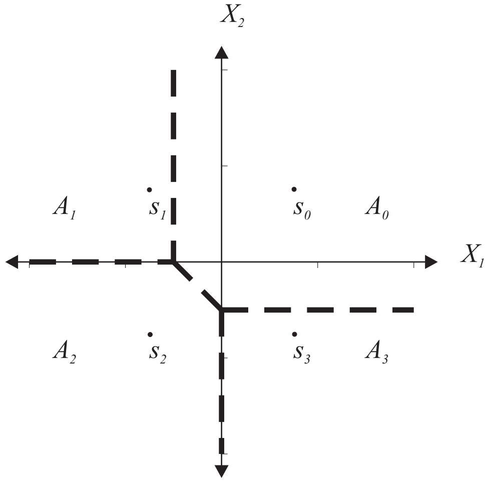
Using the parameters

$$\sigma = 0.8, \quad E = 1, \quad p_0 = 1/2, \quad p_1 = p_2 = p_3 = 1/6, \tag{10}$$

the acceptance regions for the MAP rule are

$$A_0 = \{(x_1, x_2) | x_1 \geq -0.497, x_2 \geq -0.497, x_1 + x_2 \geq -0.497\}, \tag{11}$$
$$A_1 = \{(x_1, x_2) | x_1 \leq -0.497, x_2 \geq 0, -x_1 + x_2 \geq 0\}, \tag{12}$$
$$A_2 = \{(x_1, x_2) | x_1 \leq 0, x_2 \leq 0, x_1 + x_2 \geq -0.497\}, \tag{13}$$
$$A_3 = \{(x_1, x_2) | x_1 \geq 0, x_2 \leq -0.497, -x_1 + x_2 \geq 0\}. \tag{14}$$

Here is a sketch of these acceptance regions:



Note that the boundary between A_1 and A_3 defined by $-x_1 + x_2 \geq 0$ plays no role because of the high value of p_0.

Problem 11.3.9 Solution

A short answer is that the decorrelator cannot be the same as the optimal maximum likelihood (ML) detector. If they were the same, that means we have reduced the 2^k comparisons of the optimal detector to a linear transformation followed by k single bit comparisons.

However, as this is not a satisfactory answer, we will build a simple example with $k = 2$ users and precessing gain $n = 2$ to show the difference between the ML detector and the decorrelator. In particular, suppose user 1 transmits with code vector $\mathbf{S}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}'$ and user transmits with code vector $\mathbf{S}_2 = \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix}'$ In addition, we assume that the users powers are $p_1 = p_2 = 1$. In this case, $\mathbf{P} = \mathbf{I}$ and

$$\mathbf{S} = \begin{bmatrix} 1 & \cos\theta \\ 0 & \sin\theta \end{bmatrix}. \tag{1}$$

For the ML detector, there are four hypotheses corresponding to each possible transmitted bit of each user. Using H_i to denote the hypothesis that $\mathbf{X} = \mathbf{x}_i$, we have

$$\mathbf{X} = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{X} = \mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \tag{2}$$

$$\mathbf{X} = \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \mathbf{X} = \mathbf{x}_4 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}. \tag{3}$$

When $\mathbf{X} = \mathbf{x}_i$, $\mathbf{Y} = \mathbf{y}_i + \mathbf{N}$ where $\mathbf{y}_i = \mathbf{S}\mathbf{x}_i$. Thus under hypothesis H_i, $\mathbf{Y} = \mathbf{y}_i + \mathbf{N}$ is a Gaussian $(\mathbf{y}_i, \sigma^2\mathbf{I})$ random vector with PDF

$$f_{\mathbf{Y}|H_i}(\mathbf{y}) = \frac{1}{2\pi\sigma^2} e^{-(\mathbf{y}-\mathbf{y}_i)'(\sigma^2\mathbf{I})^{-1}(\mathbf{y}-\mathbf{y}_i)/2} = \frac{1}{2\pi\sigma^2} e^{-\|\mathbf{y}-\mathbf{y}_i\|^2/2\sigma^2}. \tag{4}$$

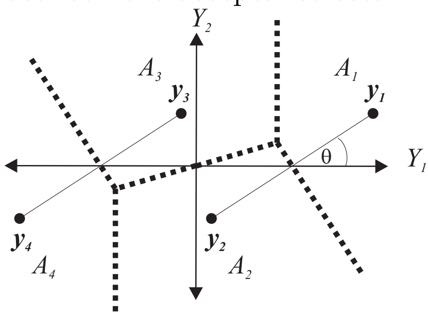
With the four hypotheses equally likely, the MAP and ML detectors are the same and minimize the probability of error. From Theorem 11.8, this decision rule is

$$\mathbf{y} \in A_m \text{ if } f_{\mathbf{Y}|H_m}(\mathbf{y}) \geq f_{\mathbf{Y}|H_j}(\mathbf{y}) \text{ for all } j. \tag{5}$$

This rule simplifies to

$$\mathbf{y} \in A_m \text{ if } \|\mathbf{y} - \mathbf{y}_m\| \leq \|\mathbf{y} - \mathbf{y}_j\| \text{ for all } j. \tag{6}$$

It is useful to show these acceptance sets graphically. In this plot, the area around $\mathbf{y}_i$ is the acceptance set A_i and the dashed lines are the boundaries between the acceptance sets.



$$\mathbf{y}_1 = \begin{bmatrix} 1 + \cos\theta \\ \sin\theta \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} -1 + \cos\theta \\ \sin\theta \end{bmatrix},$$

$$\mathbf{y}_2 = \begin{bmatrix} 1 - \cos\theta \\ -\sin\theta \end{bmatrix}, \quad \mathbf{y}_4 = \begin{bmatrix} -1 - \cos\theta \\ -\sin\theta \end{bmatrix}.$$

The probability of a correct decision is

$$\mathrm{P}[C] = \frac{1}{4} \sum_{i=1}^{4} \int_{A_i} f_{\mathbf{Y}|H_i}(\mathbf{y})\, d\mathbf{y}. \tag{7}$$

Even though the components of $\mathbf{Y}$ are conditionally independent given H_i, the four integrals $\int_{A_i} f_{\mathbf{Y}|H_i}(\mathbf{y})\, d\mathbf{y}$ cannot be represented in a simple form. Moreoever, they cannot even be represented by the $\Phi(\cdot)$ function. Note that the probability of a correct decision is the probability that the bits X_1 and X_2 transmitted by both users are detected correctly.

The probability of a bit error is still somewhat more complex. For example if $X_1 = 1$, then hypotheses H_1 and H_3 are equally likely. The detector guesses $\hat{X}_1 = 1$ if $\mathbf{Y} \in A_1 \cup A_3$. Given $X_1 = 1$, the conditional probability of a correct decision on this bit is

$$\begin{aligned} \mathrm{P}\left[\hat{X}_1 = 1 | X_1 = 1\right] &= \frac{1}{2}\mathrm{P}\left[\mathbf{Y} \in A_1 \cup A_3 | H_1\right] + \frac{1}{2}\mathrm{P}\left[\mathbf{Y} \in A_1 \cup A_3 | H_3\right] \\ &= \frac{1}{2}\int_{A_1 \cup A_3} f_{\mathbf{Y}|H_1}(\mathbf{y})\ d\mathbf{y} + \frac{1}{2}\int_{A_1 \cup A_3} f_{\mathbf{Y}|H_3}(\mathbf{y})\ d\mathbf{y}. \end{aligned} \tag{8}$$

By comparison, the decorrelator does something simpler. Since $\mathbf{S}$ is a square invertible matrix,

$$(\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}' = \mathbf{S}^{-1}(\mathbf{S}')^{-1}\mathbf{S}' = \mathbf{S}^{-1} = \frac{1}{\sin\theta}\begin{bmatrix} 1 & -\cos\theta \\ 0 & 1 \end{bmatrix}. \tag{9}$$

We see that the components of $\tilde{\mathbf{Y}} = \mathbf{S}^{-1}\mathbf{Y}$ are

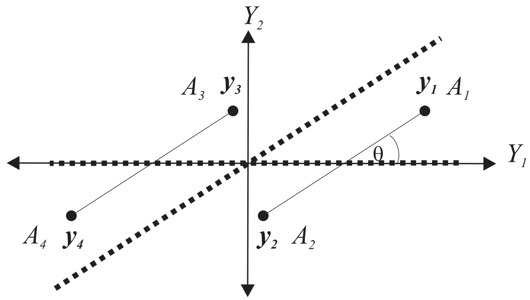
$$\tilde{Y}_1 = Y_1 - \frac{\cos\theta}{\sin\theta}Y_2, \qquad \tilde{Y}_2 = \frac{Y_2}{\sin\theta}. \tag{10}$$

Assuming (as in earlier sketch) that $0 < \theta < \pi/2$, the decorrelator bit decisions are

$$\hat{X}_1 = \operatorname{sgn}(\tilde{Y}_1) = \operatorname{sgn}\left(Y_1 - \frac{\cos\theta}{\sin\theta}Y_2\right), \tag{11}$$

$$\hat{X}_2 = \operatorname{sgn}(\tilde{Y}_2) = \operatorname{sgn}\left(\frac{Y_2}{\sin\theta}\right) = \operatorname{sgn}(Y_2). \tag{12}$$

Graphically, these regions are:



Because we chose a coordinate system such that $\mathbf{S}_1$ lies along the x-axis, the effect of the decorrelator on the rule for bit X_2 is particularly easy to understand. For bit X_2, we just check whether the vector $\mathbf{Y}$ is in the upper half plane. Generally, the boundaries of the decorrelator decision regions are determined by straight lines, they are easy to implement and probability of error is easy to calculate. However, these regions are suboptimal in terms of probability of error.

Problem 11.4.1 Solution

Under hypothesis H_i, the conditional PMF of X is

$$P_{X|H_i}(x) = \begin{cases} (1-p_i)p_i^{x-1}/(1-p_i^{20}) & x = 1,2,\ldots,20, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $p_0 = 0.99$ and $p_1 = 0.9$. It follows that for $x_0 = 0,1,\ldots,19$ that

$$\begin{aligned} \mathrm{P}\left[X > x_0|H_i\right] &= \frac{1-p_i}{1-p_i^{20}} \sum_{x=x_0+1}^{20} p_i^{x-1} \\ &= \frac{1-p_i}{1-p_i^{20}} \left[p_i^{x_0} + \cdots + p_i^{19}\right] \\ &= \frac{p_i^{x_0}(1-p_i)}{1-p_i^{20}} \left[1 + p_i + \cdots + p_i^{19-x_0}\right] \\ &= \frac{p_i^{x_0}(1-p_i^{20-x_0})}{1-p_i^{20}} = \frac{p_i^{x_0} - p_i^{20}}{1-p_i^{20}}. \end{aligned} \tag{2}$$

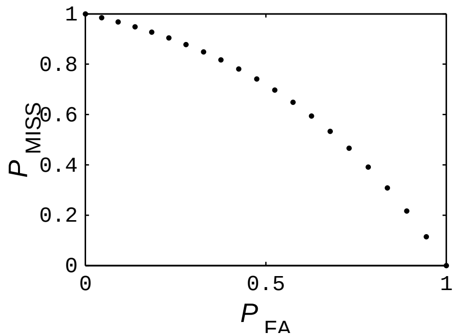
We note that the above formula is also correct for $x_0 = 20$. Using this formula, the false alarm and miss probabilities are

$$P_{\rm FA} = \mathrm{P}\left[X > x_0 | H_0\right] = \frac{p_0^{x_0} - p_0^{20}}{1 - p_0^{20}}, \tag{3}$$

$$P_{\rm MISS} = 1 - \mathrm{P}\left[X > x_0 | H_1\right] = \frac{1 - p_1^{x_0}}{1 - p_1^{20}}. \tag{4}$$

The MATLAB program `rocdisc(p0,p1)` returns the false alarm and miss probabilities and also plots the ROC. Here is the program and the output for `rocdisc(0.9,0.99)`:

```
function [PFA,PMISS]=rocdisc(p0,p1);
x=0:20;
PFA= (p0.^x-p0^(20))/(1-p0^(20));
PMISS= (1.0-(p1.^x))/(1-p1^(20));
plot(PFA,PMISS,'k.');
xlabel('\itP_{\rm FA}');
ylabel('\itP_{\rm MISS}');
```



From the receiver operating curve, we learn that we have a fairly lousy sensor. No matter how we set the threshold x_0, either the false alarm probability or the miss probability (or both!) exceed 0.5.

Problem 11.4.3 Solution

With $v = 1.5$ and $d = 0.5$, it appeared in Example 11.14 that $T = 0.5$ was best among the values tested. However, it also seemed likely the error probability P_e would decrease for larger values of T. To test this possibility we use sqdistor with 100,000 transmitted bits by trying the following:

```
>> T=[0.4:0.1:1.0];Pe=sqdistor(1.5,0.5,100000,T);
>> [Pmin,Imin]=min(Pe);T(Imin)
ans =
   0.80000000000000
```

Thus among $\{0.4, 0.5, \cdots, 1.0\}$, it appears that $T = 0.8$ is best. Now we test values of T in the neighborhood of 0.8:

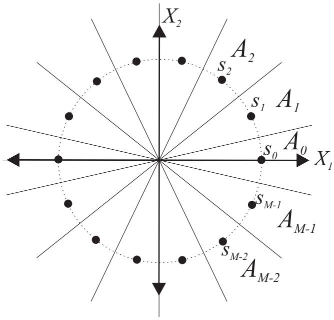
```
>> T=[0.70:0.02:0.9];Pe=sqdistor(1.5,0.5,100000,T);
>>[Pmin,Imin]=min(Pe);T(Imin)
ans =
   0.78000000000000
```

This suggests that $T = 0.78$ is best among these values. However, inspection of the vector `Pe` shows that all values are quite close. If we repeat this experiment a few times, we obtain:

```
>> T=[0.70:0.02:0.9];Pe=sqdistor(1.5,0.5,100000,T);
>> [Pmin,Imin]=min(Pe);T(Imin)
ans =
   0.78000000000000
>> T=[0.70:0.02:0.9];Pe=sqdistor(1.5,0.5,100000,T);
>> [Pmin,Imin]=min(Pe);T(Imin)
ans =
   0.80000000000000
>> T=[0.70:0.02:0.9];Pe=sqdistor(1.5,0.5,100000,T);
>> [Pmin,Imin]=min(Pe);T(Imin)
ans =
   0.76000000000000
>> T=[0.70:0.02:0.9];Pe=sqdistor(1.5,0.5,100000,T);
>> [Pmin,Imin]=min(Pe);T(Imin)
ans =
   0.78000000000000
```

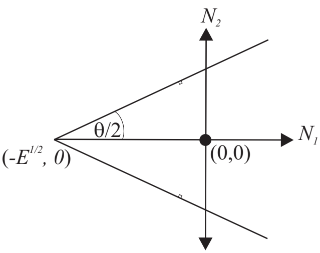
This suggests that the best value of T is in the neighborhood of 0.78. If someone were paying you to find the best T, you would probably want to do more testing. The only useful lesson here is that when you try to optimize parameters using simulation results, you should repeat your experiments to get a sense of the variance of your results.

Problem 11.4.5 Solution



In the solution to Problem 11.3.5, we found that the signal constellation and acceptance regions shown in the adjacent figure. We could solve this problem by a general simulation of an M-PSK system. This would include a random sequence of data sysmbols, mapping symbol i to vector $\mathbf{s}_i$, adding the noise vector $\mathbf{N}$ to produce the receiver output $\mathbf{X} = \mathbf{s}_i + \mathbf{N}$.

However, we are only asked to find the probability of symbol error, but not the probability that symbol i is decoded as symbol j at the receiver. Because of the symmetry of the signal constellation and the acceptance regions, the probability of symbol error is the same no matter what symbol is transmitted.



Thus it is simpler to assume that $\mathbf{s}_0$ is transmitted every time and check that the noise vector $\mathbf{N}$ is in the pie slice around $\mathbf{s}_0$. In fact by translating $\mathbf{s}_0$ to the origin, we obtain the "pie slice" geometry shown in the figure. Because the lines marking the boundaries of the pie slice have slopes $\pm \tan\theta/2$.

The pie slice region is given by the constraints

$$N_2 \leq \tan(\theta/2)\left[N_1 + \sqrt{E}\right], \qquad N_2 \geq -\tan(\theta/2)\left[N_1 + \sqrt{E}\right]. \tag{1}$$

We can rearrange these inequalities to express them in vector form as

$$\begin{bmatrix} -\tan\theta/2 & 1 \\ -\tan\theta/2 & -1 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sqrt{E}\tan\theta/2. \tag{2}$$

Finally, since each N_i has variance σ^2, we define the Gaussian $(\mathbf{0}, \mathbf{I})$ random vector $\mathbf{Z} = \mathbf{N}/\sigma$ and write our constraints as

$$\begin{bmatrix} -\tan\theta/2 & 1 \\ -\tan\theta/2 & -1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sqrt{\gamma}\tan\theta/2, \tag{3}$$

where $\gamma = E/\sigma^2$ is the signal to noise ratio of the system.

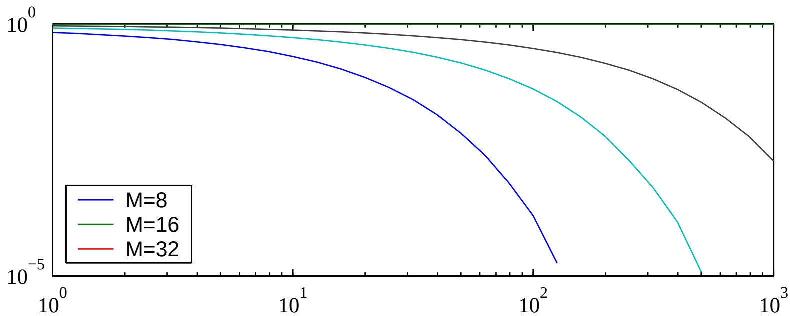
The MATLAB "simulation" simply generates many pairs $\begin{bmatrix} Z_1 & Z_2 \end{bmatrix}'$ and checks what fraction meets these constraints. the function mpsksim(M,snr,n) simulates the M-PSK system with SNR snr for n bit transmissions. The script mpsktest graphs the symbol error probability for $M = 8, 16, 32$.

```
function Pe=mpsksim(M,snr,n);
%Problem 8.4.5 Solution:
%Pe=mpsksim(M,snr,n)
%n bit M-PSK simulation
t=tan(pi/M);
A =[-t 1; -t -1];
Z=randn(2,n);
PC=zeros(length(snr));
for k=1:length(snr),
    B=(A*Z)<=t*sqrt(snr(k));
    PC(k)=sum(min(B))/n;
end
Pe=1-PC;
```

```
%mpsktest.m;
snr=10.^((0:30)/10);
n=500000;
Pe8=mpsksim(8,snr,n);
Pe16=mpsksim(16,snr,n);
Pe32=mpsksim(32,snr,n);
loglog(snr,Pe8,snr,Pe16,snr,Pe32);
legend('M=8','M=16','M=32',3);
```

In mpsksim, each column of the matrix Z corresponds to a pair of noise variables $\begin{bmatrix} Z_1 & Z_2 \end{bmatrix}'$. The code B=(A*Z)<=t*sqrt(snr(k)) checks whether each pair of noise variables is in the pie slice region. That is, B(1,j) and B(2,j) indicate if the ith pair meets the first and second constraints. Since min(B) operates on each column of B, min(B) is a row vector indicating which pairs of noise variables passed the test.

