16:332:542 **Information Theory and Coding** **Final Examination**

May 5, 2005

This is an 180 minute exam. Please answer the following questions in the notebooks provided. This is a closed book test. Make sure that you have included your name, personal 4 digit code (unrelated to your RU ID digits) and signature in each book used (*5 points*). Read each question carefully. All statements must be justified. Computations should be simplified as much as possible.

1. *20 points* Let $X_1, Z_1, Z_2, \ldots$ be iid Bernoulli random variables which take values 0 and 1 with equal probability. Define the sequence of random variables X_i as

$$X_{i+1} = X_i + Z_i, \qquad i = 1, 2, \ldots, n-1.$$

   Find the mutual information $I(X_1; X_2, X_3, \ldots, X_n)$.

2. *35 points* Let Z take values 0 and 1 with probabilities $1-p$ and p. Let X, which is independent of Z, take values $1, 2, \ldots, n$ with probabilities $\mathbf{q} = [q_1, q_2, \ldots, q_n]$. Let

$$Y = XZ.$$

   (a) *10 points* Find the entropy of Y in terms of the entropies of X and Z.

   (b) *10 points* Find the p and $\mathbf{q}$ that maximize $H(Y)$.

   (c) *15 points* Suppose X and Y are the input and output of a discrete memoryless channel. For fixed p, what is the capacity $C(p)$ of the channel? What value of p maximizes $C(p)$?

3. *40 points* A discrete memoryless multiple access channel has inputs X_1 and X_2 and output $Y = X_1 + X_2$. The inputs X_1 and X_2 both use alphabet $\mathcal{X} = \{0, 1, 2\}$; the output Y has alphabet $\mathcal{Y} = \{0, 1, \ldots, 4\}$.

   (a) *20 points* Under the assumption that each X_i uses equiprobable inputs, find and sketch the (R_1, R_2) region of achievable rates for this 2-user MAC.

   (b) *20 points* Suppose user 1 and user 2 collaborate and act as single transmitter of rate R with input $\mathbf{X} = (X_1, X_2)$ and output Y. What is the capacity of the channel? What input distribution achieves capacity?

4. *20 points* Consider two parallel channels with independent Gaussian noise Z_1 and Z_2 with variances $N_1 = 1$ and $N_2 = 2$. The signalling is

$$Y_1 = X_1 + Z_1$$
$$Y_2 = X_2 + Z_2$$

   The transmitter is subject to the power constraint $E\left[X_1^2 + X_2^2\right] \le \overline{P}$. Find and sketch the capacity $C(\overline{P})$ of this channel as a function of $\overline{P}$.

5. *55 points* Every coding theorem we proved this semester included a converse that was proven using the Fano bound. For example, in the case of a discrete, memoryless channel, for any sequence of $(2^{nR}, n)$ codes with message index X, codewords $X^n(W)$, receiver output Y^n, decoding function $g(Y^n)$, and error probability $P_e^{(n)} = P\left[W \neq g(Y^n)\right]$, the proof used these steps:

$$\begin{aligned} nR &= H(W) && (1) \\ &= H(W|Y^n) + I(W;Y^n) && (2) \\ &\leq H(W|Y^n) + I(X^n(W);Y^n) && (3) \\ &\leq 1 + P_e^{(n)} nR + I(X^n(W);Y^n) && (4) \\ &\leq 1 + P_e^{(n)} nR + nC && (5) \end{aligned}$$

(a) *25 points* For each of the above steps, (1) through (5), there is a specific reason that step holds. Given a precise justification for each of the five steps above.

(b) *10 points* Explain how step (5) implies a converse to the coding theorem.

(c) *20 points* For one of the above five steps, the correct reason is simply "the Fano bound" or "Fano's inequality." *Derive the Fano bound as used in the above five step proof.* Hint: The proof defines the error event

$$E = \begin{cases} 1 & g(Y^n) \neq W \\ 0 & g(Y^n) = W \end{cases}$$

and then expands $H(E, W|Y^n)$ in two different ways.