332:542 **Information Theory and Coding** **Examination 1**

March 24, 2005

This is an 80 minute exam. You may have an additional 100 minutes to answer the following questions in the notebooks provided. The exam is closed book. Make sure that you have included your name, your personal random 4 digit code number and signature in each book used (*5 points*). Read each question carefully. All statements must be justified. Computations should be simplified as much as possible.

1. *25 points* Each day, it rains (event $R = 1$) or not (event $R = 0$). A TV station subscribes to a weather forecasting service which delivers a prediction: $Q = 1$ if the prediction is rain, or $Q = 0$ if no rain. Each day, the TV weatherman makes the weather announcement $A = Q$. Fortunately, Q and R are not independent and have the following PMF

$P_{R,Q}(r, q)$	$q = 1$	$q = 0$
$r = 1$	$1/8$	$1/16$
$r = 0$	$3/16$	$10/16$

   (a) *10 points* A student observes that the weatherman is correct with probability $12/16$ but could be correct with probability $13/16$ by always making the weather announcement $A = 0$, corresponding to "no rain." The student applies for the weatherman's job, but the boss, who is an information theorist, turns him down. Why?

   (b) *10 points* The prediction Q is based on a maximum likelihood (ML) hypothesis test (using some unspecified observations X) as to whether $R = 0$ or $R = 1$. For what values of $p = P(R = 1)$ does the weatherman's announcement $A = Q$ also maximize the probability $P(C)$ of a correct prediction based on Q?

   (c) *5 points* Was it necessary in the preceding step to specify that the prediction Q was based on maximum likelihood?

2. *30 points* Let X, Y, Z be an ensemble of discrete random variables. In each of the following problems, there exists an equality or inequality between the two quantities. Fill in the blank $\underline{\quad ? \quad}$ with the appropriate relationship ($\leq$, $=$, or $\geq$) and justify the correctness of that relationship.

   (a) $I(X, Y; Z)$ $\underline{\quad ? \quad}$ $I(X; Z)$

   (b) $H(X|Z)$ $\underline{\quad ? \quad}$ $H(X, Y|Z)$

   (c) $H(X, Z) - H(X)$ $\underline{\quad ? \quad}$ $H(X, Y, Z) - H(X, Y)$

3. *30 points* The process $X_1, X_2, \ldots$ is an iid Bernoulli (p) random sequence. Let $R_n = (X_1 + \cdots + X_n)/n$ denote the success rate of the process.

   (a) In terms of R_n, characterize the set $A_\epsilon^{(n)}$ of typical sequences.

   (b) When $p = 1/2$, what sequences are typical?

   (c) For what values of $p > 1/2$, if any, does $A_\epsilon^{(n)}$ include the most probable sequence? For such p, does $A_\epsilon^{(n)}$ include the most probably sequence *for all* $\epsilon > 0$?

4. *20 points* Consider the code $\{0, 10, 01\}$ for a ternary source. Justify your answers to the following questions:

   (a) *5 points* Is the code instantaneous?

   (b) *5 points* Is the code nonsingular?

   (c) *5 points* Is the code uniquely decodable?

(d) *5 points* Is there an instantaneous code with the same codeword lengths? If so, find an example of such a code.

5. *50 points* The outcome of a roulette wheel is either red $X = 1$ or black $X = 0$, equiprobably and independently from spin to spin. By observing the ball until the last instant that bets can be placed, a gambler can predict X with some accuracy. Given the gambler's prediction, $Y = 0$ or $Y = 1$, conditional probabilities for X are given by

$$P_{X|Y}(1|1) = P_{X|Y}(0|0) = 3/4.$$

(a) Calculate the mutual information $I(X;Y)$.

(b) The gambler has some initial capital C_0. On each spin, she bets a fraction $1 - q$ of her total capital on the predicted color and a fraction q on the other color. Let $Z_n = 1$ if the gambler's prediction is correct on trial n. After N spins, the gamblers capital is the random variable C_N. Express C_N in terms of $Z_1, \ldots, Z_N$.

(c) Find q_C^*, the value of q that maximizes the expected value $E[C_N]$.

(d) Define the rate of growth as

$$R_N = \frac{1}{N} \log_2 \frac{C_N}{C_0}.$$

Find q_R^*, the value of q that maximizes the expected value $E[R_N]$. For $q = q_R^*$, compare $E[R_N]$ and $I(X;Y)$.

(e) If you were the gambler, would you use $q = q_R^*$ or $q = q_C^*$? Explain why.