Here is the output of mpsktest:



The curves for $M = 8$ and $M = 16$ end prematurely because for high SNR, the error rate is so low that no errors are generated in 500,000 symbols. In this case, the measured P_e is zero and since $\log 0 = -\infty$, the `loglog` function simply ignores the zero values.

Problem 11.4.7 Solution

For the CDMA system of Problem 11.3.9, the received signal resulting from the transmissions of k users was given by

$$\mathbf{Y} = \mathbf{S}\mathbf{P}^{1/2}\mathbf{X} + \mathbf{N}, \tag{1}$$

where $\mathbf{S}$ is an $n \times k$ matrix with ith column $\mathbf{S}_i$ and $\mathbf{P}^{1/2} = \text{diag}[\sqrt{p_1}, \ldots, \sqrt{p_k}]$ is a $k \times k$ diagonal matrix of received powers, and $\mathbf{N}$ is a Gaussian $(\mathbf{0}, \sigma^2\mathbf{I})$ Gaussian noise vector.

(a) When $\mathbf{S}$ has linearly independent columns, $\mathbf{S}'\mathbf{S}$ is invertible. In this case, the decorrelating detector applies a transformation to $\mathbf{Y}$ to generate

$$\tilde{\mathbf{Y}} = (\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}'\mathbf{Y} = \mathbf{P}^{1/2}\mathbf{X} + \tilde{\mathbf{N}}, \tag{2}$$

where $\tilde{\mathbf{N}} = (\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}'\mathbf{N}$ is still a Gaussian noise vector with expected value $\text{E}[\tilde{\mathbf{N}}] = \mathbf{0}$. Decorrelation separates the signals in that the ith component of $\tilde{\mathbf{Y}}$ is

$$\tilde{Y}_i = \sqrt{p_i}X_i + \tilde{N}_i. \tag{3}$$

This is the same as a single user-receiver output of the binary communication system of Example 11.6. The single-user decision rule $\hat{X}_i = \text{sgn}\,(\tilde{Y}_i)$ for the transmitted bit X_i has probability of error

$$P_{e,i} = \mathrm{P}\left[\tilde{Y}_i > 0 | X_i = -1\right]$$
$$= \mathrm{P}\left[-\sqrt{p_i} + \tilde{N}_i > 0\right] = Q\left(\sqrt{\frac{p_i}{\mathrm{Var}[\tilde{N}_i]}}\right). \tag{4}$$

However, since $\tilde{\mathbf{N}} = \mathbf{AN}$ where $\mathbf{A} = (\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}'$, Theorem 8.11 tells us that $\tilde{\mathbf{N}}$ has covariance matrix $\mathbf{C}_{\tilde{\mathbf{N}}} = \mathbf{A}\mathbf{C}_{\mathbf{N}}\mathbf{A}'$. We note that the general property that $(\mathbf{B}^{-1})' = (\mathbf{B}')^{-1}$ implies that $\mathbf{A}' = \mathbf{S}((\mathbf{S}'\mathbf{S})')^{-1} = \mathbf{S}(\mathbf{S}'\mathbf{S})^{-1}$. These facts imply

$$\mathbf{C}_{\tilde{\mathbf{N}}} = (\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}'(\sigma^2\mathbf{I})\mathbf{S}(\mathbf{S}'\mathbf{S})^{-1} = \sigma^2(\mathbf{S}'\mathbf{S})^{-1}. \tag{5}$$

Note that $\mathbf{S}'\mathbf{S}$ is called the correlation matrix since its i,jth entry is $\mathbf{S}_i'\mathbf{S}_j$ is the correlation between the signal of user i and that of user j. Thus $\mathrm{Var}[\tilde{N}_i] = \sigma^2(\mathbf{S}'\mathbf{S})^{-1}_{ii}$ and the probability of bit error for user i is for user i is

$$P_{e,i} = Q\left(\sqrt{\frac{p_i}{\mathrm{Var}[\tilde{N}_i]}}\right) = Q\left(\sqrt{\frac{p_i}{(\mathbf{S}'\mathbf{S})^{-1}_{ii}}}\right). \tag{6}$$

To find the probability of error for a randomly chosen but, we average over the bits of all users and find that

$$P_e = \frac{1}{k}\sum_{i=1}^{k} P_{e,i} = \frac{1}{k}\sum_{i=1}^{k} Q\left(\sqrt{\frac{p_i}{(\mathbf{S}'\mathbf{S})^{-1}_{ii}}}\right). \tag{7}$$

(b) When $\mathbf{S}'\mathbf{S}$ is not invertible, the detector flips a coin to decide each bit. In this case, $P_{e,i} = 1/2$ and thus $P_e = 1/2$.

(c) When $\mathbf{S}$ is chosen randomly, we need to average over all possible matrices $\mathbf{S}$ to find the average probability of bit error. However, there are 2^{kn}

possible matrices **S** and averaging over all of them is too much work. Instead, we randomly generate m matrices **S** and estimate the average P_e by averaging over these m matrices.

A function `berdecorr` uses this method to evaluate the decorrelator BER. The code has a lot of lines because it evaluates the BER using `m` signal sets for each combination of users `k` and SNRs `snr`. However, because the program generates signal sets and calculates the BER asssociated with each, there is no need for the simulated transmission of bits. Thus the program runs quickly. Since there are only 2^n distinct columns for matrix **S**, it is quite possible to generate signal sets that are not linearly independent. In this case, `berdecorr` assumes the "flip a coin" rule is used. Just to see whether this rule dominates the error probability, we also display counts of how often **S** is rank deficient.

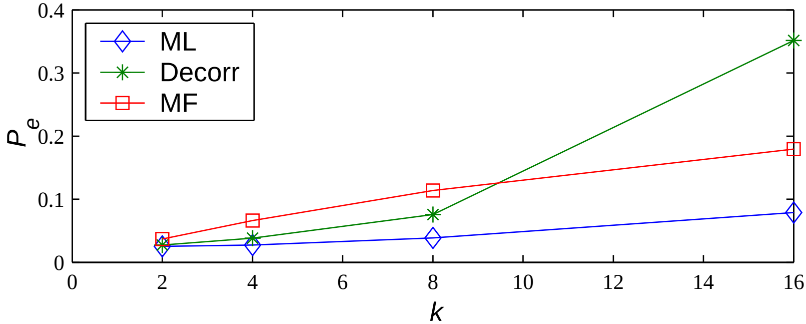
Here is the (somewhat tedious) code:

```
function Pe=berdecorr(n,k,snr,m);
%Problem 8.4.7 Solution: R-CDMA with decorrelation
%proc gain=n, users=k, average Pe for m signal sets
count=zeros(1,length(k)); %counts rank<k signal sets
Pe=zeros(length(k),length(snr)); snr=snr(:)';
for mm=1:m,
   for i=1:length(k),
   S=randomsignals(n,k(i)); R=S'*S;
   if (rank(R)<k(i))
      count(i)=count(i)+1;
      Pe(i,:)=Pe(i,:)+0.5*ones(1,length(snr));
   else
      G=diag(inv(R));
      Pe(i,:)=Pe(i,:)+sum(qfunction(sqrt((1./G)*snr)))/k(i);
   end
end
end
disp('Rank deficiency count:');disp(k);disp(count);
Pe=Pe/m;
```

Running `berdecorr` with processing gains $n = 16$ and $n = 32$ yields the following output:

```
>> k=[1 2 4 8 16 32];
>> pe16=berdecorr(16,k,4,10000);
Rank deficiency count:
  1      2      4      8      16     32
  0      2      2      12     454    10000
>> pe16'
ans =
  0.0228 0.0273 0.0383 0.0755 0.3515 0.5000
>> pe32=berdecorr(32,k,4,10000);
Rank deficiency count:
  1      2      4      8      16     32
  0      0      0      0      0      0
>> pe32'
ans =
  0.0228 0.0246 0.0290 0.0400 0.0771 0.3904
>>
```

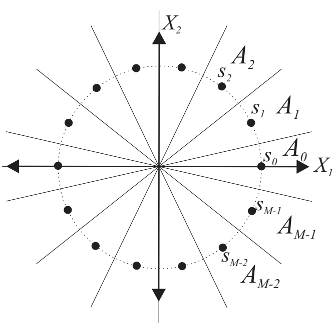
As you might expect, the BER increases as the number of users increases. This occurs because the decorrelator must suppress a large set of interferers. Also, in generating 10,000 signal matrices $\mathbf{S}$ for each value of k we see that rank deficiency is fairly uncommon, however it occasionally occurs for processing gain $n = 16$, even if $k = 4$ or $k = 8$. Finally, here is a plot of these same BER statistics for $n = 16$ and $k \in \{2, 4, 8, 16\}$. Just for comparison, on the same graph is the BER for the matched filter detector and the maximum likelihood detector found in Problem 11.4.6.



We see from the graph that the decorrelator is better than the matched filter for a small number of users. However, when the number of users k is large (relative to the processing gain n), the decorrelator suffers because it must suppress all interfering users. Finally, we note that these conclusions are specific to this scenario when all users have equal SNR. When some users have very high SNR, the decorrelator is good for the low-SNR user because it zeros out the interference from the high-SNR user.

Problem 11.4.9 Solution

(a) For the M-PSK communication system with additive Gaussian noise, A_j denoted the hypothesis that signal $\mathbf{s}_j$ was transmitted. The solution to Problem 11.3.5 derived the MAP decision rule



$$\mathbf{x} \in A_m \text{ if } \|\mathbf{x} - \mathbf{s}_m\|^2 \leq \|\mathbf{x} - \mathbf{s}_j\|^2 \text{ for all } j.$$

In terms of geometry, the interpretation is that all vectors $\mathbf{x}$ closer to $\mathbf{s}_m$ than to any other signal $\mathbf{s}_j$ are assigned to A_m. In this problem, the signal constellation (i.e., the set of vectors $\mathbf{s}_i$) is the set of vectors on the circle of radius E. The acceptance regions are the "pie slices" around each signal vector.

We observe that

$$\|\mathbf{x} - \mathbf{s}_j\|^2 = (\mathbf{x} - \mathbf{s}_j)'(\mathbf{x} - \mathbf{s}_j) = \mathbf{x}'\mathbf{x} - 2\mathbf{x}'\mathbf{s}_j + \mathbf{s}_j'\mathbf{s}'. \tag{1}$$

Since all the signals are on the same circle, $\mathbf{s}_j'\mathbf{s}_j$ is the same for all j. Also, $\mathbf{x}'\mathbf{x}$ is the same for all j. Thus

$$\min_j \|\mathbf{x} - \mathbf{s}_j\|^2 = \min_j -\mathbf{x}'\mathbf{s}_j = \max_j \mathbf{x}'\mathbf{s}_j. \tag{2}$$

Since $\mathbf{x}'\mathbf{s}_j = \|\mathbf{x}\| \|\mathbf{s}_j\| \cos\phi$ where ϕ is the angle between $\mathbf{x}$ and $\mathbf{s}_j$. Thus maximizing $\mathbf{x}'\mathbf{s}_j$ is equivalent to minimizing the angle between $\mathbf{x}$ and $\mathbf{s}_j$.

(b) In Problem 11.4.5, we estimated the probability of symbol error without building a complete simulation of the M-PSK system. In this problem, we need to build a more complete simulation to determine the probabilities P_{ij}. By symmetry, it is sufficient to transmit $\mathbf{s}_0$ repeatedly and count how often the receiver guesses $\mathbf{s}_j$. This is done by the functionp=mpskerr(M,snr,n).

```
function p=mpskerr(M,snr,n);
%Problem 8.4.5 Solution:
%Pe=mpsksim(M,snr,n)
%n bit M-PSK simulation
t=(2*pi/M)*(0:(M-1));
S=sqrt(snr)*[cos(t);sin(t)];
X=repmat(S(:,1),1,n)+randn(2,n);
[y,e]=max(S'*X);
p=countequal(e-1,(0:(M-1)))'/n;
```

Note that column i of `S` is the signal $\mathbf{s}_{i-1}$. The kth column of `X` corresponds to $\mathbf{X}_k = \mathbf{s}_0 + \mathbf{N}_k$, the received signal for the kth transmission. Thus `y(k)` corresponds to $\max_j \mathbf{X}_k'\mathbf{s}_j$ and `e(k)` reports the receiver decision for the kth transmission. The vector `p` calculates the relative frequency of each receiver decision.

The next step is to translate the vector $\begin{bmatrix} P_{00} & P_{01} & \cdots & P_{0,M-1} \end{bmatrix}'$ (corresponding to `p` in MATLAB) into an entire matrix $\mathbf{P}$ with elements P_{ij}.

The symmetry of the phase rotiation dictates that each row of $\mathbf{P}$ should be a one element cyclic rotation of the previous row. Moreover, by symmetry we observe that $P_{01} = P_{0,M-1}$, $P_{02} = P_{0,M-2}$ and so on. However, because `p` is derived from a simulation experiment, it will exhibit this symmetry only approximately.

```
function P=mpskmatrix(p);
M=length(p);
r=[0.5 zeros(1,M-2)];
A=toeplitz(r)+...
    hankel(fliplr(r));
A=[zeros(1,M-1);A];
A=[[1; zeros(M-1,1)] A];
P=toeplitz(A*(p(:)));
```

Our ad hoc (and largely unjustified) solution is to take the average of estimates of probabilities that symmetry says should be identical. (Why this is might be a good thing to do would make an interesting exam problem.) In `mpskmatrix(p)`, the matrix `A` implements the averaging. The code will become clear by examining the matrices `A` and the output `P`.

(c) The next step is to determine the effect of the mapping of bits to transmission vectors $\mathbf{s}_j$. The matrix $\mathbf{D}$ with i, jth element d_{ij} that indicates the number of bit positions in which the bit string assigned to $\mathbf{s}_i$ differs from the bit string assigned to $\mathbf{s}_j$. In this case, the integers provide a compact representation of this mapping. For example the binary mapping is

$\mathbf{s}_0$	$\mathbf{s}_1$	$\mathbf{s}_2$	$\mathbf{s}_3$	$\mathbf{s}_4$	$\mathbf{s}_5$	$\mathbf{s}_6$	$\mathbf{s}_7$
000	001	010	011	100	101	110	111
0	1	2	3	4	5	6	7

The Gray mapping is

$\mathbf{s}_0$	$\mathbf{s}_1$	$\mathbf{s}_2$	$\mathbf{s}_3$	$\mathbf{s}_4$	$\mathbf{s}_5$	$\mathbf{s}_6$	$\mathbf{s}_7$
000	001	011	010	110	111	101	100
0	1	3	2	6	7	5	4

Thus the binary mapping can be represented by a vector

$$\mathbf{c}_1 = \begin{bmatrix} 0 & 1 & \cdots & 7 \end{bmatrix}', \tag{3}$$

while the Gray mapping is described by

$$\mathbf{c}_2 = \begin{bmatrix} 0 & 1 & 3 & 2 & 6 & 7 & 5 & 4 \end{bmatrix}'. \tag{4}$$

```
function D=mpskdist(c);
L=length(c);m=log2(L);
[C1,C2]=ndgrid(c,c);
B1=dec2bin(C1,m);
B2=dec2bin(C2,m);
D=reshape(sum((B1~=B2),2),L,L);
```

The function `D=mpskdist(c)` translates the mapping vector `c` into the matrix `D` with entries d_{ij}. The method is to generate grids `C1` and `C2` for the pairs of integers, convert each integer into a length $\log_2 M$ binary string, and then to count the number of bit positions in which each pair differs.

Given matrices $\mathbf{P}$ and $\mathbf{D}$, the rest is easy. We treat BER as as a finite random variable that takes on value d_{ij} with probability P_{ij}. the expected value of this finite random variable is the expected number of bit errors. Note that the BER is a "rate" in that

$$\text{BER} = \frac{1}{M} \sum_i \sum_j P_{ij} d_{ij}. \tag{5}$$

is the expected number of bit errors per transmitted symbol.

```
function Pb=mpskmap(c,snr,n);
M=length(c);
D=mpskdist(c);
Pb=zeros(size(snr));
for i=1:length(snr),
    p=mpskerr(M,snr(i),n);
    P=mpskmatrix(p);
    Pb(i)=finiteexp(D,P)/M;
end
```

Given the integer mapping vector `c`, we estimate the BER of the a mapping using just one more function `Pb=mpskmap(c,snr,n)`. First we calculate the matrix `D` with elements dij. Next, for each value of the vector **snr**, we use n transmissions to estimate the probabilities P_{ij}. Last, we calculate the expected number of bit errors per transmission.

(d) We evaluate the binary mapping with the following commands:

```
>> c1=0:7;
>>snr=[4     8     16     32     64];
>>Pb=mpskmap(c1,snr,1000000);
>> Pb
Pb =
    0.7640    0.4878    0.2198    0.0529    0.0038
```

(e) Here is the performance of the Gray mapping:

```
>> c2=[0 1 3 2 6 7 5 4];
>>snr=[4     8     16     32     64];
>>Pg=mpskmap(c2,snr,1000000);
>> Pg
Pg =
    0.4943    0.2855    0.1262    0.0306    0.0023
```

Experimentally, we observe that the BER of the binary mapping is higher than the BER of the Gray mapping by a factor in the neighborhood of 1.5 to 1.7

In fact, this approximate ratio can be derived by a quick and dirty analysis. For high SNR, suppose that that $\mathbf{s}_i$ is decoded as $\mathbf{s}_{i+1}$ or $\mathbf{s}_{i-1}$ with probability $q = P_{i,i+1} = P_{i,i-1}$ and all other types of errors are negligible. In this case, the BER formula based on this approximation corresponds to summing the matrix D for the first off-diagonals and the corner elements. Here are the calculations:

```
>> D=mpskdist(c1);
>> sum(diag(D,1))+sum(diag(D,-1))+D(1,8)+D(8,1)
ans =
    28
>> DG=mpskdist(c2);
>> sum(diag(DG,1))+sum(diag(DG,-1))+DG(1,8)+DG(8,1)
ans =
    16
```

Thus in high SNR, we would expect

$$\text{BER(binary)} \approx 28q/M, \qquad \text{BER(Gray)} \approx 16q/M. \tag{6}$$

The ratio of BERs is $28/16 = 1.75$. Experimentally, we found at high SNR that the ratio of BERs was $0.0038/0.0023 = 1.65$, which seems to be in the right ballpark.

