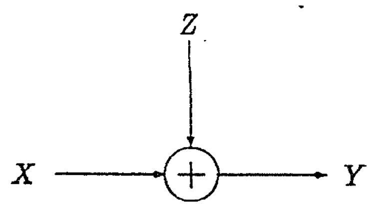
Chapter 10

The Gaussian Channel

1. *A mutual information game.* Consider the following channel:



Throughout this problem we shall constrain the signal power

$$EX = 0, \qquad EX^2 = P, \tag{10.1}$$

and the noise power

$$EZ = 0, \qquad EZ^2 = N, \tag{10.2}$$

and assume that X and Z are independent. The channel capacity is given by $I(X;X+Z)$.

Now for the game. The noise player chooses a distribution on Z to minimize $I(X;X+Z)$, while the signal player chooses a distribution on X to maximize $I(X;X+Z)$.

Letting $X^* \sim \mathcal{N}(0,P), Z^* \sim \mathcal{N}(0,N)$, show that X^* and Z^* satisfy the saddlepoint conditions

$$I(X;X+Z^*) \le I(X^*;X^*+Z^*) \le I(X^*;X^*+Z). \tag{10.3}$$

Thus

$$\min_Z \max_X I(X;X+Z) = \max_X \min_Z I(X;X+Z) \tag{10.4}$$

$$= \frac{1}{2}\log\left(1+\frac{P}{N}\right), \tag{10.5}$$

and the game has a value. In particular, a deviation from normal for either player worsens the mutual information from that player's standpoint. Can you discuss the implications of this?

Note: Part of the proof hinges on the entropy power inequality from Section 16.7, which states that if $\mathbf{X}$ and $\mathbf{Y}$ are independent random n-vectors with densities, then

$$e^{\frac{2}{n}h(\mathbf{X}+\mathbf{Y})} \geq e^{\frac{2}{n}h(\mathbf{X})} + e^{\frac{2}{n}h(\mathbf{Y})}. \tag{10.6}$$

Solution: *A mutual information game.*

Let X and Z be random variables with $EX = 0$, $EX^2 = P$, $EZ = 0$ and $EZ^2 = N$. Let $X^* \sim \mathcal{N}(0, P)$ and $Z^* \sim \mathcal{N}(0, N)$. Then as proved in class,

$$\begin{aligned}
I(X; X+Z^*) &= h(X+Z^*) - h(X+Z^*|X) && (10.7)\\
&= h(X+Z^*) - h(Z^*) && (10.8)\\
&\leq h(X^*+Z^*) - h(Z^*) && (10.9)\\
&= I(X^*; X^*+Z^*), && (10.10)
\end{aligned}$$

where the inequality follows from the fact that given the variance, the entropy is maximized by the normal.

To prove the other inequality, we use the entropy power inequality,

$$2^{2h(X+Z)} \leq 2^{2h(X)} + 2^{2h(Z)}. \tag{10.11}$$

Let

$$g(Z) = \frac{2^{2h(Z)}}{2\pi e}. \tag{10.12}$$

Then

$$\begin{aligned}
I(X^*; X^*+Z) &= h(X^*+Z) - h(X^*+Z|X^*) && (10.13)\\
&= h(X^*+Z) - h(Z) && (10.14)\\
&\geq \frac{1}{2}\log\left(2^{2h(X^*)} + 2^{2h(Z)}\right) - h(Z) && (10.15)\\
&= \frac{1}{2}\log\left((2\pi e)P + (2\pi e)g(Z)\right) - \frac{1}{2}\log(2\pi e)g(Z) && (10.16)\\
&= \frac{1}{2}\log\left(1 + \frac{P}{g(Z)}\right), && (10.17)
\end{aligned}$$

where the inequality follows from the entropy power inequality. Now $1 + \frac{P}{g(Z)}$ is a decreasing function of $g(Z)$, it is minimized when $g(Z)$ is maximum, which occurs when $h(Z)$ is maximized, i.e., when Z is normal. In this case, $g(Z^*) = N$ and we have the following inequality,

$$I(X^*; X^*+Z) \geq I(X^*; X^*+Z^*). \tag{10.18}$$

Combining the two inequalities, we have

$$I(X;X+Z^*) \leq I(X^*;X^*+Z^*) \leq I(X^*;X^*+Z). \tag{10.19}$$

Hence, using these inequalities, it follows directly that

$$\begin{aligned}
\min_Z \max_X I(X;X+Z) &\leq \max_X I(X;X+Z^*) && (10.20)\\
&= I(X^*;X^*+Z^*) && (10.21)\\
&= \min_Z I(X^*;X^*+Z) && (10.22)\\
&\leq \max_X \min_Z I(X^*;X^*+Z). && (10.23)
\end{aligned}$$

