Water Filling May Not Good Neighbors Make

Otilia Popescu & Christopher Rose
Department of Electrical and
Computer Engineering
WINLAB, Rutgers University
73 Brett Rd.,Piscataway, NJ 08854-8060
Email: otilia/crose@winlab.rutgers.edu

***Abstract*— Consider a wireless system with multiple and independent user-base pairings over some region. Each user tries to greedily optimize its performance, and eventually a simultaneous water-filling fixed point is reached. Here we seek to analytically understand properties of different water-filling fixed points. In so doing we show that water-filling alone does not generally result in optimum resource sharing and in some cases is a poor solution. We close by suggesting dynamic strategies for performance enhancement.**

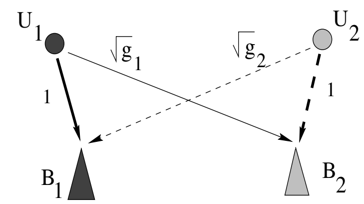

Fig. 1. A system with two transmitters and two receivers

I. Introduction

A multi-base system consists of a collection of transmitters (users) and receivers (bases) distributed over a given region. Users are assigned to particular bases, comparable to the usual cellular system. As in all such systems, despite assignment of users to bases, the spectrum is shared by everyone and thus everyone will interfere with one another to some extent.

We call a multi-base system *collaborative* if the information received at all bases is collectively available for decoding. We note that a similar collaborative approach has been used previously for systems with multiple receivers [1], [2]. In this scenario the social optimum, corresponding to the maximum sum capacity, is known and can be achieved with greedy procedures such as interference avoidance and iterative water-filling [3], [4].

On the other hand, when no cooperation is assumed, we call the system *non-collaborative*, and a given user is decoded at its associated base under interference from transmissions by other users intended for other bases. Such systems with Gaussian channels, multiple transmitters and multiple receivers and no cooperation are called *interference channels* (an example is depicted in Figure 1). Unfortunately, complete characterization of the capacity region for the interference channel has eluded information theorists for the past few decades [5]. Furthermore, the level of cooperation (codebook sharing) necessary to achieve capacity even in known cases could be problematic in a practical setting. We make no attempt to address these thorny issues here. Rather, we simplify the problem in the usual way by treating interference as Gaussian noise.

In [6] "simultaneous water-filling" was shown to be a Nash equilibrium for this problem. Here we analyze simultaneous water-filling in detail and derive relationships between the physical distribution of users and bases (characterized by user-base gains), the set of possible fixed points. Further, we identify feasible morphologies for user spectral distributions and also show that water-filling does not necessarily result in efficient resource sharing. We close by suggesting self-interest driven methods to move the system toward better performance.

II. Problem Statement

We consider the simplest mutually interfering communication system with two users (transmitters) and two base stations (receivers), as depicted in Figure 1. We assume an arbitrary signal space of dimension N in which each user $\ell = 1, 2$ has a codeword matrix $\mathbf{S}_\ell$ whose columns are individual codewords $\mathbf{s}_j^{(\ell)}$, $j = 1, ...N$. We assume that each user uses as many codewords as signal space dimensions to allow transmit covariance matrices $\mathbf{X}_\ell = \mathbf{S}_\ell \mathbf{S}_\ell^\top$ to span the entire signal space. Thus, $\mathbf{S}_\ell$ is $N \times N$ in general. An N-dimensional vector of information symbols $\mathbf{b}_\ell$ is transmitted by each user ℓ. Each codeword has unit norm and this implies the following trace constraints $\text{Tr}[\mathbf{X}_1] = \text{Tr}[\mathbf{S}_1\mathbf{S}_1^\top] = P_1 = N$ and $\text{Tr}[\mathbf{X}_2] = \text{Tr}[\mathbf{S}_2\mathbf{S}_2^\top] = P_2 = N$, which also represent user power constraints.