Problem Solutions – Chapter 12

Problem 12.1.1 Solution

First we note that the event $T > t_0$ has probability

$$\mathrm{P}[T > t_0] = \int_{t_0}^{\infty} \lambda e^{-\lambda t}\, dt = e^{-\lambda t_0}. \tag{1}$$

Given $T > t_0$, the conditional PDF of T is

$$f_{T|T>t_0}(t) = \begin{cases} \frac{f_T(t)}{\mathrm{P}[T>t_0]} & t > t_0, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} \lambda e^{-\lambda(t-t_0)} & t > t_0, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Given $T > t_0$, the minimum mean square error estimate of T is

$$\hat{T} = \mathrm{E}[T|T > t_0] = \int_{-\infty}^{\infty} t f_{T|T>t_0}(t)\ dt = \int_{t_0}^{\infty} \lambda t e^{-\lambda(t-t_0)}\, dt. \tag{3}$$

With the substitution $t' = t - t_0$, we obtain

$$\begin{aligned} \hat{T} &= \int_0^{\infty} \lambda(t_0 + t') e^{-\lambda t'}\, dt' \\ &= t_0 \underbrace{\int_0^{\infty} \lambda e^{-\lambda t'}\, dt'}_{1} + \underbrace{\int_0^{\infty} t' \lambda e^{-\lambda t'}\, dt'}_{\mathrm{E}[T]} = t_0 + \mathrm{E}[T]. \end{aligned} \tag{4}$$

Problem 12.1.3 Solution

(a) For $0 \le x \le 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy = \int_x^1 2\, dy = 2(1-x). \tag{1}$$

The complete expression of the PDF of X is

$$f_X(x) = \begin{cases} 2(1-x) & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(b) The blind estimate of X is

$$\hat{X}_B = E[X] = \int_0^1 2x(1-x)\,dx = \left(x^2 - \frac{2x^3}{3}\right)\Big|_0^1 = \frac{1}{3}. \tag{3}$$

(c) First we calculate

$$\begin{aligned} P[X > 1/2] &= \int_{1/2}^1 f_X(x)\,dx \\ &= \int_{1/2}^1 2(1-x)\,dx = (2x - x^2)\big|_{1/2}^1 = \frac{1}{4}. \end{aligned} \tag{4}$$

Now we calculate the conditional PDF of X given $X > 1/2$.

$$\begin{aligned} f_{X|X>1/2}(x) &= \begin{cases} \frac{f_X(x)}{P[X>1/2]} & x > 1/2, \\ 0 & \text{otherwise}, \end{cases} \\ &= \begin{cases} 8(1-x) & 1/2 < x \le 1, \\ 0 & \text{otherwise}. \end{cases} \end{aligned} \tag{5}$$

The minimum mean square error estimate of X given $X > 1/2$ is

$$\begin{aligned} E[X|X > 1/2] &= \int_{-\infty}^{\infty} x f_{X|X>1/2}(x)\,dx \\ &= \int_{1/2}^1 8x(1-x)\,dx = \left(4x^2 - \frac{8x^3}{3}\right)\Big|_{1/2}^1 = \frac{2}{3}. \end{aligned} \tag{6}$$

(d) For $0 \le y \le 1$, the marginal PDF of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^y 2\,dx = 2y. \tag{7}$$

The complete expression for the marginal PDF of Y is

$$f_Y(y) = \begin{cases} 2y & 0 \le y \le 1, \\ 0 & \text{otherwise}. \end{cases} \tag{8}$$

(e) The blind estimate of Y is

$$\hat{y}_B = \mathrm{E}[Y] = \int_0^1 2y^2\, dy = \frac{2}{3}. \tag{9}$$

(f) We already know that $\mathrm{P}[X > 1/2] = 1/4$. However, this problem differs from the other problems in this section because we will estimate Y based on the observation of X. In this case, we need to calculate the conditional joint PDF

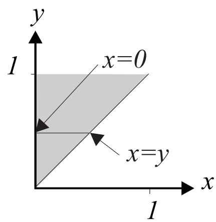
$$\begin{aligned} f_{X,Y|X>1/2}(x,y) &= \begin{cases} \frac{f_{X,Y}(x,y)}{\mathrm{P}[X>1/2]} & x > 1/2, \\ 0 & \text{otherwise}, \end{cases} \\ &= \begin{cases} 8 & 1/2 < x \le y \le 1, \\ 0 & \text{otherwise}. \end{cases} \end{aligned} \tag{10}$$

The MMSE estimate of Y given $X > 1/2$ is

$$\begin{aligned} \mathrm{E}[Y|X > 1/2] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{X,Y|X>1/2}(x,y)\, dx\, dy \\ &= \int_{1/2}^{1} y \left(\int_{1/2}^{y} 8\, dx\right) dy \\ &= \int_{1/2}^{1} y(8y-4)\, dy = \frac{5}{6}. \end{aligned} \tag{11}$$

Problem 12.1.5 Solution

(a) First we find the marginal PDF $f_Y(y)$. For $0 \le y \le 2$,



$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \int_0^y 2\, dx = 2y. \tag{1}$$

Hence, for $0 \le y \le 2$, the conditional PDF of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} 1/y & 0 \le x \le y, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(b) The optimum mean squared error estimate of X given $Y = y$ is

$$\hat{x}_M(y) = \mathrm{E}[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx = \int_0^y \frac{x}{y}\, dx = y/2. \tag{3}$$

(c) The MMSE estimator of X given Y is $\hat{X}_M(Y) = \mathrm{E}[X|Y] = Y/2$. The mean squared error is

$$\begin{aligned} e^*_{X,Y} &= \mathrm{E}\left[(X - \hat{X}_M(Y))^2\right] \\ &= \mathrm{E}\left[(X - Y/2)^2\right] = \mathrm{E}\left[X^2 - XY + Y^2/4\right]. \end{aligned} \tag{4}$$

Of course, the integral must be evaluated.

$$\begin{aligned} e^*_{X,Y} &= \int_0^1 \int_0^y 2(x^2 - xy + y^2/4)\, dx\, dy \\ &= \int_0^1 \left(2x^3/3 - x^2 y + xy^2/2\right)\Big|_{x=0}^{x=y} dy \\ &= \int_0^1 \frac{y^3}{6}\, dy = 1/24. \end{aligned} \tag{5}$$

Another approach to finding the mean square error is to recognize that the MMSE estimator is a linear estimator and thus must be the optimal linear estimator. Hence, the mean square error of the optimal linear estimator given by Theorem 12.3 must equal $e^*_{X,Y}$. That is, $e^*_{X,Y} = \mathrm{Var}[X](1 - \rho^2_{X,Y})$. However, calculation of the correlation coefficient $\rho_{X,Y}$ is at least as much work as direct calculation of $e^*_{X,Y}$.

Problem 12.1.7 Solution

We need to find the conditional estimate

$$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx. \tag{1}$$

Replacing y by Y in $E[X|Y=y]$ will yield the requested $E[X|Y]$. We start by finding $f_{Y|X}(y|x)$. Given $X=x$, $Y=x-Z$ so that

$$\begin{aligned} P[Y \le y|X=x] &= P[x-Z \le y|X=x] \\ &= P[Z \ge x-y|X=x] = 1 - F_Z(x-y). \end{aligned} \tag{2}$$

Note the last inequality follows because Z and X are independent random variables. Taking derivatives, we have

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{dP[Z \le x-y|X=x]}{dy} \\ &= \frac{d}{dy}(1 - F_Z(x-y)) = f_Z(x-y). \end{aligned} \tag{3}$$

It follows that X and Y have joint PDF

$$f_{X,Y}(x,y) = f_{Y|X}(y|x)\, f_X(x) = f_Z(x-y)\, f_X(x). \tag{4}$$

By the definition of conditional PDF,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_Z(x-y)\, f_X(x)}{f_Y(y)}, \tag{5}$$

and thus

$$\begin{aligned} E[X|Y=y] &= \int_{-\infty}^{\infty} x \frac{f_Z(x-y)\, f_X(x)}{f_Y(y)}\, dx \\ &= \frac{1}{f_Y(y)} \int_{-\infty}^{\infty} x f_Z(x-y)\, f_X(x)\, dx. \end{aligned} \tag{6}$$

Without more information, this is the simplest possible answer. Also note that the denominator $f_Y(y)$ is given by

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_{-\infty}^{\infty} f_Z(x-y)\,f_X(x)\,dx. \tag{7}$$

For a given PDF $f_Z(z)$, it is sometimes possible to compute these integrals in closed form; Gaussian Z is one such example.

Problem 12.2.1 Solution

(a) The marginal PMFs of X and Y are listed below

$$P_X(x) = \begin{cases} 1/3 & x = -1, 0, 1, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

$$P_Y(y) = \begin{cases} 1/4 & y = -3, -1, 0, 1, 3, \\ 0 & \text{otherwise}. \end{cases} \tag{2}$$

(b) No, the random variables X and Y are not independent since

$$P_{X,Y}(1,-3) = 0 \neq P_X(1)\,P_Y(-3) \tag{3}$$

(c) Direct evaluation leads to

$$\mathrm{E}[X] = 0, \qquad \mathrm{Var}[X] = 2/3, \tag{4}$$

$$\mathrm{E}[Y] = 0, \qquad \mathrm{Var}[Y] = 5. \tag{5}$$

This implies

$$\mathrm{Cov}[X,Y] = \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y] = \mathrm{E}[XY] = 7/6. \tag{6}$$

(d) From Theorem 12.3, the optimal linear estimate of X given Y is

$$\hat{X}_L(Y) = \rho_{X,Y}\frac{\sigma_X}{\sigma_Y}(Y - \mu_Y) + \mu_X = \frac{7}{30}Y + 0. \tag{7}$$

Therefore, $a^* = 7/30$ and $b^* = 0$.

(e) From the previous part, X and Y have correlation coefficient

$$\rho_{X,Y} = \text{Cov}[X,Y] / \sqrt{\text{Var}[X]\,\text{Var}[Y]} = \sqrt{49/120}. \tag{8}$$

From Theorem 12.3, the minimum mean square error of the optimum linear estimate is

$$e_L^* = \sigma_X^2(1-\rho_{X,Y}^2) = \frac{2}{3}\frac{71}{120} = \frac{71}{180}. \tag{9}$$

(f) The conditional probability mass function is

$$P_{X|Y}(x|-3) = \frac{P_{X,Y}(x,-3)}{P_Y(-3)} = \begin{cases} \frac{1/6}{1/4} = 2/3 & x=-1, \\ \frac{1/12}{1/4} = 1/3 & x=0, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

(g) The minimum mean square estimator of X given that $Y = 3$ is

$$\hat{x}_M(-3) = \text{E}[X|Y=-3] = \sum_x x P_{X|Y}(x|-3) = -2/3. \tag{11}$$

(h) The mean squared error of this estimator is

$$\begin{aligned} \hat{e}_M(-3) &= \text{E}\left[(X-\hat{x}_M(-3))^2|Y=-3\right] \\ &= \sum_x (x+2/3)^2 P_{X|Y}(x|-3) \\ &= (-1/3)^2(2/3) + (2/3)^2(1/3) = 2/9. \end{aligned} \tag{12}$$

Problem 12.2.3 Solution

The solution to this problem is to simply calculate the various quantities required for the optimal linear estimator given by Theorem 12.3. First we calculate the necessary moments of X and Y.

$$\mathrm{E}[X] = -1(1/4) + 0(1/2) + 1(1/4) = 0, \tag{1}$$
$$\mathrm{E}\left[X^2\right] = (-1)^2(1/4) + 0^2(1/2) + 1^2(1/4) = 1/2, \tag{2}$$
$$\mathrm{E}[Y] = -1(17/48) + 0(17/48) + 1(14/48) = -1/16, \tag{3}$$
$$\mathrm{E}\left[Y^2\right] = (-1)^2(17/48) + 0^2(17/48) + 1^2(14/48) = 31/48, \tag{4}$$
$$\mathrm{E}[XY] = 3/16 - 0 - 0 + 1/8 = 5/16. \tag{5}$$

The variances and covariance are

$$\mathrm{Var}[X] = \mathrm{E}\left[X^2\right] - (\mathrm{E}[X])^2 = 1/2, \tag{6}$$
$$\mathrm{Var}[Y] = \mathrm{E}\left[Y^2\right] - (\mathrm{E}[Y])^2 = 493/768, \tag{7}$$
$$\mathrm{Cov}[X,Y] = \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y] = 5/16, \tag{8}$$
$$\rho_{X,Y} = \frac{\mathrm{Cov}[X,Y]}{\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y]}} = \frac{5\sqrt{6}}{\sqrt{493}}. \tag{9}$$

By reversing the labels of X and Y in Theorem 12.3, we find that the optimal linear estimator of Y given X is

$$\hat{Y}_L(X) = \rho_{X,Y}\frac{\sigma_Y}{\sigma_X}(X - \mathrm{E}[X]) + \mathrm{E}[Y] = \frac{5}{8}X - \frac{1}{16}. \tag{10}$$

The mean square estimation error is

$$e_L^* = \mathrm{Var}[Y](1 - \rho_{X,Y}^2) = 343/768. \tag{11}$$

Problem 12.2.5 Solution

The linear mean square estimator of X given Y is

$$\hat{X}_L(Y) = \left(\frac{\mathrm{E}[XY] - \mu_X\mu_Y}{\mathrm{Var}[Y]}\right)(Y - \mu_Y) + \mu_X. \tag{1}$$

To find the parameters of this estimator, we calculate

$$f_Y(y) = \int_0^y 6(y-x)\,dx = 6xy - 3x^2\Big|_0^y = 3y^2 \qquad (0 \le y \le 1), \tag{2}$$

$$f_X(x) = \int_x^1 6(y-x)\,dy = \begin{cases} 3(1+-2x+x^2) & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The moments of X and Y are

$$\mathrm{E}[Y] = \int_0^1 3y^3\,dy = 3/4, \tag{4}$$

$$\mathrm{E}[X] = \int_0^1 3x(1-2x+x^2)\,dx = 1/4, \tag{5}$$

$$\mathrm{E}\left[Y^2\right] = \int_0^1 3y^4\,dy = 3/5, \tag{6}$$

$$\mathrm{E}\left[X^2\right] = \int_0^1 3x^2(1+-2x+x^2)\,dx = 1/10. \tag{7}$$

The correlation between X and Y is

$$\mathrm{E}[XY] = 6\int_0^1 \int_x^1 xy(y-x)\,dy\,dx = 1/5. \tag{8}$$

Putting these pieces together, the optimal linear estimate of X given Y is

$$\hat{X}_L(Y) = \left(\frac{1/5 - 3/16}{3/5 - (3/4)^2}\right)\left(Y - \frac{3}{4}\right) + \frac{1}{4} = \frac{Y}{3}. \tag{9}$$

Problem 12.2.7 Solution

From the problem statement, we learn the following facts:

$$f_R(r) = \begin{cases} e^{-r} & r \ge 0, \\ 0 & \text{otherwise,} \end{cases} \qquad f_{X|R}(x|r) = \begin{cases} re^{-rx} & x \ge 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Note that $f_{X,R}(x,r) > 0$ for all non-negative X and R. Hence, for the remainder of the problem, we assume both X and R are non-negative and we omit the usual "zero otherwise" considerations.

(a) To find $\hat{r}_M(X)$, we need the conditional PDF

$$f_{R|X}(r|x) = \frac{f_{X|R}(x|r)\, f_R(r)}{f_X(x)}. \tag{2}$$

The marginal PDF of X is

$$f_X(x) = \int_0^\infty f_{X|R}(x|r)\, f_R(r)\; dr = \int_0^\infty re^{-(x+1)r}\; dr. \tag{3}$$

We use the integration by parts formula $\int u\, dv = uv - \int v\, du$ by choosing $u = r$ and $dv = e^{-(x+1)r}\, dr$. Thus $v = -e^{-(x+1)r}/(x+1)$ and

$$\begin{aligned} f_X(x) &= \frac{-r}{x+1} e^{-(x+1)r}\Big|_0^\infty + \frac{1}{x+1}\int_0^\infty e^{-(x+1)r}\; dr \\ &= \frac{-1}{(x+1)^2} e^{-(x+1)r}\Big|_0^\infty = \frac{1}{(x+1)^2}. \end{aligned} \tag{4}$$

Now we can find the conditional PDF of R given X.

$$f_{R|X}(r|x) = \frac{f_{X|R}(x|r)\, f_R(r)}{f_X(x)} = (x+1)^2 re^{-(x+1)r}. \tag{5}$$

By comparing, $f_{R|X}(r|x)$ to the Erlang PDF shown in Appendix A, we see that given $X = x$, the conditional PDF of R is an Erlang PDF with parameters $n = 1$ and $\lambda = x + 1$. This implies

$$\mathrm{E}\left[R|X = x\right] = \frac{1}{x+1}, \qquad \mathrm{Var}\left[R|X = x\right] = \frac{1}{(x+1)^2}. \tag{6}$$

Hence, the MMSE estimator of R given X is

$$\hat{r}_M(X) = \mathrm{E}\left[R|X\right] = \frac{1}{X+1}. \tag{7}$$

(b) The MMSE estimate of X given $R = r$ is $\mathrm{E}[X|R = r]$. From the initial problem statement, we know that given $R = r$, X is exponential with expectred value $1/r$. That is, $\mathrm{E}[X|R = r] = 1/r$. Another way of writing this statement is

$$\hat{x}_M(R) = \mathrm{E}\left[X|R\right] = 1/R. \tag{8}$$

(c) Note that the expected value of X is

$$\mathrm{E}\left[X\right] = \int_0^\infty x f_X(x)\, dx = \int_0^\infty \frac{x}{(x+1)^2}\, dx = \infty. \tag{9}$$

Because $\mathrm{E}[X]$ doesn't exist, the LMSE estimate of X given R doesn't exist.

(d) Just as in part (c), because $\mathrm{E}[X]$ doesn't exist, the LMSE estimate of R given X doesn't exist.

Problem 12.2.9 Solution

These four joint PMFs are actually related to each other. In particular, completing the row sums and column sums shows that each random variable has the same marginal PMF. That is,

$$\begin{aligned} P_X(x) = P_Y(x) = P_U(x) = P_V(x) = P_S(x) = P_T(x) = P_Q(x) = P_R(x) \\ = \begin{cases} 1/3 & x = -1, 0, 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{1}$$

This implies

$$\mathrm{E}\left[X\right] = \mathrm{E}\left[Y\right] = \mathrm{E}\left[U\right] = \mathrm{E}\left[V\right] = \mathrm{E}\left[S\right] = \mathrm{E}\left[T\right] = \mathrm{E}\left[Q\right] = \mathrm{E}\left[R\right] = 0, \tag{2}$$

and that

$$\begin{aligned} \mathrm{E}\left[X^2\right] = \mathrm{E}\left[Y^2\right] = \mathrm{E}\left[U^2\right] = \mathrm{E}\left[V^2\right] \\ = \mathrm{E}\left[S^2\right] = \mathrm{E}\left[T^2\right] = \mathrm{E}\left[Q^2\right] = \mathrm{E}\left[R^2\right] = 2/3. \end{aligned} \tag{3}$$

Since each random variable has zero expected value, the second moment equals the variance. Also, the standard deviation of each random variable is $\sqrt{2/3}$. These common properties will make it much easier to answer the questions.