We have shown an inequality relationship in one direction between $\min_Z \max_X I(X;X+Z)$ and $\max_X \min_Z I(X;X+Z)$. We will now prove the inequality in the other direction is a general result for all functions of two variables.

For any function $f(a,b)$ of two variables, for all b, for any a_0,

$$f(a_0,b) \geq \min_a f(a,b). \tag{10.24}$$

Hence

$$\max_b f(a_0,b) \geq \max_b \min_a f(a,b). \tag{10.25}$$

Taking the minimum over a_0, we have

$$\min_{a_0} \max_b f(a_0,b) \geq \min_{a_0} \max_b \min_a f(a,b). \tag{10.26}$$

or

$$\min_a \max_b f(a,b) \geq \max_b \min_a f(a,b). \tag{10.27}$$

From this result,

$$\min_Z \max_X I(X;X+Z) \geq \max_X \min_Z I(X;X+Z). \tag{10.28}$$

From (10.23) and (10.28), we have

$$\begin{aligned}
\min_Z \max_X I(X;X+Z) &= \max_X \min_Z I(X;X+Z) && (10.29)\\
&= \frac{1}{2}\log\left(1+\frac{P}{N}\right). && (10.30)
\end{aligned}$$

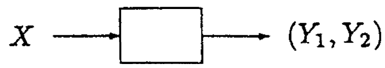
This inequality implies that we have a saddlepoint in the game, which is the value of the game. If signal player chooses X^*, the noise player cannot do any better than choosing Z^*. Similarly, any deviation by the signal player from X^* will make him do worse, if the noise player has chosen Z^*. Any deviation by either player will make him do worse.

Another implication of this result is that not only is the normal the best possible signal distribution, it is the worst possible noise distribution.

2. *A channel with two independent looks at Y.* Let Y_1 and Y_2 be conditionally independent and conditionally identically distributed given X.

   (a) Show $I(X;Y_1,Y_2) = 2I(X;Y_1) - I(Y_1;Y_2)$.

   (b) Conclude that the capacity of the channel



   is less than twice the capacity of the channel

Solution: *A channel with two independent looks at Y. Channel with two independent looks at Y .*

(a)

$$\begin{aligned} I(X;Y_1,Y_2) &= H(Y_1,Y_2) - H(Y_1,Y_2|X) && (10.31)\\ &= H(Y_1) + H(Y_2) - I(Y_1;Y_2) - H(Y_1|X) - H(Y_2|X) && (10.32)\\ &\quad \text{(since } Y_1 \text{ and } Y_2 \text{ are conditionally independent given} && (10.33)\\ &= I(X;Y_1) + I(X;Y_2) - I(Y_1;Y_2) && (10.34)\\ &= 2I(X;Y_1) - I(Y_1;Y_2) \quad \text{(since } Y_1 \text{ and } Y_2 \text{ are conditionally identically distributed)} && (10.35) \end{aligned}$$

(b) The capacity of the single look channel $X \to Y_1$ is

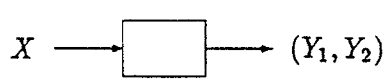
$$C_1 = \max_{p(x)} I(X;Y_1). \qquad (10.36)$$

The capacity of the channel $X \to (Y_1, Y_2)$ is

$$\begin{aligned} C_2 &= \max_{p(x)} I(X;Y_1,Y_2) && (10.37)\\ &= \max_{p(x)} 2I(X;Y_1) - I(Y_1;Y_2) && (10.38)\\ &\le \max_{p(x)} 2I(X;Y_1) && (10.39)\\ &= 2C_1. && (10.40) \end{aligned}$$

Hence, two independent looks cannot be more than twice as good as one look.

