Two State Chain

- States OFF and ON (0 and 1)
- If OFF, switch ON with prob $p$, If ON, switch OFF with prob $q$
Limiting State Probabilities

- States $i$ and $j$ with $P[V_{ij}] = 1$
- First transition time $T_{jj}$ with span 1
- $R(t) = \text{time spent in state } j \text{ over } [0, t]$

\[
\lim_{n \to \infty} P[X_n = j | X_0 = i] = \lim_{n \to \infty} P_{ij}^{(n)} = \frac{1}{E[T_{jj}]} = \lim_{t \to \infty} \frac{R(t)}{t}
\]
Example: Communicating Classes

- Positive transition probs shown. What are comm classes?
Transient and Recurrent States

- A state $j$ is either
  
  - Transient if $P[V_{jj}] < 1$
  
  - Positive Recurrent if $P[V_{jj}] = 1$ and $E[T_{jj}] < \infty$,
  
  - Null Recurrent if $P[V_{jj}] = 1$ and $E[T_{jj}] = \infty$
Class Properties II

- For a communicating class of a Markov chain, one of the following must be true:
  - All states are transient
  - All states are null recurrent
  - All states are positive recurrent
- Comm classes represent system modes
Irreducible Markov Chain

- A Markov chain is *irreducible* if there is only one communicating class.

- The system has only one mode

- If irreducible then

\[
\lim_{n \to \infty} P_{ij}^{(n)} = \pi_j
\]
Periodic and Aperiodic States

- State $i$ has *period* $d$ if $d$ is the largest integer such that $P_{ii}^{(n)} = 0$ whenever $n$ is not divisible by $d$.

- If $d = 1$, then state $i$ is called *aperiodic*.

- All states in the same comm. class have same period.
Aperiodic Irreducible Chain

- For an aperiodic irreducible Markov chain, either
  - States are all transient or all null recurrent, 
    \[ \lim_{n \to \infty} P^{(n)}_{ij} = 0. \]  
    No stationary probs
  - All states are positive recurrent, and
    \[ p_j = \lim_{n \to \infty} P^n_{ij} > 0 \]
    are unique stationary probs satisfying
    \[ \pi_j = \sum_i \pi_i P_{ij} \]
Finite State Irreducible Aperiodic Chain

- For an irreducible, aperiodic, finite MC,

\[
\lim_{n \to \infty} P^n = \begin{bmatrix}
\pi \\
\vdots \\
\pi
\end{bmatrix} = \begin{bmatrix}
\pi_0 & \cdots & \pi_K \\
\vdots & \ddots & \vdots \\
\pi_0 & \cdots & \pi_K
\end{bmatrix}
\]

- \( \pi = \pi P \quad \sum_{j=0}^{K} \pi_j = 1 \)
Irreducible Periodic Chain

- For an irreducible, positive recurrent, periodic MC, $\pi_j$ are the unique nonnegative solution of

$$\pi_j = \sum_i \pi_i P_{ij} \quad \sum_j \pi_j = 1$$

- $\pi_j$ is the limiting fraction of time spent in state $j$
Continuous Time Markov Chain

- Continuous time MC \( \{X(t) | t \geq 0\} \) is a continuous time, discrete value process

- For infinitesimal \( \Delta \),

\[
P[X(t + \Delta) = j | X(t) = i] = q_{ij} \Delta \\
P[X(t + \Delta) = i | X(t) = i] = 1 - \sum_{j \neq i} q_{ij} \Delta
\]
State Departure Rate

• Only one transition in time $\Delta$ and

$$ P[X(t + \Delta) \neq i | X(t) = i] = \sum_{j \neq i} q_{i,j} \Delta $$

• In every tiny $\Delta$, a Bernoulli trial determines whether the system exits state $i$.

• Time spent in $i$ is exponential, parameter

$$ \nu_i = \sum_{j \neq i} q_{i,j} $$
Cts Time MC Departures

• When system enters \( i \), start competing Poisson processes \( N_{ik}(t) \) of rate \( q_{ik} \) for other states \( k \).

• If the process \( N_{ij}(t) \) has first arrival, then go to \( j \)

• Repeat
Cts Time MC Departures II

- Event $D_i$: depart $i$ in $(t, t + \delta]$
- Event $D_{ij}$: go to state $j$

\[ P[D_{ij} | D_i] = \frac{P[D_{ij}]}{P[D_i]} = \frac{q_{ij}\Delta}{\nu_i \Delta} = \frac{q_{ij}}{\nu_i} \]

- Wait in $i$ exponential time – parameter $\nu_i$
- Go to $j$ with prob $q_{ij}/\nu_i$
Embedded Discrete Time Markov Chain

- For a cts time MC, the embedded discrete time Markov chain has transition probabilities

\[ P_{ij} = q_{ij} / \nu_i \]

- \( \nu_i = \sum_{k \neq i} q_{ik} \).

- Embedded chain ignores time spent in states
Embedded Chain II

- Comm classes for cts time chain $=$ cmm classes for embedded chain

- Defn: Cts time MC is irreducible if embedded MC is irreducible
Chapman-Kolmogorov Equations

- let $q_{jj} = -v_j$. For a cts time MC

$$\frac{dp_j(t)}{dt} = \sum_i q_{ij}p_i(t)$$

- In terms of matrices,

$$\frac{dp(t)}{dt} = p(t)Q$$
Proof of Differential Eqns

\[ p_j(t + \Delta) = (1 - v_j \Delta) p_j(t) + \sum_{i \neq j} q_{ij} \Delta p_i(t) \]

or

\[ p_j(t + \Delta) - p_j(t) = \sum_i q_{ij} \Delta p_i(t) \]

Divide by \( \Delta \), let \( \Delta \to 0 \)
Limiting State Probabilities

- For a cts time MC, \( \lim_{t \to \infty} p_j(t) = p_j \), where the \( p_j \) are the unique solution to

\[
\sum_i q_{ij} p_i = 0 \quad \sum_j p_j = 1
\]

- Rate of transitions into \( i \) equals rate out of \( i \)
Birth-Death Process

- $q_{ij} = 0$ for $|i - j| > 1$

**Queues: in state $i$**

- $\lambda_i =$ arrival rate
- $\mu_i =$ service rate
Birth Death Limiting State Probs

Limiting State Probabilities satisfy

\[ p_{i-1} \lambda_{i-1} = p_i \mu_i \quad \sum_{i=0}^{\infty} p_i = 1 \]
Example: $M/M/1$ queue

- Poisson arrivals, exponential service, 1 server

\[ p_i = \rho p_{i-1} \text{ with } \rho = \lambda/\mu. \]

- Thus \( p_i = (1 - \rho)\rho^i \).
$M/M/\infty$ Quee

- Poisson arrivals, exponential service, $\infty$ servers

- $p_n = (\rho/n) p_{n-1}$ with $\rho = \lambda/\mu$.

- $p_n = (\rho^n/n!) p_0$, $p_0 = e^{-\rho}$