(a) Random variables X and Y are independent since for all x and y,

$$P_{X,Y}(x, y) = P_X(x)\, P_Y(y)\,. \tag{4}$$

Since each other pair of random variables has the same marginal PMFs as X and Y but a different joint PMF, all of the other pairs of random variables must be dependent. Since X and Y are independent, $\rho_{X,Y} = 0$. For the other pairs, we must compute the covariances.

$$\text{Cov}\,[U, V] = \text{E}\,[UV] = (1/3)(-1) + (1/3)(-1) = -2/3, \tag{5}$$
$$\text{Cov}\,[S, T] = \text{E}\,[ST] = 1/6 - 1/6 + 0 + -1/6 + 1/6 = 0, \tag{6}$$
$$\text{Cov}\,[Q, R] = \text{E}\,[QR] = 1/12 - 1/6 - 1/6 + 1/12 = -1/6 \tag{7}$$

The correlation coefficient of U and V is

$$\rho_{U,V} = \frac{\text{Cov}\,[U, V]}{\sqrt{\text{Var}[U]}\sqrt{\text{Var}[V]}} = \frac{-2/3}{\sqrt{2/3}\sqrt{2/3}} = -1 \tag{8}$$

In fact, since the marginal PMF's are the same, the denominator of the correlation coefficient will be 2/3 in each case. The other correlation coefficients are

$$\rho_{S,T} = \frac{\text{Cov}\,[S, T]}{2/3} = 0, \qquad \rho_{Q,R} = \frac{\text{Cov}\,[Q, R]}{2/3} = -1/4. \tag{9}$$

(b) From Theorem 12.3, the least mean square linear estimator of U given V is

$$\hat{U}_L(V) = \rho_{U,V}\frac{\sigma_U}{\sigma_V}(V - \text{E}\,[V]) + \text{E}\,[U] = \rho_{U,V} V = -V. \tag{10}$$

Similarly for the other pairs, all expected values are zero and the ratio of the standard deviations is always 1. Hence,

$$\hat{X}_L(Y) = \rho_{X,Y} Y = 0, \tag{11}$$

$$\hat{S}_L(T) = \rho_{S,T} T = 0, \tag{12}$$

$$\hat{Q}_L(R) = \rho_{Q,R} R = -R/4. \tag{13}$$

From Theorem 12.3, the mean square errors are

$$e_L^*(X,Y) = \text{Var}[X](1-\rho_{X,Y}^2) = 2/3, \tag{14}$$

$$e_L^*(U,V) = \text{Var}[U](1-\rho_{U,V}^2) = 0, \tag{15}$$

$$e_L^*(S,T) = \text{Var}[S](1-\rho_{S,T}^2) = 2/3, \tag{16}$$

$$e_L^*(Q,R) = \text{Var}[Q](1-\rho_{Q,R}^2) = 5/8. \tag{17}$$

Problem 12.3.1 Solution

In this case, the joint PDF of X and R is

$$\begin{aligned} f_{X,R}(x,r) &= f_{X|R}(x|r)\, f_R(r) \\ &= \begin{cases} \frac{1}{r_0\sqrt{128\pi}} e^{-(x+40+40\log_{10} r)^2/128} & 0 \le r \le r_0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{1}$$

From Theorem 12.5, the MAP estimate of R given $X = x$ is the value of r that maximizes $f_{X|R}(x|r) f_R(r)$. Since R has a uniform PDF over $[0, 1000]$,

$$\hat{r}_{\text{MAP}}(x) = \arg\max_{0\le r} f_{X|R}(x|r)\, f_R(r) = \arg\max_{0\le r\le 1000} f_{X|R}(x|r) \tag{2}$$

Hence, the maximizing value of r is the same as for the ML estimate in Quiz 12.3 unless the maximizing r exceeds 1000 m. In this case, the maximizing value is $r = 1000$ m. From the solution to Quiz 12.3, the resulting ML estimator is

$$\hat{r}_{\text{ML}}(x) = \begin{cases} 1000 & x < -160, \\ (0.1)10^{-x/40} & x \ge -160. \end{cases} \tag{3}$$

Problem 12.3.3 Solution

Both parts (a) and (b) rely on the conditional PDF of R given $N = n$. When dealing with situations in which we mix continuous and discrete random variables, its often helpful to start from first principles.

$$\begin{aligned}
f_{R|N}(r|n)\ dr &= \mathrm{P}[r < R \leq r + dr|N = n] \\
&= \frac{\mathrm{P}[r < R \leq r + dr, N = n]}{\mathrm{P}[N = n]} \\
&= \frac{\mathrm{P}[N = n|R = r]\,\mathrm{P}[r < R \leq r + dr]}{\mathrm{P}[N = n]}. \qquad (1)
\end{aligned}$$

In terms of PDFs and PMFs, we have

$$f_{R|N}(r|n) = \frac{P_{N|R}(n|r)\, f_R(r)}{P_N(n)}. \qquad (2)$$

To find the value of n that maximizes $f_{R|N}(r|n)$, we need to find the denominator $P_N(n)$.

$$\begin{aligned}
P_N(n) &= \int_{-\infty}^{\infty} P_{N|R}(n|r)\, f_R(r)\ dr \\
&= \int_0^{\infty} \frac{(rT)^n e^{-rT}}{n!} \mu e^{-\mu r}\ dr \\
&= \frac{\mu T^n}{n!(\mu + T)} \int_0^{\infty} r^n (\mu + T) e^{-(\mu+T)r}\ dr \\
&= \frac{\mu T^n}{n!(\mu + T)} \mathrm{E}[X^n]. \qquad (3)
\end{aligned}$$

where X is an exponential random variable with expected value $1/(\mu + T)$.There are several ways to derive the nth moment of an exponential random variable including integration by parts. In Example 9.4, the MGF is used to show that $\mathrm{E}[X^n] = n!/(\mu + T)^n$. Hence, for $n \geq 0$,

$$P_N(n) = \frac{\mu T^n}{(\mu + T)^{n+1}}. \qquad (4)$$

Finally, the conditional PDF of R given N is

$$f_{R|N}(r|n) = \frac{P_{N|R}(n|r)\, f_R(r)}{P_N(n)}$$
$$= \frac{\frac{(rT)^n e^{-rT}}{n!}\mu e^{-\mu r}}{\frac{\mu T^n}{(\mu+T)^{n+1}}} = \frac{(\mu+T)^{n+1} r^n e^{-(\mu+T)r}}{n!}. \tag{5}$$

(a) The MMSE estimate of R given $N = n$ is the conditional expected value $\mathrm{E}[R|N = n]$. Given $N = n$, the conditional PDF oF R is that of an Erlang random variable or order $n + 1$. From Appendix A, we find that $\mathrm{E}[R|N = n] = (n + 1)/(\mu + T)$. The MMSE estimate of R given N is

$$\hat{R}_M(N) = \mathrm{E}[R|N] = \frac{N+1}{\mu+T}. \tag{6}$$

(b) The MAP estimate of R given $N = n$ is the value of r that maximizes $f_{R|N}(r|n)$.

$$\hat{R}_{\mathrm{MAP}}(n) = \arg\max_{r\geq 0} f_{R|N}(r|n)$$
$$= \arg\max_{r\geq 0} \frac{(\mu+T)^{n+1}}{n!} r^n e^{-(\mu+T)r}. \tag{7}$$

By setting the derivative with respect to r to zero, we obtain the MAP estimate

$$\hat{R}_{\mathrm{MAP}}(n) = \frac{n}{\mu+T}. \tag{8}$$

(c) The ML estimate of R given $N = n$ is the value of R that maximizes $P_{N|R}(n|r)$. That is,

$$\hat{R}_{\mathrm{ML}}(n) = \arg\max_{r\geq 0} \frac{(rT)^n e^{-rT}}{n!}. \tag{9}$$

Seting the derivative with respect to r to zero yields

$$\hat{R}_{\mathrm{ML}}(n) = n/T. \tag{10}$$

Problem 12.4.1 Solution

(a) Since $Y_1 = X + N_1$, we see that

$$D_1 = Y_1 - X = (X + N_1) - X = N_1. \tag{1}$$

Thus $\mathrm{E}[D_1] = \mathrm{E}[N_1] = 0$ and $\mathrm{E}[D_1^2] = \mathrm{E}[N_1^2]$. Since $\mathrm{E}[N_1] = 0$, we know that $\mathrm{E}[N_1^2] = \mathrm{Var}[N_1] = 1$. That is, $\mathrm{E}[D_1^2] = 1$.

(b) Note that

$$\begin{aligned} Y_3 = \frac{Y_1}{2} + \frac{Y_2}{2} &= \frac{X + N_1}{2} + \frac{X + N_2}{2} \\ &= X + \frac{N_1}{2} + \frac{N_2}{2}. \end{aligned} \tag{2}$$

It follows that

$$D_3 = Y_3 - X = \frac{N_1}{2} + \frac{N_2}{2}. \tag{3}$$

Since N_1 and N_2 are independent Gaussian random variables, D_3 is Gaussian with expected value and variance

$$\mathrm{E}[D_3] = \frac{\mathrm{E}[N_1]}{2} + \frac{\mathrm{E}[N_2]}{2} = 0, \tag{4}$$

$$\mathrm{Var}[D_3] = \frac{\mathrm{Var}[N_1]}{4} + \frac{\mathrm{Var}[N_2]}{4} = \frac{1}{4} + \frac{4}{4} = \frac{5}{4}. \tag{5}$$

Since $\mathrm{E}[D_3] = 0$, D_3 has second moment $\mathrm{E}[D_3^2] = \mathrm{Var}[D_3^2] = 5/4$. In terms of expected squared error, the estimator Y_3 is worse than the estimator Y_1. Even though Y_3 gets to average two noisy observations Y_1 and Y_2, the large variance of N_2 makes Y_2 a lousy estimate. As a result, including Y_2 as part of the estimate Y_3 is worse than just using the estimate of Y_1 by itself.

(c) In this problem,

$$\begin{aligned} Y_4 &= aY_1 + (1-a)Y_2 \\ &= a(X + N_1) + (1-a)(X + N_2) \\ &= X + aN_1 + (1-a)N_2. \end{aligned} \tag{6}$$

This implies

$$D_4 = Y_4 - X = aN_1 + (1-a)N_2. \tag{7}$$

Thus the error D_4 is a linear combination of the errors N_1 and N_2. Since N_1 and N_2 are independent, $\mathrm{E}[D_4] = 0$ and

$$\begin{aligned} \mathrm{Var}[D_4] &= a^2 \mathrm{Var}[N_1] + (1-a)^2 \mathrm{Var}[N_2] \\ &= a^2 + 4(1-a)^2. \end{aligned} \tag{8}$$

Since $\mathrm{E}[D_4] = 0$, the second moment of the error is simply

$$\mathrm{E}\left[D_4^2\right] = \mathrm{Var}[D_4] = a^2 + 4(1-a)^2. \tag{9}$$

Since $\mathrm{E}[D_4^2]$ is a quadratic function in a, we can choose a to minimize the error. In this case, taking the derivative with respect to a and setting it equal to zero yields $2a - 8(1-a) = 0$, implying $a = 0.8$. Although the problem does not request this, it is interesting to note that for $a = 0.8$, the expected squared error is $\mathrm{E}[D_4^2] = 0.80$, which is significantly less than the error obtained by using either just Y_1 or an average of Y_1 and Y_2.

Problem 12.4.3 Solution

From the problem statement, we learn for vectors $\mathbf{X} = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}'$ and $\mathbf{Y} = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}'$ that

$$\mathrm{E}[\mathbf{X}] = \mathbf{0}, \qquad \mathbf{R_X} = \begin{bmatrix} 1 & 3/4 & 1/2 \\ 3/4 & 1 & 3/4 \\ 1/2 & 3/4 & 1 \end{bmatrix}, \tag{1}$$

and

$$\mathbf{Y} = \mathbf{A}\mathbf{X} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{X}. \tag{2}$$

(a) Since $\mathrm{E}[\mathbf{Y}] = \mathbf{A}\,\mathrm{E}[\mathbf{X}] = \mathbf{0}$, we can apply Theorem 12.6 which states that the minimum mean square error estimate of X_1 is $\hat{X}_1(\mathbf{Y}) = \mathbf{R}_{X_1\mathbf{Y}}\mathbf{R}_{\mathbf{Y}}^{-1}\mathbf{Y}$ where $\hat{\mathbf{a}} =$. The rest of the solution is just calculation. (We note that even in the case of a 3×3 matrix, its convenient to use MATLAB with `format rat` mode to perform the calculations and display the results as nice fractions.) From Theorem 8.8,

$$\begin{aligned} \mathbf{R}_{\mathbf{Y}} = \mathbf{A}\mathbf{R}_{\mathbf{X}}\mathbf{A}' &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3/4 & 1/2 \\ 3/4 & 1 & 3/4 \\ 1/2 & 3/4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 7/2 & 3 \\ 3 & 7/2 \end{bmatrix}. \end{aligned} \tag{3}$$

In addition, since $\mathbf{R}_{X_1\mathbf{Y}} = \mathrm{E}[X_1\mathbf{Y}'] = \mathrm{E}[X_1\mathbf{X}'\mathbf{A}'] = \mathrm{E}[X_1\mathbf{X}']\mathbf{A}'$,

$$\begin{aligned} \mathbf{R}_{X_1\mathbf{Y}} &= \begin{bmatrix} \mathrm{E}\left[X_1^2\right] & \mathrm{E}\left[X_1X_2\right] & \mathrm{E}\left[X_1X_3\right] \end{bmatrix} \mathbf{A}' \\ &= \begin{bmatrix} R_{\mathbf{X}}(1,1) & R_{\mathbf{X}}(2,1) & R_{\mathbf{X}}(3,1) \end{bmatrix} \mathbf{A}' \\ &= \begin{bmatrix} 1 & 3/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7/4 & 5/4 \end{bmatrix}. \end{aligned} \tag{4}$$

Finally,

$$\begin{aligned} \mathbf{R}_{X_1\mathbf{Y}}\mathbf{R}_{\mathbf{Y}}^{-1} &= \begin{bmatrix} 7/4 & 5/4 \end{bmatrix} \begin{bmatrix} 14/13 & -12/13 \\ -12/13 & 14/13 \end{bmatrix} \\ &= \begin{bmatrix} 19/26 & -7/26 \end{bmatrix}. \end{aligned} \tag{5}$$

Thus the linear MMSE estimator of X_1 given $\mathbf{Y}$ is

$$\begin{aligned} \hat{X}_1(\mathbf{Y}) = \mathbf{R}_{X_1\mathbf{Y}}\mathbf{R}_{\mathbf{Y}}^{-1}\mathbf{Y} &= \frac{19}{26}Y_1 - \frac{7}{26}Y_2 \\ &= 0.7308Y_1 - 0.2692Y_2. \end{aligned} \tag{6}$$

(b) By Theorem 12.6(b), the mean squared error of the optimal estimator is

$$e_L^* = \mathrm{Var}[X_1] - \hat{\mathbf{a}}'\mathbf{R}_{\mathbf{Y}X_1}$$
$$= R_{\mathbf{X}}(1,1) - \mathbf{R}'_{\mathbf{Y}X_1}\mathbf{R}_{\mathbf{Y}}^{-1}\mathbf{R}_{\mathbf{Y}X_1}$$
$$= 1 - \begin{bmatrix} 7/4 & 5/4 \end{bmatrix} \begin{bmatrix} 14/13 & -12/13 \\ -12/13 & 14/13 \end{bmatrix} \begin{bmatrix} 7/4 \\ 5/4 \end{bmatrix} = \frac{3}{52}. \tag{7}$$

(c) We can estimate random variable X_1 based on the observation of random variable Y_1 using Theorem 12.3. Note that Theorem 12.3 is just a special case of Theorem 12.7 in which the observation is a random vector. In any case, from Theorem 12.3, the optimum linear estimate is $\hat{X}_1(Y_1) = a^*Y_1 + b^*$ where

$$a^* = \frac{\mathrm{Cov}\left[X_1, Y_1\right]}{\mathrm{Var}[Y_1]}, \qquad b^* = \mu_{X_1} - a^*\mu_{Y_1}. \tag{8}$$

Since $Y_1 = X_1 + X_2$, we see that

$$\mu_{X_1} = \mathrm{E}\left[X_1\right] = 0, \tag{9}$$
$$\mu_{Y_1} = \mathrm{E}\left[Y_1\right] = \mathrm{E}\left[X_1\right] + \mathrm{E}\left[X_2\right] = 0. \tag{10}$$

These facts, along with $\mathbf{R}_{\mathbf{X}}$ and $\mathbf{R}_{\mathbf{Y}}$ from part (a), imply

$$\mathrm{Cov}\left[X_1, Y_1\right] = \mathrm{E}\left[X_1 Y_1\right]$$
$$= \mathrm{E}\left[X_1(X_1 + X_2)\right]$$
$$= R_{\mathbf{X}}(1,1) + R_{\mathbf{X}}(1,2) = 7/4, \tag{11}$$
$$\mathrm{Var}[Y_1] = \mathrm{E}\left[Y_1^2\right] = R_{\mathbf{Y}}(1,1) = 7/2 \tag{12}$$

Thus

$$a^* = \frac{\mathrm{Cov}\left[X_1, Y_1\right]}{\mathrm{Var}[Y_1]} = \frac{7/4}{7/2} = \frac{1}{2}, \tag{13}$$
$$b^* = \mu_{X_1} - a^*\mu_{Y_1} = 0. \tag{14}$$

Thus the optimum linear estimate of X_1 given Y_1 is

$$\hat{X}_1(Y_1) = \frac{1}{2} Y_1. \tag{15}$$

From Theorem 12.3(a), the mean square error of this estimator is

$$e_L^* = \sigma_{X_1}^2 (1 - \rho_{X_1,Y_1}^2). \tag{16}$$

Since X_1 and Y_1 have zero expected value, $\sigma_{X_1}^2 = R_{\mathbf{X}}(1,1) = 1$ and $\sigma_{Y_1}^2 = R_{\mathbf{Y}}(1,1) = 7/2$. Also, since $\text{Cov}[X_1, Y_1] = 7/4$, we see that

$$\rho_{X_1,Y_1} = \frac{\text{Cov}\,[X_1, Y_1]}{\sigma_{X_1}\sigma_{Y_1}} = \frac{7/4}{\sqrt{7/2}} = \sqrt{\frac{7}{8}}. \tag{17}$$

Thus $e_L^* = 1 - (\sqrt{7/8})^2 = 1/8$. Note that $1/8 > 3/52$. As we would expect, the estimate of X_1 based on just Y_1 has larger mean square error than the estimate based on both Y_1 and Y_2.