3. *The two-look Gaussian channel.*



Consider the ordinary Shannon Gaussian channel with two correlated looks at X, i.e., $Y = (Y_1, Y_2)$, where

$$Y_1 = X + Z_1 \tag{10.41}$$
$$Y_2 = X + Z_2 \tag{10.42}$$

with a power constraint P on X, and $(Z_1, Z_2) \sim \mathcal{N}_2(\mathbf{0}, K)$, where

$$K = \begin{bmatrix} N & N\rho \\ N\rho & N \end{bmatrix}. \tag{10.43}$$

Find the capacity C for

(a) $\rho = 1$

(b) $\rho = 0$

(c) $\rho = -1$

Solution: *The two look Gaussian channel.*

It is clear that the input distribution that maximizes the capacity is $X \sim \mathcal{N}(0, P)$. Evaluating the mutual information for this distribution,

$$\begin{aligned} C_2 &= \max I(X; Y_1, Y_2) && (10.44)\\ &= h(Y_1, Y_2) - h(Y_1, Y_2|X) && (10.45)\\ &= h(Y_1, Y_2) - h(Z_1, Z_2|X) && (10.46)\\ &= h(Y_1, Y_2) - h(Z_1, Z_2) && (10.47) \end{aligned}$$

Now since

$$(Z_1, Z_2) \sim \mathcal{N}\left(0, \begin{bmatrix} N & N\rho \\ N\rho & N \end{bmatrix}\right), \tag{10.48}$$

we have

$$h(Z_1, Z_2) = \frac{1}{2}\log(2\pi e)^2 |K_Z| = \frac{1}{2}\log(2\pi e)^2 N^2(1-\rho^2). \tag{10.49}$$

Since $Y_1 = X + Z_1$, and $Y_2 = X + Z_2$, we have

$$(Y_1, Y_2) \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} P+N & P+\rho N \\ P+\rho N & P+N \end{bmatrix}\right), \tag{10.50}$$

and

$$h(Y_1, Y_2) = \frac{1}{2}\log(2\pi e)^2 |K_Y| = \frac{1}{2}\log(2\pi e)^2 (N^2(1-\rho^2) + 2PN(1-\rho)). \tag{10.51}$$

Hence the capacity is

$$\begin{aligned} C_2 &= h(Y_1, Y_2) - h(Z_1, Z_2) && (10.52)\\ &= \frac{1}{2}\log\left(1 + \frac{2P}{N(1+\rho)}\right). && (10.53) \end{aligned}$$

(a) $\rho = 1$. In this case, $C = \frac{1}{2}\log(1+\frac{P}{N})$, which is the capacity of a single look channel. This is not surprising, since in this case $Y_1 = Y_2$.

(b) $\rho = 0$. In this case,

$$C = \frac{1}{2}\log\left(1+\frac{2P}{N}\right), \tag{10.54}$$

which corresponds to using twice the power in a single look. The capacity is the same as the capacity of the channel $X \to (Y_1 + Y_2)$.

(c) $\rho = -1$. In this case, $C = \infty$, which is not surprising since if we add Y_1 and Y_2, we can recover X exactly.

Note that the capacity of the above channel in all cases is the same as the capacity of the channel $X \to Y_1 + Y_2$.

4. *Parallel channels and waterfilling.* Consider a pair of parallel Gaussian channels, i.e.,

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \tag{10.55}$$

where

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim \mathcal{N}\left(0, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}\right), \tag{10.56}$$

and there is a power constraint $E(X_1^2 + X_2^2) \le 2P$. Assume that $\sigma_1^2 > \sigma_2^2$. At what power does the channel stop behaving like a single channel with noise variance σ_2^2, and begin behaving like a pair of channels?

Solution: *Parallel channels and waterfilling.* By the result of Section 10.4, it follows that we will put all the signal power into the channel with less noise until the total power of noise + signal in that channel equals the noise power in the other channel. After that, we will split any additional power evenly between the two channels.

Thus the combined channel begins to behave like a pair of parallel channels when the signal power is equal to the difference of the two noise powers, i.e., when $2P = \sigma_1^2 - \sigma_2^2$.