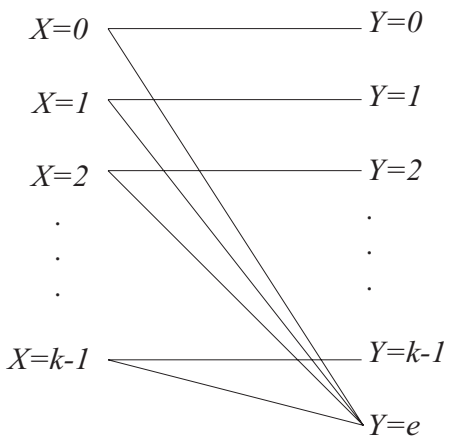
16:332:542 **Information Theory and Coding** **Final Examination**

May 8, 2002

This is an 180 minute exam. Please answer the following four questions in the notebooks provided. You are permitted to look at the Cover& Thomas text but not other materials. Make sure that you have included your name, ID number (last 4 digits only) and signature in each book used (*5 points*). Read each question carefully. All statements must be justified. Computations should be simplified as much as possible.

1. *20 points* Consider the following k input and $k+1$ output discrete erasure channel:



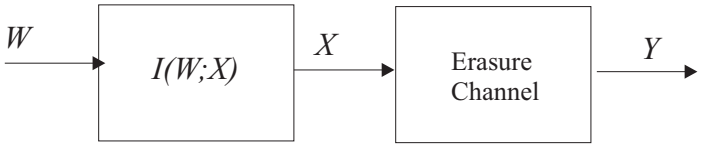
   (a) For a given input distribution $p(x)$, what is the mutual information $I(X;Y)$? (Express your answer in terms of $H(X)$)

   (b) Define the random variable E as

$$E = \begin{cases} 1 & Y = e \\ 0 & \text{otherwise} \end{cases}$$

   What are $H(Y, E)$ and $H(Y, E|X)$?

   (c) Suppose an arbitrary j input, k output channel from W to X is followed in cascade by the X, Y erasure channel from part (a) as follows:



   What is $I(W;Y)$? Your answer should be expressed in terms of $I(W;X)$. Hint: consider the auxiliary random variable E.

2. *20 points* Consider a channel consisting of two parallel AWGN channels with inputs X_1, X_2 and outputs

$$Y_1 = X_1 + Z_1$$
$$Y_2 = X_2 + Z_2$$

   The noises Z_1 and Z_2 are independent and have variances N_1 and N_2 with $N_1 < N_2$. However, we are constrained to use the same symbol on both channels, i.e. $X_1 = X_2 = X$, where X is constrained to have power $E\left[X^2\right] = P$.

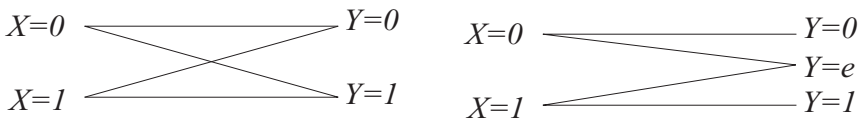
(a) Suppose at the receiver, we combine the outputs to produce $Y = Y_1 + Y_2$? What is the capacity C_1 of channel with input X and output Y? What type of signaling achieves this capacity?

(b) Suppose the receiver can view both outputs Y_1 and Y_2. What is the capacity C_2 of this system? Does the optimal signaling change from part (a)?

(c) Suppose the receiver must combine the two received signals to produce $Y' = \alpha Y_1 + (1-\alpha)Y_2$ where $0 \le \alpha \le 1$. However, as the receiver designer, you can choose the best α for combining. What is the capacity C' of this system with input X and output Y'? Is there a loss in capacity relative to C_2?

(d) Suppose the transmitter, is still constrained to transmit the same signal on both channels, but can choose how much power to use on each channel. That is, for constants a and b, $X_1 = aX$ and $X_2 = bX$. Subject to a constraint that the total transmitted power is bounded by $2P$, what are the optimal a and b and coresponding capacity C'' of the system with outputs (Y_1, Y_2)?

3. Consider the binary symmetric channel and the binary erasure channel shown below:

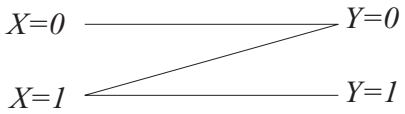


BSC BEC

(a) Find the capacity $C_{\text{BSC}}(\epsilon)$ of the BSC and $C_E(\delta)$ of the erasure channel.

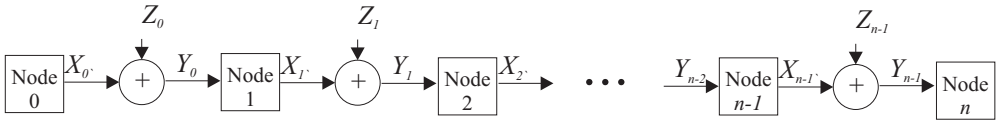
(b) When $\delta = \epsilon$, use the data processing theorem to prove that the BEC has higher capacity than the BSC.

(c) Consider the Z-channel:



Use the data processing theorem to find an upper bound and a lower bound to the capacity $C_Z(\alpha)$ of the z channel. Express these bounds in terms of $C_{\text{BSC}}(\cdot)$ and $C_E(\cdot)$.

4. Suppose we have a wireless network with n hops. For $i = 0, 1, \ldots, n-1$, node i transmits coded messages at rate R_i to node $i+1$ over an AWGN channel with noise variance N_i:



Assume each node transmits in an orthogonal channel. Node i decodes messages the transmission of node $i-1$ and forwards to node $i+1$. Note that the nodes may or may not use different coding strategies. Node i transmits at power P_i and the multihop system is subject to the constraint $\sum_{i=0}^{n-1} P_i = P$.

(a) In terms of P_i and N_i, what is the capacity C_i of the channel i from node i to node $i+1$? (Yes, this is a gift.)

(b) For a given set of powers $P_0, \ldots, P_n$, what is the capacity of C of the multihop communication system from node 0 to node n? Express your answer in terms of C_i. Explain your answer in terms of the end-to-end data rate R.

(c) What is the optimal power allocation $P_0, \ldots, P_{n-1}$? What is the corresponding channel capacity C?