Problem 12.4.5 Solution

The key to this problem is to write $\mathbf{Y}$ in terms of $\mathbf{Q}$. First we observe that

$$\begin{aligned} Y_1 &= q_0 + 1q_1 + 1^2 q_2 + Z_1, && (1)\\ Y_2 &= q_0 + 2q_1 + 2^2 q_2 + Z_2, && (2)\\ &\vdots \quad \vdots \\ Y_n &= q_0 + nq_1 + n^2 q_2 + Z_n. && (3) \end{aligned}$$

In terms of the vector $\mathbf{Q}$, we can write

$$\mathbf{Y} = \underbrace{\begin{bmatrix} 1 & 1 & 1^2 \\ 1 & 2 & 2^2 \\ \vdots & \vdots & \vdots \\ 1 & n & n^2 \end{bmatrix}}_{\mathbf{K}_n} \mathbf{Q} + \mathbf{Z} = \mathbf{K}_n \mathbf{Q} + \mathbf{Z}. \tag{4}$$

From the problem statement we know that $\mathrm{E}[\mathbf{Q}] = \mathbf{0}$, $\mathrm{E}[\mathbf{Z}] = \mathbf{0}$, $\mathbf{R_Q} = \mathbf{I}$, and $\mathbf{R_Z} = \mathbf{I}$. Applying Theorem 12.8 as expressed in Equation (12.77), we obtain

$$\hat{\mathbf{Q}}_L(\mathbf{Y}) = \mathbf{R_{QY}}\mathbf{R_Y}^{-1}\mathbf{Y}. \tag{5}$$

Since $\mathbf{Q}$ and the noise $\mathbf{Z}$ are independent,

$$\mathrm{E}\left[\mathbf{Q}\mathbf{Z}'\right] = \mathrm{E}\left[\mathbf{Q}\right]\mathrm{E}\left[\mathbf{Z}'\right] = \mathbf{0}. \tag{6}$$

This implies

$$\begin{aligned}
\mathbf{R_{QY}} &= \mathrm{E}\left[\mathbf{Q}\mathbf{Y}'\right] \\
&= \mathrm{E}\left[\mathbf{Q}(\mathbf{K}_n\mathbf{Q} + \mathbf{Z})'\right] \\
&= \mathrm{E}\left[\mathbf{Q}\mathbf{Q}'\mathbf{K}_n' + \mathbf{Q}\mathbf{Z}'\right] = \mathbf{R_Q}\mathbf{K}_n'.
\end{aligned} \tag{7}$$

Again using (6), we have that

$$\begin{aligned}
\mathbf{R_Y} &= \mathrm{E}\left[\mathbf{Y}\mathbf{Y}'\right] \\
&= \mathrm{E}\left[(\mathbf{K}_n\mathbf{Q} + \mathbf{Z})(\mathbf{K}_n\mathbf{Q} + \mathbf{Z})'\right] \\
&= \mathrm{E}\left[(\mathbf{K}_n\mathbf{Q} + \mathbf{Z})(\mathbf{Q}'\mathbf{K}_n' + \mathbf{Z}')\right] \\
&= \mathbf{K}_n\mathrm{E}\left[\mathbf{Q}\mathbf{Q}'\right]\mathbf{K}_n' + \mathbf{K}_n\mathrm{E}\left[\mathbf{Q}\mathbf{Z}'\right] + \mathrm{E}\left[\mathbf{Z}\mathbf{Q}'\right]\mathbf{K}_n' + \mathrm{E}\left[\mathbf{Z}\mathbf{Z}'\right] \\
&= \mathbf{K}_n\mathbf{K}_n' + \mathbf{I}.
\end{aligned} \tag{8}$$

It follows that

$$\hat{\mathbf{Q}} = \mathbf{R_{QY}}\mathbf{R_Y}^{-1}\mathbf{Y} = \mathbf{K}_n'(\mathbf{K}_n\mathbf{K}_n' + \mathbf{I})^{-1}\mathbf{Y}. \tag{9}$$

Problem 12.4.7 Solution

From Theorem 12.6, we know that the minimum mean square error estimate of X given $\mathbf{Y}$ is $\hat{X}_L(\mathbf{Y}) = \hat{\mathbf{a}}'\mathbf{Y}$, where $\hat{\mathbf{a}} = \mathbf{R_Y}^{-1}\mathbf{R}_{\mathbf{Y}X}$. In this problem, $\mathbf{Y}$ is simply a scalar Y and $\hat{\mathbf{a}}$ is a scalar $\hat{a}$. Since $\mathrm{E}[Y] = 0$,

$$\mathbf{R_Y} = \mathrm{E}\left[\mathbf{Y}\mathbf{Y}'\right] = \mathrm{E}\left[Y^2\right] = \sigma_Y^2. \tag{1}$$

Similarly,

$$\mathbf{R}_{\mathbf{Y}X} = \mathrm{E}\left[\mathbf{Y}X\right] = \mathrm{E}\left[YX\right] = \mathrm{Cov}\left[X,Y\right]. \tag{2}$$

It follows that

$$\hat{a} = \mathbf{R}_{\mathbf{Y}}^{-1}\mathbf{R}_{\mathbf{Y}X} = \left(\sigma_Y^2\right)^{-1}\mathrm{Cov}\left[X,Y\right] = \frac{\sigma_X}{\sigma_Y}\frac{\mathrm{Cov}\left[X,Y\right]}{\sigma_X\sigma_Y} = \frac{\sigma_X}{\sigma_Y}\rho_{X,Y}. \tag{3}$$

Problem 12.4.9 Solution

(a) In this case, we use the observation $\mathbf{Y}$ to estimate each X_i. Since $\mathrm{E}[X_i] = 0$,

$$\mathrm{E}\left[\mathbf{Y}\right] = \sum_{j=1}^{k}\mathrm{E}\left[X_j\right]\sqrt{p_j}\mathbf{S}_j + \mathrm{E}\left[\mathbf{N}\right] = \mathbf{0}. \tag{1}$$

Thus, Theorem 12.6 tells us that the MMSE linear estimate of X_i is $\hat{X}_i(\mathbf{Y}) = \mathbf{R}_{X_i\mathbf{Y}}\mathbf{R}_{\mathbf{Y}}^{-1}\mathbf{Y}$. First we note that

$$\mathbf{R}_{X_i\mathbf{Y}} = \mathrm{E}\left[X_i\mathbf{Y}'\right] = \mathrm{E}\left[X_i\left(\sum_{j=1}^{k}X_j\sqrt{p_j}\mathbf{S}'_j + \mathbf{N}'\right)\right] \tag{2}$$

Since $\mathbf{N}$ and X_i are independent, $\mathrm{E}[X_i\mathbf{N}'] = \mathrm{E}[X_i]\,\mathrm{E}[\mathbf{N}'] = \mathbf{0}$. Because X_i and X_j are independent for $i \neq j$, $\mathrm{E}[X_iX_j] = \mathrm{E}[X_i]\,\mathrm{E}[X_j] = 0$ for $i \neq j$. In addition, $\mathrm{E}[X_i^2] = 1$, and it follows that

$$\mathbf{R}_{X_i\mathbf{Y}} = \sum_{j=1}^{k}\mathrm{E}\left[X_iX_j\right]\sqrt{p_j}\mathbf{S}'_j + \mathrm{E}\left[X_i\mathbf{N}'\right] = \sqrt{p_i}\mathbf{S}'_i. \tag{3}$$

For the same reasons,

$$\begin{aligned}
\mathbf{R}_{\mathbf{Y}} &= \mathrm{E}\left[\mathbf{Y}\mathbf{Y}'\right] \\
&= \mathrm{E}\left[\left(\sum_{j=1}^{k}\sqrt{p_j}X_j\mathbf{S}_j + \mathbf{N}\right)\left(\sum_{l=1}^{k}\sqrt{p_l}X_l\mathbf{S}'_l + \mathbf{N}'\right)\right] \\
&= \sum_{j=1}^{k}\sum_{l=1}^{k}\sqrt{p_jp_l}\,\mathrm{E}\left[X_jX_l\right]\mathbf{S}_j\mathbf{S}'_l \\
&\quad + \sum_{j=1}^{k}\sqrt{p_j}\underbrace{\mathrm{E}\left[X_j\mathbf{N}\right]}_{=0}\mathbf{S}_j + \sum_{l=1}^{k}\sqrt{p_l}\underbrace{\mathrm{E}\left[X_l\mathbf{N}'\right]}_{=0}\mathbf{S}'_l + \mathrm{E}\left[\mathbf{N}\mathbf{N}'\right] \\
&= \sum_{j=1}^{k}p_j\mathbf{S}_j\mathbf{S}'_j + \sigma^2\mathbf{I}. \qquad (4)
\end{aligned}$$

Now we use a linear algebra identity. For a matrix $\mathbf{S}$ with columns $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_k$, and a diagonal matrix $\mathbf{P} = \mathrm{diag}[p_1, p_2, \ldots, p_k]$,

$$\sum_{j=1}^{k}p_j\mathbf{S}_j\mathbf{S}'_j = \mathbf{S}\mathbf{P}\mathbf{S}'. \qquad (5)$$

Although this identity may be unfamiliar, it is handy in manipulating correlation matrices. (Also, if this is unfamiliar, you may wish to work out an example with $k = 2$ vectors of length 2 or 3.) Thus,

$$\mathbf{R}_{\mathbf{Y}} = \mathbf{S}\mathbf{P}\mathbf{S}' + \sigma^2\mathbf{I}, \qquad (6)$$

and

$$\mathbf{R}_{X_i\mathbf{Y}}\mathbf{R}_{\mathbf{Y}}^{-1} = \sqrt{p_i}\mathbf{S}'_i\left(\mathbf{S}\mathbf{P}\mathbf{S}' + \sigma^2\mathbf{I}\right)^{-1}. \qquad (7)$$

Recall that if $\mathbf{C}$ is symmetric, then $\mathbf{C}^{-1}$ is also symmetric. This implies the MMSE estimate of X_i given $\mathbf{Y}$ is

$$\hat{X}_i(\mathbf{Y}) = \mathbf{R}_{X_i\mathbf{Y}}\mathbf{R}_{\mathbf{Y}}^{-1}\mathbf{Y} = \sqrt{p_i}\mathbf{S}'_i\left(\mathbf{S}\mathbf{P}\mathbf{S}' + \sigma^2\mathbf{I}\right)^{-1}\mathbf{Y}. \qquad (8)$$

(b) We observe that $\mathbf{V} = (\mathbf{SPS}' + \sigma^2\mathbf{I})^{-1}\mathbf{Y}$ is a vector that does not depend on which bit X_i that we want to estimate. Since $\hat{X}_i = \sqrt{p_i}\mathbf{S'}_i\mathbf{V}$, we can form the vector of estimates

$$\hat{\mathbf{X}} = \begin{bmatrix} \hat{X}_1 \\ \vdots \\ \hat{X}_k \end{bmatrix} = \begin{bmatrix} \sqrt{p_1}\mathbf{S'}_1\mathbf{V} \\ \vdots \\ \sqrt{p_k}\mathbf{S'}_k\mathbf{V} \end{bmatrix} = \begin{bmatrix} \sqrt{p_1} & & \\ & \ddots & \\ & & \sqrt{p_k} \end{bmatrix} \begin{bmatrix} \mathbf{S'}_1 \\ \vdots \\ \mathbf{S'}_k \end{bmatrix} \mathbf{V}$$

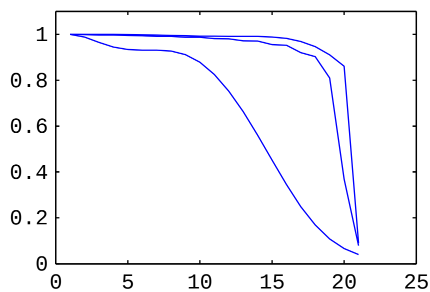
$$= \mathbf{P}^{1/2}\mathbf{S}'\mathbf{V}$$

$$= \mathbf{P}^{1/2}\mathbf{S}'\left(\mathbf{SPS}' + \sigma^2\mathbf{I}\right)^{-1}\mathbf{Y}. \tag{9}$$

Problem 12.5.1 Solution

This problem can be solved using the function `mse` defined in Example 12.10. All we need to do is define the correlation structure of the vector $\mathbf{X} = \begin{bmatrix} X_1 & \cdots & X_{21} \end{bmatrix}'$. Just as in Example 12.10, we do this by defining just the first row of the correlation matrix. Here are the commands we need, and the resulting plot.

```
r1=sinc(0.1*(0:20)); mse(r1);
hold on;
r5=sinc(0.5*(0:20)); mse(r5);
r9=sinc(0.9*(0:20)); mse(r9);
```



Although the plot lacks labels, there are three curves for the mean square error MSE(n) corresponding to $\phi_0 \in \{0.1, 0.5, 0.9\}$. Keep in mind that MSE($n$) is the MSE of the linear estimate of X_{21} using random variables $X_1, \ldots, X_n$.

If you run the commands, you'll find that the $\phi_0 = 0.1$ yields the lowest mean square error while $\phi_0 = 0.9$ results in the highest mean square error. When $\phi_0 = 0.1$, random variables X_n for $n = 10, 11, \ldots, 20$ are increasingly correlated with X_{21}. The result is that the MSE starts to decline rapidly for $n > 10$. As ϕ_0 increases, fewer observations X_n are correlated with X_{21}.

The result is the MSE is simply worse as ϕ_0 increases. For example, when $\phi_0 = 0.9$, even X_{20} has only a small correlation with X_{21}. We only get a good estimate of X_{21} at time $n = 21$ when we observe $X_{21} + W_{21}$.

Problem 12.5.3 Solution

The solution to this problem is almost the same as the solution to Example 12.10, except perhaps the MATLAB code is somewhat simpler. As in the example, let $\mathbf{W}^{(n)}$, $\mathbf{X}^{(n)}$, and $\mathbf{Y}^{(n)}$ denote the vectors, consisting of the first n components of $\mathbf{W}$, $\mathbf{X}$, and $\mathbf{Y}$. Just as in Examples 12.8 and 12.10, independence of $\mathbf{X}^{(n)}$ and $\mathbf{W}^{(n)}$ implies that the correlation matrix of $\mathbf{Y}^{(n)}$ is

$$\mathbf{R}_{\mathbf{Y}^{(n)}} = \mathrm{E}\left[(\mathbf{X}^{(n)} + \mathbf{W}^{(n)})(\mathbf{X}^{(n)} + \mathbf{W}^{(n)})'\right] = \mathbf{R}_{\mathbf{X}^{(n)}} + \mathbf{R}_{\mathbf{W}^{(n)}} \tag{1}$$

Note that $\mathbf{R}_{\mathbf{X}^{(n)}}$ and $\mathbf{R}_{\mathbf{W}^{(n)}}$ are the $n \times n$ upper-left submatrices of $\mathbf{R}_{\mathbf{X}}$ and $\mathbf{R}_{\mathbf{W}}$. In addition,

$$\mathbf{R}_{\mathbf{Y}^{(n)}X} = \mathrm{E}\left[\begin{bmatrix} X_1 + W_1 \\ \vdots \\ X_n + W_n \end{bmatrix} X_1\right] = \begin{bmatrix} r_0 \\ \vdots \\ r_{n-1} \end{bmatrix}. \tag{2}$$

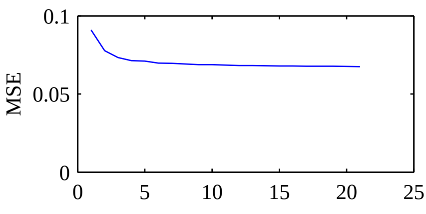
Compared to the solution of Example 12.10, the only difference in the solution is in the reversal of the vector $\mathbf{R}_{\mathbf{Y}^{(n)}X}$. The optimal filter based on the first n observations is $\hat{\mathbf{a}}^{(n)} = \mathbf{R}_{\mathbf{Y}^{(n)}}^{-1}\mathbf{R}_{\mathbf{Y}^{(n)}X}$, and the mean square error is

$$e_L^* = \mathrm{Var}[X_1] - (\hat{\mathbf{a}}^{(n)})'\mathbf{R}_{\mathbf{Y}^{(n)}X}. \tag{3}$$

```
function e=mse953(r)
N=length(r);
e=[];
for n=1:N,
    RYX=r(1:n)';
    RY=toeplitz(r(1:n))+0.1*eye(n);
    a=RY\RYX;
    en=r(1)-(a')*RYX;
    e=[e;en];
end
plot(1:N,e);
```

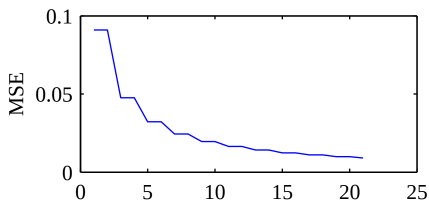
The program `mse953.m` simply calculates the mean square error e_L^*. The input is the vector `r` corresponding to the vector $\begin{bmatrix} r_0 & \cdots & r_{20} \end{bmatrix}$, which holds the first row of the Toeplitz correlation matrix $\mathbf{R_X}$. Note that $\mathbf{R}_{\mathbf{X}^{(n)}}$ is the Toeplitz matrix whose first row is the first n elements of `r`.

To plot the mean square error as a function of the number of observations, n, we generate the vector `r` and then run `mse953(r)`. For the requested cases (a) and (b), the necessary MATLAB commands and corresponding mean square estimation error output as a function of n are shown here:




```
ra=sinc(0.1*pi*(0:20));
mse953(ra)
```

(a)




```
rb=cos(0.5*pi*(0:20));
mse953(rb)
```

(b)

In comparing the results of cases (a) and (b), we see that the mean square estimation error depends strongly on the correlation structure given by $r_{|i-j|}$. For case (a), Y_1 is a noisy observation of X_1 and is highly correlated with X_1. The MSE at $n = 1$ is just the variance of W_1. Additional samples of Y_n mostly help to average the additive noise. Also, samples X_n for $n \geq 10$ have very little correlation with X_1. Thus for $n \geq 10$, the samples of Y_n result in almost no improvement in the estimate of X_1.

In case (b), $Y_1 = X_1 + W_1$, just as in case (a), is simply a noisy copy of X_1 and the estimation error is due to the variance of W_1. On the other hand, for case (b), X_5, X_9, X_{13} and X_{17} and X_{21} are completely correlated with X_1. Other samples also have significant correlation with X_1. As a result, the MSE continues to go down with increasing n.

