Bayes’ Theorem

- 

\[ P[B|A] = \frac{P[A|B]P[B]}{P[A]} \]

- For an event space \( B_1, B_2, \ldots, B_m \),

\[ P[B_i|A] = \frac{P[A|B_i]P[B_i]}{\sum_{i=1}^{m} P[A|B_i]P[B_i]} \]
Independent Events

- Events $A$ and $B$ are *independent* if and only if

$$P[AB] = P[A]P[B]$$

- Always check if you are asked!
Sequential Experiments - Example

- Two coins, one biased, one fair, but you don’t know which is which.

- Coin 1: \( P[H] = \frac{3}{4} \). Coin 2: \( P[H] = \frac{1}{2} \)

- Pick a coin at random and flip it. Let \( C_i \) denote the event that coin \( i \) is picked. What is \( P[C_1 | H] \)
Solution: Tree Diagram

\[
P[C_1|H] = \frac{P[C_1H]}{P[C_1H] + P[C_2H]} = \frac{3/8}{3/8 + 1/4} = \frac{3}{5}
\]
Definition: 2 Independent Events

Definition 1.6 Events $A$ and $B$ are independent if and only if

$$P[AB] = P[A]P[B]$$

Equivalent definitions:

Definition: 3 Independent Events

Definition 1.7 $A_1$, $A_2$, and $A_3$ are independent if and only if

- $A_1$ and $A_2$ are independent.
- $A_2$ and $A_3$ are independent.
- $A_1$ and $A_3$ are independent.
Fundamental Principle of Counting

- Experiment $A$ has $n$ possible outcomes,
- Experiment $B$ has $k$ possible outcomes,
- There are $nk$ possible outcomes when you perform both experiments.
Permutations

- \( k\)-permutation: an ordered sequence of \( k \) distinguishable objects

- \((n)_k = \text{no. of } k\text{-permutations of } n \text{ dist. objects.}\)

\[(n)_k = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}\]
Combinations

- Pick a subset of $k$ out of $n$ objects.
- Order of selection doesn’t matter
- Each subset is a $k$-combination
How Many Combinations

- \( \binom{n}{k} = \text{“} n \text{ choose } k \text{”} \)

- Two steps for a \( k \)-permutation:
  1. Choose a \( k \)-combination out of \( n \) objects.
  2. Choose a \( k \)-permutation of the \( k \) objects in the \( k \)-combination.

\[
\binom{n}{k} = \binom{n}{k} \cdot k! \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}
\]
Problem 1.8.6

A basketball team has

- 3 pure centers, 4 pure forwards, 4 pure guards
- one swingman who can play either guard or forward.

A pure player can play only the designated position. How many lineups are there (1 center, 2 forwards, 2 guards)
Problem 1.8.6 Solution

Three possibilities:

1. swingman plays guard: $N_1$ lineups
2. swingman plays forward $N_2$ lineups
3. swingman doesn’t play. $N_3$ lineups

\[ N = N_1 + N_2 + N_3 \]
\[ N_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = 72 \]
\[ N_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 72 \]
\[ N_3 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 108 \]
Multiple Outcomes

- $n$ independent trials
- $r$ possible trial outcomes $(s_1, \ldots, s_r)$
- $P[s_k] = p_k$
Multiple Outcomes (2)

- Outcome is a sequence:
  - Example: $s_3s_4s_3s_1$

  $P[s_3s_4s_3s_1] = p_3p_4p_3p_1 = p_1p_3^2p_4$

  $= p_1^{n_1}p_2^{n_2}p_3^{n_3}p_4^{n_4}$

- Prob depends on how many times each outcome occurred
Multiple Outcomes (3)

$N_i = \text{no. of time } s_i \text{ occurs}$

$$P[N_1 = n_1, \ldots, N_r = n_r] = M p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$$

$M = \text{Multinomial Coefficient}$

$$= \frac{n!}{n_1! n_2! \cdots n_r!}$$
Chapter 2: Random Variables

- Experiment: Procedure + Observations
- Observation is an outcome
- Assign a number to each outcome: Random variable
Random Variables

Three ways to get an rv:

- The rv is the observation
- The rv is a function of the observation
- The rv is a function of a rv
Discrete Random Variables

- \( S_X = \text{range of } X \) (set of possible values)
- \( X \) is discrete if \( S_X \) is countable
- Discrete rv \( X \) has PMF

\[
P_X(x) = P[X = x]
\]
PMF Properties

- $P_X(x) \geq 0$
- $\sum_{x \in S_X} P_X(x) = 1$
- For an event $B \subset S_X$,

$$P[B] = P[X \in B] = \sum_{x \in B} P_X(x)$$
Bernoulli RV

Get the phone number of a random student. Let $X = 0$ if the last digit is even. Otherwise, let $X = 1$.

$$P_X(x) = \begin{cases} 
1 - p & x = 0 \\
p & x = 1 \\
0 & \text{otherwise}
\end{cases}$$
Binomial RV

- Test $n$ circuits, each circuit is rejected with probability $p$ independent of other tests.

- $K =$ no. of rejects

- $K$ is the number of successes in $n$ trials:

\[
P_K(k) = \begin{cases} 
\binom{n}{k} p^k (1 - p)^{n-k} & k = 0, 1, \ldots, n \\
0 & \text{otherwise}
\end{cases}
\]
Geometric RV

Circuit rejected with prob $p$. $Y$ is the number of tests up to and including the first reject.

$p \quad r \quad \bullet Y=1$

$p \quad r \quad \bullet Y=2$

$p \quad r \quad \bullet Y=3$

$1-p \quad a \quad 1-p \quad a \quad 1-p \quad a \quad \ldots$
From the tree, $P[Y = 1] = p$, $P[Y = 2] = p(1 - p)$,

$$P_Y(y) = \begin{cases} 
p(1 - p)^{y-1} & y = 1, 2, \ldots \\
0 & \text{otherwise}
\end{cases}$$