Lecture 16: Central Limit Theorem

- Review: $X_1, X_2, \ldots$ iid Gaussian rv’s

- $W_n = X_1 + \cdots + X_n$ is Gaussian with
  
  $E[W_n] = n\mu_X$

  $\text{Var}[W_n] = n\sigma^2_X$

- What if $X_1, X_2, \ldots$ are not Gaussian?
Sum Of Bernoulli rv’s

- 50 flips of a fair coin: $X_i = 1$ is $H$ on flip $i$.

- $W_n$ is binomial

$$P_{W_n}(w) = \begin{cases} 
\binom{50}{w} (1/2)^{50} & w = 0, 1, \ldots, 50 \\
0 & \text{otherwise}
\end{cases}$$

- What does this look like?
Central Limit Theorem

• Standardized rv $Z_n$:

$$Z_n = \frac{W_n - E[W]}{\sigma_W} = \frac{\sum_{i=1}^{n} X_i - n\mu_X}{\sqrt{n\sigma_X^2}}$$

• $E[Z_n] = 0 \quad \text{Var}[Z_n] = 1$

• Central Limit Theorem:

$$\lim_{n \to \infty} F_{Z_n}(z) = \Phi(z)$$

• Usual Proof: Show MGF of $Z_n$ converges to Gaussian MGF
Applying the CLT

- For $W_n = X_1 + \cdots + X_n$,

\[ F_{W_n}(w) = P \left[ \sqrt{n\sigma_X^2}Z_n + n\mu_X \leq w \right] = F_{Z_n} \left( \frac{w - n\mu_X}{\sqrt{n\sigma_X^2}} \right) \]

- For large $n$, CLT says $F_{Z_n}(z) \approx \Phi(z)$.

- CLT Approximation:

\[ F_{W_n}(w) \approx \Phi \left( \frac{w - n\mu_X}{\sqrt{n\sigma_X^2}} \right) \]
CLT for Uniform RV’s

(a) $n = 1$

(b) $n = 2$

(c) $n = 3$

(d) $n = 4$
CLT for Binomial RV’s

- $n = 4, p = 1/2$
- $n = 8, p = 1/2$
- $n = 16, p = 1/2$
- $n = 32, p = 1/2$
Chap. 8: Sample Mean

- iid \( X_1, \ldots, X_n \), each with PDF \( f_X(x) \)

- The sample mean of \( X \) is the rv

\[
M_n(X) = \frac{X_1 + \cdots + X_n}{n}
\]

- Remember \( M_n(X) \) is a rv!

- \( M_n(X) \) is **not** the expected value \( E[X] \)
Mean and Variance of $M_n(X)$

- Thm:

\[ E[M_n(X)] = E[X] \quad \text{Var}[M_n(X)] = \frac{\text{Var}[X]}{n} \]

- \( \lim_{n \to \infty} \text{Var}[M_n(X)] = 0 \) suggests \( M_n(X) \to E[X] \)

- How does a seq of rv’s approach a constant?
Markov Inequality

- For nonnegative rv $X$ and $c > 0$,

$$P[X \geq c] \leq \frac{E[X]}{c}$$

- Proof: $f_X(x) = 0$ for $x < 0$ and

$$E[X] = \int_0^c x f_X(x) \, dx + \int_c^{\infty} x f_X(x) \, dx$$

$$\geq \int_c^{\infty} x f_X(x) \, dx \geq c \int_c^{\infty} f_X(x) \, dx = c P[X \geq c]$$
Markov Inequality Example

• \( X = \) height (in feet) of a random adult.

• \( E[X] = 5.5 \) feet

• Markov inequality says

\[
P[X \geq 11] \leq \frac{5.5}{11} = \frac{1}{2}
\]

• Statement is true but is so weak it sounds wrong
Chebyshev Inequality

- Let $X = (Y - \mu_Y)^2$ and apply the Markov inequality:

$$P[(Y - \mu_Y)^2 \geq c^2] \leq \frac{E[(Y - \mu_Y)^2]}{c^2}$$

- Chebyshev Inequality:

$$P[|Y - \mu_Y| \geq c] \leq \frac{\text{Var}[Y]}{c^2}$$
Chernoff Bound

- For rv $X$ and constant $c$,

$$P[X \geq c] \leq \min_{s \geq 0} e^{-sc} \Phi_X(s)$$

- Proof:

$$P[X \geq c] = \int_c^{\infty} f_X(x) \, dx = \int_{-\infty}^{\infty} u(x-c) f_X(x) \, dx$$
• For all $s \geq 0$, $u(x - c) \leq e^{s(x-c)}$, implying

$$P[X \geq c] \leq \int_{-\infty}^{\infty} e^{s(x-c)} f_X(x) \, dx$$

$$= e^{-sc} \int_{-\infty}^{\infty} e^{sx} f_X(x) \, dx$$

$$= e^{-sc} \phi_X(s)$$

• Upper bound must hold for minimizing $s$
Heights: Chebyshev

- For height $X$, $E[X] = 5.5$ and $\sigma_X = 1$ ft

$$P[X \geq 11] = P[X - \mu_X \geq 11 - \mu_X] = P[|X - \mu_X| \geq 5.5]$$

- Chebyshev:

$$P[X \geq 11] = P[|X - \mu_X| \geq 5.5]$$

$$\leq \text{Var } [X] / (5.5)^2 = 0.033 \approx 1/30$$
Heights: Chernoff

- If $X$ is $N[5.5, 1]$, $\phi_X(s) = e^{(11s+s^2)/2}$

- The Chernoff bound is

$$P[X \geq 11] \leq \min_{s \geq 0} e^{-11s} e^{(11s+s^2)/2} = \min_{s \geq 0} e^{(s^2-11s)/2}$$

- Choose $s$ to $\min h(s) = s^2 - 11s \implies s = 5.5$ and

$$P[X \geq 11] \leq e^{(s^2-11s)/2} \bigg|_{s=5.5} = e^{-(5.5)^2/2} = 2.7 \times 10^{-7}$$