Problem Solutions – Chapter 13

Problem 13.1.1 Solution

There are many correct answers to this question. A correct answer specifies enough random variables to specify the sample path exactly. One choice for an alternate set of random variables that would specify $m(t,s)$ is

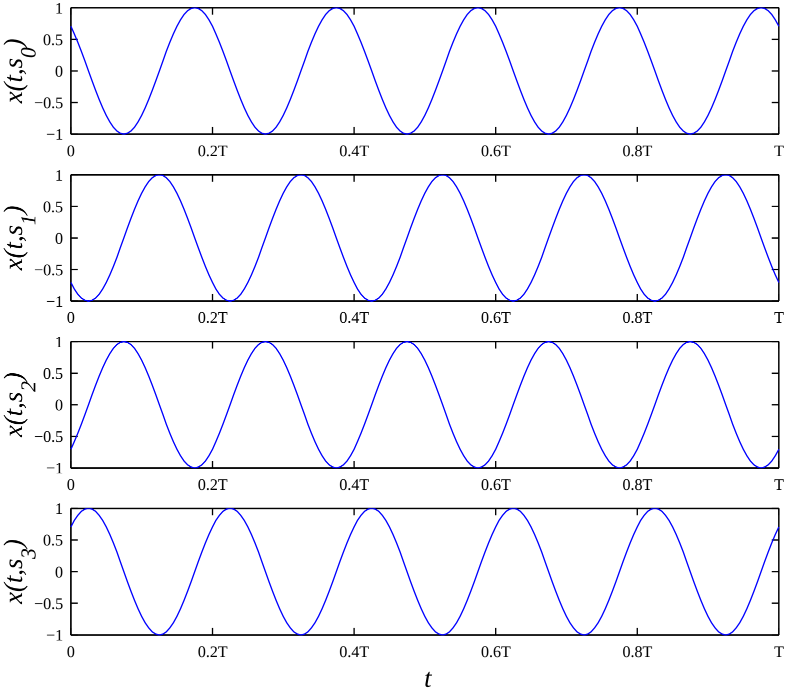
- $m(0,s)$, the number of ongoing calls at the start of the experiment
- N, the number of new calls that arrive during the experiment
- $X_1,\ldots,X_N$, the interarrival times of the N new arrivals
- H, the number of calls that hang up during the experiment
- $D_1,\ldots,D_H$, the call completion times of the H calls that hang up

Problem 13.1.3 Solution

The sample space of the underlying experiment is $S=\{s_0,s_1,s_2,s_3\}$. The four elements in the sample space are equally likely. The ensemble of sample functions is $\{x(t,s_i)|i=0,1,2,3\}$ where

$$x(t,s_i)=\cos(2\pi f_0 t+\pi/4+i\pi/2),\qquad 0\le t\le T. \tag{1}$$

For $f_0=5/T$, this ensemble is shown below.



Problem 13.1.5 Solution

The statement is *false*. As a counterexample, consider the rectified cosine waveform $X(t) = R|\cos 2\pi ft|$ of Example 13.9. When $t = \pi/2$, then $\cos 2\pi ft = 0$ so that $X(\pi/2) = 0$. Hence $X(\pi/2)$ has PDF

$$f_{X(\pi/2)}(x) = \delta(x). \tag{1}$$

That is, $X(\pi/2)$ is a discrete random variable.

Problem 13.2.1 Solution

In this problem, we start from first principles. What makes this problem fairly straightforward is that the ramp is defined for all time. That is, the ramp doesn't start at time $t = W$. Thus,

$$P[X(t) \leq x] = P[t - W \leq x] = P[W \geq t - x]. \tag{1}$$

Since $W \geq 0$, if $x \geq t$ then $\mathrm{P}[W \geq t - x] = 1$. When $x < t$,

$$\mathrm{P}\left[W \geq t - x\right] = \int_{t-x}^{\infty} f_W(w)\, dw = e^{-(t-x)}. \tag{2}$$

Combining these facts, we have

$$F_{X(t)}(x) = \mathrm{P}\left[W \geq t - x\right] = \begin{cases} e^{-(t-x)} & x < t, \\ 1 & t \leq x. \end{cases} \tag{3}$$

We note that the CDF contain no discontinuities. Taking the derivative of the CDF $F_{X(t)}(x)$ with respect to x, we obtain the PDF

$$f_{X(t)}(x) = \begin{cases} e^{x-t} & x < t, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Problem 13.2.3 Solution

Once we find the first one part in 10^4 oscillator, the number of additional tests needed to find the next one part in 10^4 oscillator once again has a geometric PMF with mean $1/p$ since each independent trial is a success with probability p. That is $T_2 = T_1 + T'$ where T' is independent and identically distributed to T_1. Thus,

$$\begin{aligned} \mathrm{E}\left[T_2|T_1 = 3\right] &= \mathrm{E}\left[T_1|T_1 = 3\right] + \mathrm{E}\left[T'|T_1 = 3\right] \\ &= 3 + \mathrm{E}\left[T'\right] = 23 \text{ minutes.} \end{aligned} \tag{1}$$

Problem 13.3.1 Solution

Each Y_k is the sum of two identical independent Gaussian random variables. Hence, each Y_k must have the same PDF. That is, the Y_k are identically distributed. Next, we observe that the sequence of Y_k is independent. To see this, we observe that each Y_k is composed of two samples of X_k that are unused by any other Y_j for $j \neq k$.

Problem 13.3.3 Solution

The number Y_k of failures between successes $k-1$ and k is exactly $y \geq 0$ iff after success $k-1$, there are y failures followed by a success. Since the Bernoulli trials are independent, the probability of this event is $(1-p)^y p$. The complete PMF of Y_k is

$$P_{Y_k}(y) = \begin{cases} (1-p)^y p & y = 0, 1, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Since this argument is valid for all k including $k = 1$, we can conclude that $Y_1, Y_2, \ldots$ are identically distributed. Moreover, since the trials are independent, the failures between successes $k-1$ and k and the number of failures between successes $k'-1$ and k' are independent. Hence, $Y_1, Y_2, \ldots$ is an iid sequence.

Problem 13.4.1 Solution

This is a very straightforward problem. The Poisson process has rate $\lambda = 4$ calls per second. When t is measured in seconds, each $N(t)$ is a Poisson random variable with mean $4t$ and thus has PMF

$$P_{N(t)}(n) = \begin{cases} \frac{(4t)^n}{n!} e^{-4t} & n = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Using the general expression for the PMF, we can write down the answer for each part.

(a) $P_{N(1)}(0) = 4^0 e^{-4}/0! = e^{-4} \approx 0.0183$.

(b) $P_{N(1)}(4) = 4^4 e^{-4}/4! = 32e^{-4}/3 \approx 0.1954$.

(c) $P_{N(2)}(2) = 8^2 e^{-8}/2! = 32e^{-8} \approx 0.0107$.

Problem 13.4.3 Solution

Since there is always a backlog an the service times are iid exponential random variables, The time between service completions are a sequence of iid exponential random variables. that is, the service completions are a Poisson process. Since the expected service time is 30 minutes, the rate of the Poisson process is $\lambda = 1/30$ per minute. Since t hours equals $60t$ minutes, the expected number serviced is $\lambda(60t)$ or $2t$. Moreover, the number serviced in the first t hours has the Poisson PMF

$$P_{N(t)}(n) = \begin{cases} \frac{(2t)^n e^{-2t}}{n!} & n = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Problem 13.4.5 Solution

Note that it matters whether $t \geq 2$ minutes. If $t \leq 2$, then any customers that have arrived must still be in service. Since a Poisson number of arrivals occur during $(0, t]$,

$$P_{N(t)}(n) = \begin{cases} (\lambda t)^n e^{-\lambda t}/n! & n = 0, 1, 2, \ldots, \\ 0 & \text{otherwise,} \end{cases} \qquad (0 \leq t \leq 2.) \tag{1}$$

For $t \geq 2$, the customers in service are precisely those customers that arrived in the interval $(t-2, t]$. The number of such customers has a Poisson PMF with mean $\lambda[t - (t-2)] = 2\lambda$. The resulting PMF of $N(t)$ is

$$P_{N(t)}(n) = \begin{cases} (2\lambda)^n e^{-2\lambda}/n! & n = 0, 1, 2, \ldots, \\ 0 & \text{otherwise,} \end{cases} \qquad (t \geq 2.) \tag{2}$$

Problem 13.4.7 Solution

(a) N_τ is a Poisson ($\alpha = 10\tau$) random variable. You should know that $\mathrm{E}[N_\tau] = 10\tau$. Thus $\mathrm{E}[N_{60}] = 10 \cdot 60 = 600$.

(b) In a $\tau = 10$ minute interval N_{10} hamburgers are sold. Thus,

$$P[N_{10} = 0] = P_{N_{10}}(0) = (100)^0 e^{-100}/0! = e^{-100}. \tag{1}$$

(c) Let t denote the time 12 noon. In this case, for $w > 0$, $W > w$ if and only if no burgers are sold in the time interval $[t, t+w]$. That is,

$$\begin{aligned} P[W > w] &= P[\text{No burgers are sold in } [t, t+w]] \\ &= P[N_w = 0] \\ &= P_{N_w}(0) = (10w)^0 e^{-10w}/0! = e^{-10w}. \end{aligned} \tag{2}$$

For $w > 0$, $F_W(w) = 1 - P[W > w] = 1 - e^{-10w}$. That is, the CDF of W is

$$F_W(w) = \begin{cases} 0 & w < 0, \\ 1 - e^{-10w} & w \geq 0. \end{cases} \tag{3}$$

Taking a derivative, we have

$$f_W(w) = \begin{cases} 0 & w < 0, \\ 10e^{-10w} & w \geq 0. \end{cases} \tag{4}$$

We see that W is an exponential $\lambda = 10$ random variable.

Problem 13.4.9 Solution

This proof is just a simplified version of the proof given for Theorem 13.3. The first arrival occurs at time $X_1 > x \geq 0$ iff there are no arrivals in the interval $(0, x]$. Hence, for $x \geq 0$,

$$P[X_1 > x] = P[N(x) = 0] = (\lambda x)^0 e^{-\lambda x}/0! = e^{-\lambda x}. \tag{1}$$

Since $P[X_1 \leq x] = 0$ for $x < 0$, the CDF of X_1 is the exponential CDF

$$F_{X_1}(x) = \begin{cases} 0 & x < 0, \\ 1 - e^{-\lambda x} & x \geq 0. \end{cases} \tag{2}$$

Problem 13.5.1 Solution

Customers entering (or not entering) the casino is a Bernoulli decomposition of the Poisson process of arrivals at the casino doors. By Theorem 13.6, customers entering the casino are a Poisson process of rate $100/2 = 50$ customers/hour. Thus in the two hours from 5 to 7 PM, the number, N, of customers entering the casino is a Poisson random variable with expected value $\alpha = 2 \cdot 50 = 100$. The PMF of N is

$$P_N(n) = \begin{cases} 100^n e^{-100}/n! & n = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Problem 13.5.3 Solution

(a) The trains (red and blue together) arrive as a Poisson process of rate $\lambda_R + \lambda_B = 0.45$ trains per minute. In one hour, the number of trains that arrive N is a Poisson ($\alpha = 27$) random variable. The PMF is

$$P_N(=) \begin{cases} 27^n e^{-27}/n! & n = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(b) Each train that arrives is a red train with probability $p = \lambda_R/(\lambda_R + \lambda_B) = 1/3$. Given that $N = 30$ trains arrive, R is conditionally a binomial $(30, p)$ random variable. The conditional PMF is

$$P_{R|N}(r|30) = \binom{30}{r} p^r (1-p)^{30-r}. \tag{2}$$

Problem 13.5.5 Solution

In an interval $(t, t + \Delta]$ with an infinitesimal Δ, let A_i denote the event of an arrival of the process $N_i(t)$. Also, let $A = A_1 \cup A_2$ denote the event of an arrival of either process. Since $N_i(t)$ is a Poisson process, the alternative model says that $\mathrm{P}[A_i] = \lambda_i \Delta$. Also, since $N_1(t) + N_2(t)$ is a Poisson process, the proposed Poisson process model says

$$\mathrm{P}[A] = (\lambda_1 + \lambda_2)\Delta. \tag{1}$$

Lastly, the conditional probability of a type 1 arrival given an arrival of either type is

$$\mathrm{P}[A_1|A] = \frac{\mathrm{P}[A_1 A]}{\mathrm{P}[A]} = \frac{\mathrm{P}[A_1]}{\mathrm{P}[A]} = \frac{\lambda_1 \Delta}{(\lambda_1 + \lambda_2)\Delta} = \frac{\lambda_1}{\lambda_1 + \lambda_2}. \tag{2}$$

This solution is something of a cheat in that we have used the fact that the sum of Poisson processes is a Poisson process without using the proposed model to derive this fact.

Problem 13.5.7 Solution

(a) The last runner's finishing time is $L = \max(R_1, \ldots, R_{10})$ and

$$\begin{aligned}
\mathrm{P}[L \leq 20] &= \mathrm{P}[\max(R_1, \ldots, R_{10}) \leq 20] \\
&= \mathrm{P}[R_1 \leq 20, R_2 \leq 20, \ldots, R_{10} \leq 20] \\
&= \mathrm{P}[R_1 \leq 20]\,\mathrm{P}[R_2 \leq 20] \cdots \mathrm{P}[R_{10} \leq 20] \\
&= (\mathrm{P}[R_1 \leq 20])^{10} \\
&= \left(1 - e^{-20\mu}\right)^{10} = \left(1 - e^{-2}\right)^{10} \approx 0.234.
\end{aligned} \tag{1}$$

(b) At the start at time zero, we can view each runner as the first arrival of an independent Poisson process of rate μ. Thus, at time zero, the arrival of the first runner can be viewed as the first arrival of a process of rate 10μ. Hence, X_1 is exponential with expected value $1/(10\mu) = 1$ and has PDF

$$f_{X_1}(x_1) = \begin{cases} e^{-x_1} & x_1 \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(c) We can view Y as the 10th arrival of a Poisson process of rate μ. Thus Y has the Erlang $(n = 10, \mu)$ PDF

$$f_Y(y) = \begin{cases} \frac{\mu^{10} y^9 e^{-\mu y}}{9!} & y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

(d) We already found the PDF of X_1. We observe that after the first runner finishes, there are still 9 runners on the course. Because each runner's time is memoryless, each runner has a residual running time that is an exponential (μ) random variable. Because these residual running times are independent X_2 is exponential with expected value $1/(9\mu) = 1/0.9$ and has PDF

$$f_{X_2}(x_2) = \begin{cases} 9\mu e^{-9\mu x_2} & x_2 \geq 0, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} 0.9 e^{-0.9 x_2} & x_2 \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Similarly, for the ith arrival, there are $10 - i + 1 = 11 - i$ runners left on the course. The interarrival time for the ith arriving runner is the same as waiting for the first arrival of a Poisson process of rate $(11 - i)\mu$. Thus X_i has PDF

$$f_{X_i}(x_i) = \begin{cases} (11 - i)\mu e^{-(11-i)\mu x_i} & x_i \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Finally, we observe that the memoryless property of the runners' exponential running times ensures that the X_i are independent random variables. Hence,

$$\begin{aligned} f_{X_1,\ldots,X_{10}}&(x_1, \ldots, x_{10}) \\ &= f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_{10}}(x_{10}) \\ &= \begin{cases} 10! \mu^{10} e^{-\mu(10x_1 + 9x_2 + \cdots + 2x_9 + x_{10})} & x_i \geq 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{6}$$

Problem 13.6.1 Solution

From the problem statement, the change in the stock price is $X(8) - X(0)$ and the standard deviation of $X(8) - X(0)$ is 1/2 point. In other words, the variance of $X(8) - X(0)$ is $\text{Var}[X(8) - X(0)] = 1/4$. By the definition of Brownian motion. $\text{Var}[X(8) - X(0)] = 8\alpha$. Hence $\alpha = 1/32$.

Problem 13.6.3 Solution

We need to verify that $Y(t) = X(ct)$ satisfies the conditions given in Definition 13.10. First we observe that $Y(0) = X(c \cdot 0) = X(0) = 0$. Second, we note that since $X(t)$ is Brownian motion process implies that $Y(t) - Y(s) = X(ct) - X(cs)$ is a Gaussian random variable. Further, $X(ct) - X(cs)$ is independent of $X(t')$ for all $t' \leq cs$. Equivalently, we can say that $X(ct) - X(cs)$ is independent of $X(c\tau)$ for all $\tau \leq s$. In other words, $Y(t) - Y(s)$ is independent of $Y(\tau)$ for all $\tau \leq s$. Thus $Y(t)$ is a Brownian motion process.

Problem 13.6.5 Solution

Recall that the vector $\mathbf{X}$ of increments has independent components $X_n = W_n - W_{n-1}$. Alternatively, each W_n can be written as the sum

$$W_1 = X_1, \tag{1}$$

$$W_2 = X_1 + X_2, \tag{2}$$

$$\vdots$$

$$W_k = X_1 + X_2 + \cdots + X_k. \tag{3}$$

In terms of matrices, $\mathbf{W} = \mathbf{A}\mathbf{X}$ where $\mathbf{A}$ is the lower triangular matrix

$$\mathbf{A} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & \cdots & \cdots & 1 \end{bmatrix}. \tag{4}$$

Since $\text{E}[\mathbf{W}] = \mathbf{A}\,\text{E}[\mathbf{X}] = \mathbf{0}$, it folows from Theorem 8.11 that

$$f_{\mathbf{W}}(\mathbf{w}) = \frac{1}{|\det(\mathbf{A})|} f_{\mathbf{X}}\left(\mathbf{A}^{-1}\mathbf{w}\right). \tag{5}$$

Since $\mathbf{A}$ is a lower triangular matrix, $\det(\mathbf{A})$ is the product of its diagonal entries. In this case, $\det(\mathbf{A}) = 1$. In addition, reflecting the fact that each $X_n = W_n - W_{n-1}$,

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ 0 & -1 & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^{-1}\mathbf{W} = \begin{bmatrix} W_1 \\ W_2 - W_1 \\ W_3 - W_2 \\ \vdots \\ W_k - W_{k-1} \end{bmatrix}. \tag{6}$$

Combining these facts with the observation that $f_{\mathbf{X}}(\mathbf{x}) = \prod_{n=1}^{k} f_{X_n}(x_n)$, we can write

$$f_{\mathbf{W}}(\mathbf{w}) = f_{\mathbf{X}}\left(\mathbf{A}^{-1}\mathbf{w}\right) = \prod_{n=1}^{k} f_{X_n}(w_n - w_{n-1}), \tag{7}$$

which completes the missing steps in the proof of Theorem 13.8.

Problem 13.7.1 Solution

The discrete time autocovariance function is

$$C_X[m,k] = \mathrm{E}\left[(X_m - \mu_X)(X_{m+k} - \mu_X)\right]. \tag{1}$$

For $k = 0$, $C_X[m,0] = \mathrm{Var}[X_m] = \sigma_X^2$. For $k \neq 0$, X_m and X_{m+k} are independent so that

$$C_X[m,k] = \mathrm{E}\left[(X_m - \mu_X)\right]\mathrm{E}\left[(X_{m+k} - \mu_X)\right] = 0. \tag{2}$$

Thus the autocovariance of X_n is

$$C_X[m,k] = \begin{cases} \sigma_X^2 & k = 0, \\ 0 & k \neq 0. \end{cases} \tag{3}$$

Problem 13.7.3 Solution

In this problem, the daily temperature process results from

$$C_n = 16\left[1 - \cos\frac{2\pi n}{365}\right] + 4X_n, \tag{1}$$

where X_n is an iid random sequence of $N[0,1]$ random variables. The hardest part of this problem is distinguishing between the process C_n and the covariance function $C_C[k]$.

(a) The expected value of the process is

$$\begin{aligned} E[C_n] &= 16\,E\left[1 - \cos\frac{2\pi n}{365}\right] + 4\,E[X_n] \\ &= 16\left[1 - \cos\frac{2\pi n}{365}\right]. \end{aligned} \tag{2}$$

(b) Note that (1) and (2) imply

$$C_n - E[C_n] = 4X_n. \tag{3}$$

This implies that the autocovariance of C_n is

$$\begin{aligned} C_C[m,k] &= E[(C_m - E[C_m])(C_{m+k} - E[C_{m+k}])] \\ &= 16\,E[X_m X_{m+k}] = \begin{cases} 16 & k = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{4}$$

(c) A model of this type may be able to capture the mean and variance of the daily temperature. However, one reason this model is overly simple is because day to day temperatures are uncorrelated. A more realistic model might incorporate the effects of “heat waves” or “cold spells” through correlated daily temperatures.

Problem 13.7.5 Solution

This derivation of the Poisson process covariance is almost identical to the derivation of the Brownian motion autocovariance since both rely on the use of independent increments. From the definition of the Poisson process, we know that $\mu_N(t) = \lambda t$. When $\tau \geq 0$, we can write

$$\begin{aligned} C_N(t,\tau) &= \mathrm{E}\left[N(t)N(t+\tau)\right] - (\lambda t)[\lambda(t+\tau)] \\ &= \mathrm{E}\left[N(t)[(N(t+\tau)-N(t))+N(t)]\right] - \lambda^2 t(t+\tau) \\ &= \mathrm{E}\left[N(t)[N(t+\tau)-N(t)]\right] + \mathrm{E}\left[N^2(t)\right] - \lambda^2 t(t+\tau). \end{aligned} \tag{1}$$

By the definition of the Poisson process, $N(t+\tau) - N(t)$ is the number of arrivals in the interval $[t, t+\tau)$ and is independent of $N(t)$ for $\tau > 0$. This implies

$$\begin{aligned} \mathrm{E}\left[N(t)[N(t+\tau)-N(t)]\right] &= \mathrm{E}\left[N(t)\right]\mathrm{E}\left[N(t+\tau)-N(t)\right] \\ &= \lambda t[\lambda(t+\tau) - \lambda t]. \end{aligned} \tag{2}$$

Note that since $N(t)$ is a Poisson random variable, $\mathrm{Var}[N(t)] = \lambda t$. Hence

$$\mathrm{E}\left[N^2(t)\right] = \mathrm{Var}[N(t)] + (\mathrm{E}\left[N(t)\right]^2 = \lambda t + (\lambda t)^2. \tag{3}$$

Therefore, for $\tau \geq 0$,

$$C_N(t,\tau) = \lambda t[\lambda(t+\tau) - \lambda t) + \lambda t + (\lambda t)^2 - \lambda^2 t(t+\tau) = \lambda t. \tag{4}$$

If $\tau < 0$, then we can interchange the labels t and $t+\tau$ in the above steps to show $C_N(t,\tau) = \lambda(t+\tau)$. For arbitrary t and τ, we can combine these facts to write

$$C_N(t,\tau) = \lambda \min(t, t+\tau). \tag{5}$$

Problem 13.7.7 Solution

Since the X_n are independent,

$$\mathrm{E}\left[Y_n\right] = \mathrm{E}\left[X_{n-1}X_n\right] = \mathrm{E}\left[X_{n-1}\right]\mathrm{E}\left[X_n\right] = 0. \tag{1}$$

Thus the autocovariance function is

$$C_Y[n,k] = \mathrm{E}[Y_n Y_{n+k}] = \mathrm{E}[X_{n-1}X_n X_{n+k-1}X_{n+k}]\,. \tag{2}$$

To calculate this expectation, what matters is whether any of the four terms in the product are the same. This reduces to five cases:

1. $n+k-1 > n$, or equivalently $k > 1$:

   In this case, we have

   $$n-1 < n < n+k-1 < n+k, \tag{3}$$

   implying that X_{n-1}, X_n, X_{n+k-1} and X_{n+k} are independent. It follows that

   $$\begin{aligned} \mathrm{E}[X_{n-1}X_n X_{n+k-1}X_{n+k}] &= \mathrm{E}[X_{n-1}]\,\mathrm{E}[X_n]\,\mathrm{E}[X_{n+k-1}]\,\mathrm{E}[X_{n+k}] \\ &= 0. \end{aligned} \tag{4}$$

2. $n+k < n-1$, or equivalently $k < -1$:

   In this case, we have

   $$n+k-1 < n+k < n-1 < n, \tag{5}$$

   implying that X_{n+k-1} X_{n+k}, X_{n-1}, and X_n are independent. It follows that

   $$\begin{aligned} \mathrm{E}[X_{n-1}X_n X_{n+k-1}X_{n+k}] &= \mathrm{E}[X_{n-1}]\,\mathrm{E}[X_n]\,\mathrm{E}[X_{n+k-1}]\,\mathrm{E}[X_{n+k}] \\ &= 0. \end{aligned} \tag{6}$$

3. $k = -1$:

   In this case, we have

   $$\begin{aligned} \mathrm{E}[X_{n-1}X_n X_{n+k-1}X_{n+k}] &= \mathrm{E}\left[X_{n-1}^2 X_n X_{n-2}\right] \\ &= \mathrm{E}\left[X_{n-1}^2\right]\mathrm{E}[X_n]\,\mathrm{E}[X_{n-2}] = 0. \end{aligned} \tag{7}$$

4. $k = 0$:

   In this case, we have

$$\begin{aligned} \mathrm{E}\left[X_{n-1}X_nX_{n+k-1}X_{n+k}\right] &= \mathrm{E}\left[X_{n-1}^2X_n^2\right] \\ &= \mathrm{E}\left[X_{n-1}^2\right]\mathrm{E}\left[X_n^2\right] = 9. \end{aligned} \tag{8}$$

5. $k = 1$:

   In this case, we have

$$\begin{aligned} \mathrm{E}\left[X_{n-1}X_nX_{n+k-1}X_{n+k}\right] &= \mathrm{E}\left[X_{n-1}X_n^2X_{n+1}\right] \\ &= \mathrm{E}\left[X_{n-1}\right]\mathrm{E}\left[X_n^2\right]\mathrm{E}\left[X_{n+1}\right] = 0. \end{aligned} \tag{9}$$

Combining these case, we find that

$$C_Y\left[n,k\right] = \begin{cases} 9 & k = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Problem 13.8.1 Solution

For a set of samples $Y(t_1), \ldots, Y(t_k)$, we observe that $Y(t_j) = X(t_j + a)$. This implies

$$f_{Y(t_1),\ldots,Y(t_k)}(y_1, \ldots, y_k) = f_{X(t_1+a),\ldots,X(t_k+a)}(y_1, \ldots, y_k). \tag{1}$$

Thus,

$$f_{Y(t_1+\tau),\ldots,Y(t_k+\tau)}(y_1, \ldots, y_k) = f_{X(t_1+\tau+a),\ldots,X(t_k+\tau+a)}(y_1, \ldots, y_k). \tag{2}$$

Since $X(t)$ is a stationary process,

$$f_{X(t_1+\tau+a),\ldots,X(t_k+\tau+a)}(y_1, \ldots, y_k) = f_{X(t_1+a),\ldots,X(t_k+a)}(y_1, \ldots, y_k). \tag{3}$$

This implies

$$\begin{aligned} f_{Y(t_1+\tau),\ldots,Y(t_k+\tau)}(y_1, \ldots, y_k) &= f_{X(t_1+a),\ldots,X(t_k+a)}(y_1, \ldots, y_k) \\ &= f_{Y(t_1),\ldots,Y(t_k)}(y_1, \ldots, y_k). \end{aligned} \tag{4}$$

We can conclude that $Y(t)$ is a stationary process.

Problem 13.8.3 Solution

For an arbitrary set of samples $Y(t_1), \ldots, Y(t_k)$, we observe that $Y(t_j) = X(at_j)$. This implies

$$f_{Y(t_1),\ldots,Y(t_k)}(y_1, \ldots, y_k) = f_{X(at_1),\ldots,X(at_k)}(y_1, \ldots, y_k). \quad (1)$$

Thus,

$$f_{Y(t_1+\tau),\ldots,Y(t_k+\tau)}(y_1, \ldots, y_k) = f_{X(at_1+a\tau),\ldots,X(at_k+a\tau)}(y_1, \ldots, y_k). \quad (2)$$

We see that a time offset of τ for the $Y(t)$ process corresponds to an offset of time $\tau' = a\tau$ for the $X(t)$ process. Since $X(t)$ is a stationary process,

$$\begin{aligned} f_{Y(t_1+\tau),\ldots,Y(t_k+\tau)}(y_1, \ldots, y_k) &= f_{X(at_1+\tau'),\ldots,X(at_k+\tau')}(y_1, \ldots, y_k) \\ &= f_{X(at_1),\ldots,X(at_k)}(y_1, \ldots, y_k) \\ &= f_{Y(t_1),\ldots,Y(t_k)}(y_1, \ldots, y_k). \end{aligned} \quad (3)$$

We can conclude that $Y(t)$ is a stationary process.

Problem 13.8.5 Solution

Since $Y_n = X_{kn}$,

$$f_{Y_{n_1+l},\ldots,Y_{n_m+l}}(y_1, \ldots, y_m) = f_{X_{kn_1+kl},\ldots,X_{kn_m+kl}}(y_1, \ldots, y_m) \quad (1)$$

Stationarity of the X_n process implies

$$\begin{aligned} f_{X_{kn_1+kl},\ldots,X_{kn_m+kl}}(y_1, \ldots, y_m) &= f_{X_{kn_1},\ldots,X_{kn_m}}(y_1, \ldots, y_m) \\ &= f_{Y_{n_1},\ldots,Y_{n_m}}(y_1, \ldots, y_m). \end{aligned} \quad (2)$$

We combine these steps to write

$$f_{Y_{n_1+l},\ldots,Y_{n_m+l}}(y_1, \ldots, y_m) = f_{Y_{n_1},\ldots,Y_{n_m}}(y_1, \ldots, y_m). \quad (3)$$

Thus Y_n is a stationary process.

Problem 13.8.7 Solution

Since $g(\cdot)$ is an unspecified function, we will work with the joint CDF of $Y(t_1+\tau),\ldots,Y(t_n+\tau)$. To show $Y(t)$ is a stationary process, we will show that for all τ,

$$F_{Y(t_1+\tau),\ldots,Y(t_n+\tau)}(y_1,\ldots,y_n) = F_{Y(t_1),\ldots,Y(t_n)}(y_1,\ldots,y_n)\,. \tag{1}$$

By taking partial derivatives with respect to $y_1,\ldots,y_n$, it should be apparent that this implies that the joint PDF $f_{Y(t_1+\tau),\ldots,Y(t_n+\tau)}(y_1,\ldots,y_n)$ will not depend on τ. To proceed, we write

$$\begin{aligned}
&F_{Y(t_1+\tau),\ldots,Y(t_n+\tau)}(y_1,\ldots,y_n)\\
&\qquad = \mathrm{P}\left[Y(t_1+\tau)\le y_1,\ldots,Y(t_n+\tau)\le y_n\right]\\
&\qquad = \mathrm{P}\Bigg[\underbrace{g(X(t_1+\tau))\le y_1,\ldots,g(X(t_n+\tau))\le y_n}_{A_\tau}\Bigg]\,.
\end{aligned} \tag{2}$$

In principle, we can calculate $\mathrm{P}[A_\tau]$ by integrating $f_{X(t_1+\tau),\ldots,X(t_n+\tau)}(x_1,\ldots,x_n)$ over the region corresponding to event A_τ. Since $X(t)$ is a stationary process,

$$f_{X(t_1+\tau),\ldots,X(t_n+\tau)}(x_1,\ldots,x_n) = f_{X(t_1),\ldots,X(t_n)}(x_1,\ldots,x_n)\,. \tag{3}$$

This implies $\mathrm{P}[A_\tau]$ does not depend on τ. In particular,

$$\begin{aligned}
F_{Y(t_1+\tau),\ldots,Y(t_n+\tau)}(y_1,\ldots,y_n) &= \mathrm{P}\left[A_\tau\right]\\
&= \mathrm{P}\left[g(X(t_1))\le y_1,\ldots,g(X(t_n))\le y_n\right]\\
&= F_{Y(t_1),\ldots,Y(t_n)}(y_1,\ldots,y_n)\,.
\end{aligned} \tag{4}$$

Problem 13.9.1 Solution

The autocorrelation function $R_X(\tau)=\delta(\tau)$ is mathematically valid in the sense that it meets the conditions required in Theorem 13.12. That is,

$$R_X(\tau)=\delta(\tau)\ge 0, \tag{1}$$

$$R_X(\tau)=\delta(\tau)=\delta(-\tau)=R_X(-\tau), \tag{2}$$

$$R_X(\tau)\le R_X(0)=\delta(0). \tag{3}$$

However, for a process $X(t)$ with the autocorrelation $R_X(\tau) = \delta(\tau)$, Definition 13.16 says that the average power of the process is

$$E\left[X^2(t)\right] = R_X(0) = \delta(0) = \infty. \tag{4}$$

Processes with infinite average power cannot exist in practice.

Problem 13.9.3 Solution

TRUE: First we observe that $E[Y_n] = E[X_n] - E[X_{n-1}] = 0$, which doesn't depend on n. Second, we verify that

$$\begin{aligned} C_Y[n,k] &= E[Y_n Y_{n+k}] \\ &= E[(X_n - X_{n-1})(X_{n+k} - X_{n+k-1})] \\ &= E[X_n X_{n+k}] - E[X_n X_{n+k-1}] \\ &\quad - E[X_{n-1} X_{n+k}] - E[X_{n-1} X_{n+k-1}] \\ &= C_X[k] - C_X[k-1] - C_X[k+1] + C_X[k], \end{aligned} \tag{1}$$

which doesn't depend on n. Hence Y_n is WSS.

Problem 13.9.5 Solution

For $k \neq 0$, X_n and X_{n+k} are independent so that

$$R_X[n,k] = E[X_n X_{n+k}] = E[X_n] E[X_{n+k}] = \mu^2. \tag{1}$$

For $k = 0$,

$$R_X[n,0] = E[X_n X_n] = E\left[X_n^2\right] = \sigma^2 + \mu^2. \tag{2}$$

Combining these expressions, we obtain

$$R_X[n,k] = R_X[k] = \mu^2 + \sigma^2 \delta[k], \tag{3}$$

where $\delta[k] = 1$ if $k = 0$ and is otherwise zero.

Problem 13.9.7 Solution

FALSE: The autocorrelation of Y_n is

$$\begin{aligned} R_Y[n,k] &= \mathrm{E}[Y_n Y_{n+k}] \\ &= \mathrm{E}\left[(X_n + (-1)^{n-1}X_{n-1})(X_{n+k} + (-1)^{n+k-1}X_{n+k-1})\right] \\ &= \mathrm{E}[X_n X_{n+k}] + \mathrm{E}\left[(-1)^{n-1}X_{n-1}X_{n+k}\right] \\ &\quad + \mathrm{E}\left[X_n(-1)^{n+k-1}X_{n+k-1}\right] + \mathrm{E}\left[(-1)^{2n+k-2}X_{n-1}X_{n+k-1}\right] \\ &= R_X[k] + (-1)^{n-1}R_X[k+1] \\ &\qquad + (-1)^{n+k-1}R_X[k+1] + (-1)^k R_X[k] \\ &= [1+(-1)^k](R_X[k] + (-1)^{n-1}R_X[k+1]), \end{aligned} \tag{1}$$

which depends on n.

Problem 13.9.9 Solution

(a) In the problem statement, we are told that $X(t)$ has average power equal to 1. By Definition 13.16, the average power of $X(t)$ is $\mathrm{E}[X^2(t)] = 1$.

(b) Since Θ has a uniform PDF over $[0, 2\pi]$,

$$f_\Theta(\theta) = \begin{cases} 1/(2\pi) & 0 \le \theta \le 2\pi, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The expected value of the random phase cosine is

$$\begin{aligned} \mathrm{E}[\cos(2\pi f_c t + \Theta)] &= \int_{-\infty}^{\infty} \cos(2\pi f_c t + \theta) f_\Theta(\theta)\, d\theta \\ &= \int_0^{2\pi} \cos(2\pi f_c t + \theta)\frac{1}{2\pi} d\theta \\ &= \frac{1}{2\pi}\sin(2\pi f_c t + \theta)\Big|_0^{2\pi} \\ &= \frac{1}{2\pi}(\sin(2\pi f_c t + 2\pi) - \sin(2\pi f_c t)) = 0. \end{aligned} \tag{2}$$

(c) Since $X(t)$ and Θ are independent,

$$\begin{aligned} \mathrm{E}[Y(t)] &= \mathrm{E}[X(t)\cos(2\pi f_c t + \Theta)] \\ &= \mathrm{E}[X(t)]\,\mathrm{E}[\cos(2\pi f_c t + \Theta)] = 0. \end{aligned} \tag{3}$$

Note that the mean of $Y(t)$ is zero no matter what the mean of $X(t)$ since the random phase cosine has zero mean.

(d) Independence of $X(t)$ and Θ results in the average power of $Y(t)$ being

$$\begin{aligned} \mathrm{E}\left[Y^2(t)\right] &= \mathrm{E}\left[X^2(t)\cos^2(2\pi f_c t + \Theta)\right] \\ &= \mathrm{E}\left[X^2(t)\right]\mathrm{E}\left[\cos^2(2\pi f_c t + \Theta)\right] \\ &= \mathrm{E}\left[\cos^2(2\pi f_c t + \Theta)\right]. \end{aligned} \tag{4}$$

Note that we have used the fact from part (a) that $X(t)$ has unity average power. To finish the problem, we use the trigonometric identity $\cos^2\phi = (1 + \cos 2\phi)/2$. This yields

$$\mathrm{E}\left[Y^2(t)\right] = \mathrm{E}\left[\frac{1}{2}\left(1 + \cos(2\pi(2f_c)t + \Theta)\right)\right] = 1/2. \tag{5}$$

Note that $\mathrm{E}[\cos(2\pi(2f_c)t + \Theta)] = 0$ by the argument given in part (b) with $2f_c$ replacing f_c.

Problem 13.9.11 Solution

The solution to this problem is essentially the same as the proof of Theorem 13.13 except integrals are replaced by sums. First we verify that $\overline{X}_m$ is unbiased:

$$\begin{aligned} \mathrm{E}\left[\overline{X}_m\right] &= \frac{1}{2m+1}\mathrm{E}\left[\sum_{n=-m}^{m} X_n\right] \\ &= \frac{1}{2m+1}\sum_{n=-m}^{m}\mathrm{E}[X_n] = \frac{1}{2m+1}\sum_{n=-m}^{m}\mu_X = \mu_X. \end{aligned} \tag{1}$$

To show consistency, it is sufficient to show that $\lim_{m\to\infty} \text{Var}[\overline{X}_m] = 0$. First, we observe that $\overline{X}_m - \mu_X = \frac{1}{2m+1}\sum_{n=-m}^{m}(X_n - \mu_X)$. This implies

$$\begin{aligned}
\text{Var}[\overline{X}(T)] &= \text{E}\left[\left(\frac{1}{2m+1}\sum_{n=-m}^{m}(X_n - \mu_X)\right)^2\right] \\
&= \text{E}\left[\frac{1}{(2m+1)^2}\left(\sum_{n=-m}^{m}(X_n - \mu_X)\right)\left(\sum_{n'=-m}^{m}(X_{n'} - \mu_X)\right)\right] \\
&= \frac{1}{(2m+1)^2}\sum_{n=-m}^{m}\sum_{n'=-m}^{m}\text{E}\left[(X_n - \mu_X)(X_{n'} - \mu_X)\right] \\
&= \frac{1}{(2m+1)^2}\sum_{n=-m}^{m}\sum_{n'=-m}^{m} C_X\left[n' - n\right]. \qquad (2)
\end{aligned}$$

We note that

$$\begin{aligned}
\sum_{n'=-m}^{m} C_X\left[n' - n\right] &\leq \sum_{n'=-m}^{m}\left|C_X\left[n' - n\right]\right| \\
&\leq \sum_{n'=-\infty}^{\infty}\left|C_X\left[n' - n\right]\right| = \sum_{k=-\infty}^{\infty}\left|C_X(k)\right| < \infty. \qquad (3)
\end{aligned}$$

Hence there exists a constant K such that

$$\text{Var}[\overline{X}_m] \leq \frac{1}{(2m+1)^2}\sum_{n=-m}^{m} K = \frac{K}{2m+1}. \qquad (4)$$

Thus $\lim_{m\to\infty} \text{Var}[\overline{X}_m] \leq \lim_{m\to\infty}\frac{K}{2m+1} = 0$.

Problem 13.10.1 Solution

(a) Since $X(t)$ and $Y(t)$ are independent processes,

$$\text{E}\left[W(t)\right] = \text{E}\left[X(t)Y(t)\right] = \text{E}\left[X(t)\right]\text{E}\left[Y(t)\right] = \mu_X\mu_Y. \qquad (1)$$

In addition,

$$\begin{aligned} R_W(t,\tau) &= \mathrm{E}\left[W(t)W(t+\tau)\right] \\ &= \mathrm{E}\left[X(t)Y(t)X(t+\tau)Y(t+\tau)\right] \\ &= \mathrm{E}\left[X(t)X(t+\tau)\right]\mathrm{E}\left[Y(t)Y(t+\tau)\right] \\ &= R_X(\tau)R_Y(\tau). \end{aligned} \tag{2}$$

We can conclude that $W(t)$ is wide sense stationary.

(b) To examine whether $X(t)$ and $W(t)$ are jointly wide sense stationary, we calculate

$$R_{WX}(t,\tau) = \mathrm{E}\left[W(t)X(t+\tau)\right] = \mathrm{E}\left[X(t)Y(t)X(t+\tau)\right]. \tag{3}$$

By independence of $X(t)$ and $Y(t)$,

$$R_{WX}(t,\tau) = \mathrm{E}\left[X(t)X(t+\tau)\right]\mathrm{E}\left[Y(t)\right] = \mu_Y R_X(\tau). \tag{4}$$

Since $W(t)$ and $X(t)$ are both wide sense stationary and since $R_{WX}(t,\tau)$ depends only on the time difference τ, we can conclude from Definition 13.18 that $W(t)$ and $X(t)$ are jointly wide sense stationary.

Problem 13.10.3 Solution

(a) $Y(t)$ has autocorrelation function

$$\begin{aligned} R_Y(t,\tau) &= \mathrm{E}\left[Y(t)Y(t+\tau)\right] \\ &= \mathrm{E}\left[X(t-t_0)X(t+\tau-t_0)\right] \\ &= R_X(\tau). \end{aligned} \tag{1}$$

(b) The cross correlation of $X(t)$ and $Y(t)$ is

$$\begin{aligned} R_{XY}(t,\tau) &= \mathrm{E}\left[X(t)Y(t+\tau)\right] \\ &= \mathrm{E}\left[X(t)X(t+\tau-t_0)\right] \\ &= R_X(\tau-t_0). \end{aligned} \tag{2}$$

(c) We have already verified that $R_Y(t,\tau)$ depends only on the time difference τ. Since $\mathrm{E}[Y(t)] = \mathrm{E}[X(t-t_0)] = \mu_X$, we have verified that $Y(t)$ is wide sense stationary.

(d) Since $X(t)$ and $Y(t)$ are wide sense stationary and since we have shown that $R_{XY}(t,\tau)$ depends only on τ, we know that $X(t)$ and $Y(t)$ are jointly wide sense stationary.

Comment: This problem is badly designed since the conclusions don't depend on the specific $R_X(\tau)$ given in the problem text. (Sorry about that!)

Problem 13.11.1 Solution

For the $X(t)$ process to be stationary, we must have $f_{X(t_1)}(x) = f_{X(t_2)}(x)$. Since $X(t_1)$ and $X(t_2)$ are both Gaussian and zero mean, this requires that

$$\sigma_1^2 = \mathrm{Var}[X(t_1)] = \mathrm{Var}[X(t_2)] = \sigma_2^2. \tag{1}$$

In addition the correlation coefficient of $X(t_1)$ and $X(t_2)$ must satisfy

$$\left|\rho_{X(t_1),X(t_2)}\right| \leq 1. \tag{2}$$

This implies

$$\rho_{X(t_1),X(t_2)} = \frac{\mathrm{Cov}\left[X(t_1), X(t_2)\right]}{\sigma_1\sigma_2} = \frac{1}{\sigma_2^2} \leq 1. \tag{3}$$

Thus $\sigma_1^2 = \sigma_2^2 \geq 1$.

Problem 13.11.3 Solution

Writing $Y(t+\tau) = \int_0^{t+\tau} N(v)\,dv$ permits us to write the autocorrelation of $Y(t)$ as

$$\begin{aligned} R_Y(t,\tau) = \mathrm{E}\left[Y(t)Y(t+\tau)\right] &= \mathrm{E}\left[\int_0^t \int_0^{t+\tau} N(u)N(v)\,dv\,du\right] \\ &= \int_0^t \int_0^{t+\tau} \mathrm{E}\left[N(u)N(v)\right]\,dv\,du \\ &= \int_0^t \int_0^{t+\tau} \alpha\delta(u-v)\,dv\,du. \end{aligned} \tag{1}$$

At this point, it matters whether $\tau \geq 0$ or if $\tau < 0$. When $\tau \geq 0$, then v ranges from 0 to $t+\tau$ and at some point in the integral over v we will have $v = u$. That is, when $\tau \geq 0$,

$$R_Y(t,\tau) = \int_0^t \alpha\, du = \alpha t. \tag{2}$$

When $\tau < 0$, then we must reverse the order of integration. In this case, when the inner integral is over u, we will have $u = v$ at some point so that

$$R_Y(t,\tau) = \int_0^{t+\tau} \int_0^t \alpha\delta(u-v)\, du\, dv = \int_0^{t+\tau} \alpha\, dv = \alpha(t+\tau). \tag{3}$$

Thus we see the autocorrelation of the output is

$$R_Y(t,\tau) = \alpha \min\{t, t+\tau\}. \tag{4}$$

Perhaps surprisingly, $R_Y(t,\tau)$ is what we found in Example 13.19 to be the autocorrelation of a Brownian motion process. In fact, Brownian motion is the integral of the white noise process.

Problem 13.11.5 Solution

Let $\mathbf{W} = \begin{bmatrix} W(t_1) & W(t_2) & \cdots & W(t_n) \end{bmatrix}'$ denote a vector of samples of a Brownian motion process. To prove that $W(t)$ is a Gaussian random process, we must show that $\mathbf{W}$ is a Gaussian random vector. To do so, let

$$\begin{aligned} \mathbf{X} &= \begin{bmatrix} X_1 & \cdots & X_n \end{bmatrix}' \\ &= \begin{bmatrix} W(t_1) & W(t_2) - W(t_1) & \cdots & W(t_n) - W(t_{n-1}) \end{bmatrix}' \end{aligned} \tag{1}$$

denote the vector of increments. By the definition of Brownian motion, $X_1, \ldots, X_n$ is a sequence of independent Gaussian random variables. Thus $\mathbf{X}$ is a Gaussian random vector. Finally,

$$\mathbf{W} = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \end{bmatrix} = \begin{bmatrix} X_1 \\ X_1 + X_2 \\ \vdots \\ X_1 + \cdots + X_n \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & \cdots & \cdots & 1 \end{bmatrix}}_{\mathbf{A}} \mathbf{X}. \tag{2}$$

Since $\mathbf{X}$ is a Gaussian random vector and $\mathbf{W} = \mathbf{AX}$ with $\mathbf{A}$ a rank n matrix, Theorem 8.11 implies that $\mathbf{W}$ is a Gaussian random vector.

Problem 13.12.1 Solution

From the instructions given in the problem, the program `noisycosine.m` will generate the four plots.

```
n=1000; t=0.001*(-n:n);
w=gaussrv(0,0.01,(2*n)+1);
%Continuous Time, Continuous Value
xcc=2*cos(2*pi*t) + w';
plot(t,xcc);
xlabel('\it t');ylabel('\it X_{cc}(t)');
axis([-1 1 -3 3]);
figure; %Continuous Time, Discrete Value
xcd=round(xcc); plot(t,xcd);
xlabel('\it t');ylabel('\it X_{cd}(t)');
axis([-1 1 -3 3]);
figure; %Discrete time, Continuous Value
ts=subsample(t,100); xdc=subsample(xcc,100);
plot(ts,xdc,'b.');
xlabel('\it t');ylabel('\it X_{dc}(t)');
axis([-1 1 -3 3]);
figure; %Discrete Time, Discrete Value
xdd=subsample(xcd,100); plot(ts,xdd,'b.');
xlabel('\it t');ylabel('\it X_{dd}(t)');
axis([-1 1 -3 3]);
```

In `noisycosine.m`, we use a function `subsample.m` to obtain the discrete time sample functions. In fact, `subsample` is hardly necessary since it's such a simple one-line MATLAB function:

```
function y=subsample(x,n)
%input x(1), x(2) ...
%output y(1)=x(1), y(2)=x(1+n), y(3)=x(2n+1)
y=x(1:n:length(x));
```

However, we use it just to make `noisycosine.m` a little more clear.

Problem 13.12.3 Solution

In this problem, our goal is to find out the average number of ongoing calls in the switch. Before we use the approach of Problem 13.12.2, its worth a moment to consider the physical situation. In particular, calls arrive as a Poisson process of rate $\lambda = 100$ call/minute and each call has duration of *exactly* one minute. As a result, if we inspect the system at an arbitrary time t at least one minute past initialization, the number of calls at the switch will be exactly the number of calls N_1 that arrived in the previous minute. Since calls arrive as a Poisson proces of rate $\lambda = 100$ calls/minute. N_1 is a Poisson random variable with $\mathrm{E}[N_1] = 100$.

In fact, this should be true for every inspection time t. Hence it should surprising if we compute the time average and find the time average number in the queue to be something other than 100. To check out this quickie analysis, we use the method of Problem 13.12.2. However, unlike Problem 13.12.2, we cannot directly use the function `simswitch.m` because the call duration are no longer exponential random variables. Instead, we must modify `simswitch.m` for the deterministic one minute call durations, yielding the function `simswitchd.m`:

```
function M=simswitchd(lambda,T,t)
%Poisson arrivals, rate lambda
%Deterministic (T) call duration
%For vector t of times
%M(i) = no. of calls at time t(i)
s=poissonarrivals(lambda,max(t));
y=s+T;
A=countup(s,t);
D=countup(y,t);
M=A-D;
```

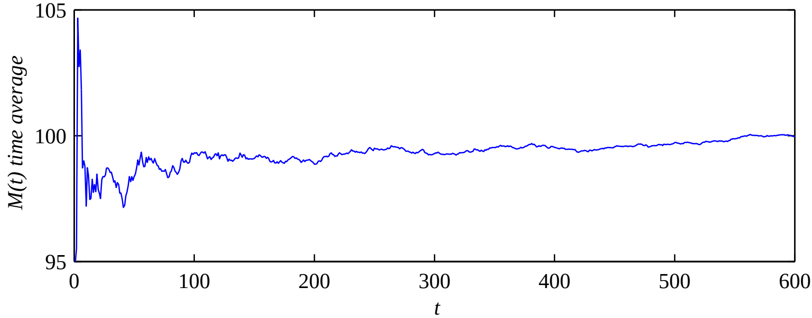
Note that if you compare `simswitch.m` in the text with `simswitchd.m` here, two changes occurred. The first is that the exponential call durations are replaced by the deterministic time T. The other change is that `count(s,t)` is replaced by `countup(s,t)`. In fact, `n=countup(x,y)` does exactly the same thing as `n=count(x,y)`; in both cases, `n(i)` is the number of elements less than or equal to $y(i)$. The difference is that `countup` requires that the vectors `x` and `y` be nondecreasing.

Now we use the same procedure as in Problem 13.12.2 and form the time average

$$\overline{M}(T) = \frac{1}{T} \sum_{t=1}^{T} M(t). \tag{1}$$

```
>> t=(1:600)';
>> M=simswitchd(100,1,t);
>> Mavg=cumsum(M)./t;
>> plot(t,Mavg);
```

We form and plot the time average using these commands will yield a plot vaguely similar to that shown below.



We used the word "vaguely" because at $t = 1$, the time average is simply the number of arrivals in the first minute, which is a Poisson ($\alpha = 100$) random variable which has not been averaged. Thus, the left side of the graph will be random for each run. As expected, the time average appears to be converging to 100.

Problem 13.12.5 Solution

Following the problem instructions, we can write the function `newarrivals.m`. For convenience, here are `newarrivals` and `poissonarrivals` side by side.

```
function s=newarrivals(lam,T)
%Usage s=newarrivals(lam,T)
%Returns Poisson arrival times
%s=[s(1) ... s(n)] over [0,T]
n=poissonrv(lam*T,1);
s=sort(T*rand(n,1));
```

```
function s=poissonarrivals(lam,T)
%arrival times s=[s(1) ... s(n)]
%  s(n)<= T < s(n+1)
n=ceil(1.1*lam*T);
s=cumsum(exponentialrv(lam,n));
while (s(length(s))< T),
  s_new=s(length(s))+ ...
    cumsum(exponentialrv(lam,n));
  s=[s; s_new];
end
s=s(s<=T);
```

Clearly the code for `newarrivals` is shorter, more readable, and perhaps, with the help of Problem 13.5.8, more logical than `poissonarrivals`. Unfortunately this doesn't mean the code runs better. Here are some `cputime` comparisons:

```
>> t=cputime;s=poissonarrivals(1,100000);t=cputime-t
t =
    0.1110
>> t=cputime;s=newarrivals(1,100000);t=cputime-t
t =
    0.5310
>> t=cputime;poissonrv(100000,1);t=cputime-t
t =
    0.5200
>>
```

Unfortunately, these results were highly repeatable. The function `poissonarrivals` generated 100,000 arrivals of a rate 1 Poisson process required roughly 0.1 seconds of cpu time. The same task took `newarrivals` about 0.5 seconds, or roughly 5 times as long! In the `newarrivals` code, the culprit is the way `poissonrv` generates a single Poisson random variable with expected value 100,000. In this case, `poissonrv` generates the first 200,000 terms of the Poisson PMF! This required calculation is so large that it dominates the work need to generate 100,000 uniform random numbers. In fact, this suggests that a more efficient way to generate a Poisson (α) random variable N is to generate arrivals of a rate α Poisson process until the Nth arrival is after time 1.

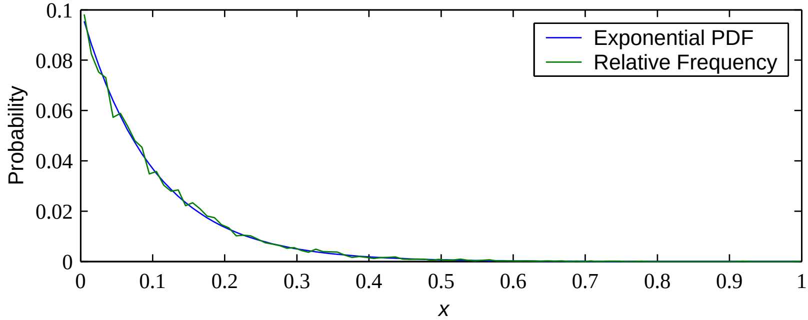
Problem 13.12.7 Solution

In this problem, we start with the `simswitch.m` code to generate the vector of departure times `y`. We then construct the vector `I` of inter-departure times. The command `hist,20` will generate a 20 bin histogram of the departure times. The fact that this histogram resembles an exponential PDF suggests that perhaps it is reasonable to try to match the PDF of an exponential (μ) random variable against the histogram.

In most problems in which one wants to fit a PDF to measured data, a key issue is how to choose the parameters of the PDF. In this problem, choosing μ is simple. Recall that the switch has a Poisson arrival process of rate λ so interarrival times are exponential (λ) random variables. If $1/\mu < 1/\lambda$, then the average time between departures from the switch is less than the average time between arrivals to the switch. In this case, calls depart the switch faster than they arrive which is impossible because each departing call was an arriving call at an earlier time. Similarly, if $1/\mu > 1/\lambda$, then calls would be departing from the switch more slowly than they arrived. This can happen to an overloaded switch; however, it's impossible in this system because each arrival departs after an exponential time. Thus the only possibility is that $1/\mu = 1/\lambda$. In the program `simswitchdepart.m`, we plot a histogram of departure times for a switch with arrival rate λ against the scaled exponential (λ) PDF $\lambda e^{-\lambda x} b$ where b is the histogram bin size. Here is the code:

```
function I=simswitchdepart(lambda,mu,T)
%Usage: I=simswitchdepart(lambda,mu,T)
%Poisson arrivals, rate lambda
%Exponential (mu) call duration
%Over time [0,T], returns I,
%the vector of inter-departure times
%M(i) = no. of calls at time t(i)
s=poissonarrivals(lambda,T);
y=s+exponentialrv(mu,length(s));
y=sort(y);
n=length(y);
I=y-[0; y(1:n-1)]; %interdeparture times
imax=max(I);b=ceil(n/100);
id=imax/b; x=id/2:id:imax;
pd=hist(I,x); pd=pd/sum(pd);
px=exponentialpdf(lambda,x)*id;
plot(x,px,x,pd);
xlabel('\it x');ylabel('Probability');
legend('Exponential PDF','Relative Frequency');
```

Here is an example of the `simswitchdepart(10,1,1000)` output:



As seen in the figure, the match is quite good. Although this is not a carefully designed statistical test of whether the inter-departure times are exponential random variables, it is enough evidence that one may want to pursue whether such a result can be proven.

In fact, the switch in this problem is an example of an $M/M/\infty$ queuing system for which it has been shown that not only do the inter-departure have an exponential distribution, but the steady-state departure process is a Poisson process. For the curious reader, details can be found, for example, in the text *Stochastic Processes: Theory for Applications* by Gallager.