Furthermore, we normalize the system and assume that each user is received at the base to which it is assigned with unit power, and that all the codewords of a given user are affected by the same channel, which is flat and characterized by gain factor g_ℓ. Hence, the received vectors at each base station are

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{S}_1\mathbf{b}_1 + \sqrt{g_2}\mathbf{U}_{21}\mathbf{S}_2\mathbf{b}_2 + \mathbf{n}_1 \\ \mathbf{r}_2 &= \sqrt{g_1}\mathbf{U}_{21}\mathbf{S}_1\mathbf{b}_1 + \mathbf{S}_2\mathbf{b}_2 + \mathbf{n}_2 \end{aligned} \tag{1}$$

where g_1 is the gain corresponding to user 1's signal at base 2, g_2 is the gain corresponding to user 2's signal at base 1, and the unitary matrices $\mathbf{U}_{ij}$ convert the *bases*[1] between the two users' signal spaces. For simplicity (but with no loss of generality) we will assume $\mathbf{U}_{ij} = \mathbf{I}$. The noise at each base is

[1]Bases as in basis sets.

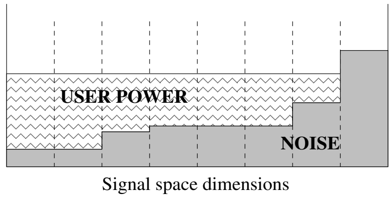

Fig. 2. Single user water-filling

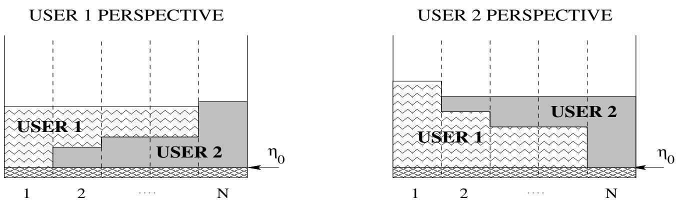

Fig. 3. Simultaneous water-filling

assumed white, with the same covariance matrix $E[\mathbf{n}_1\mathbf{n}_1^\top] = E[\mathbf{n}_2\mathbf{n}_2^\top] = \eta_0\mathbf{I}_N$.

To evaluate the performance of the system we treat the other user as Gaussian interference and thereby obtain capacity expressions for individual users as,

$$\begin{array}{l} C_1 = \frac{1}{2}\log|\mathbf{R}_1| - \frac{1}{2}\log|g_2\mathbf{S}_2\mathbf{S}_2^\top + \eta_0\mathbf{I}_N| \\ C_2 = \frac{1}{2}\log|\mathbf{R}_2| - \frac{1}{2}\log|g_1\mathbf{S}_1\mathbf{S}_1^\top + \eta_0\mathbf{I}_N| \end{array} \tag{2}$$

where $\mathbf{R}_1$ and $\mathbf{R}_2$ are the covariance matrices at each base and are expressed as $\mathbf{R}_1 = \mathbf{S}_1\mathbf{S}_1^\top + g_2\mathbf{S}_2\mathbf{S}_2^\top + \eta_0\mathbf{I}_N$ and $\mathbf{R}_2 = g_1\mathbf{S}_1\mathbf{S}_1^\top + \mathbf{S}_2\mathbf{S}_2^\top + \eta_0\mathbf{I}_N$ respectively.

III. Simultaneous Water-filling Structures

Our aim is to have systems operate without centralized control, so we prefer iterative algorithms which can be applied greedily by individual users. It was proved in [6] that an iterative water filling procedure converges to a fixed point which is a Nash equilibrium for the system. These points have the property that users are simultaneously water-filled from their own perspectives. In general this equilibrium point is not unique [6] and the algorithm may converge to different equilibria.

There are a variety of algorithms which can be used to achieve such simultaneously water-filled solutions. In our simulations we used an interference avoidance algorithm [7], [8]. However, in this paper we will focus not on the specific algorithm, since many distributed algorithms will result in simultaneously water-filled solutions. Rather, we concentrate on the water-filling structures themselves and their performance.

At this point it is useful to review the notion of water-filling. From a single user's point of view, we will regard interference from other users as Gaussian noise. That user's capacity will be maximized when its energy is distributed over the signal space in a water-filling manner as illustrated in Figure 2.

In a two-user system, the simultaneous water-filling structure is illustrated in Figure 3, where each user water-fills in the signal space that it sees. In our fixed point analysis we examine

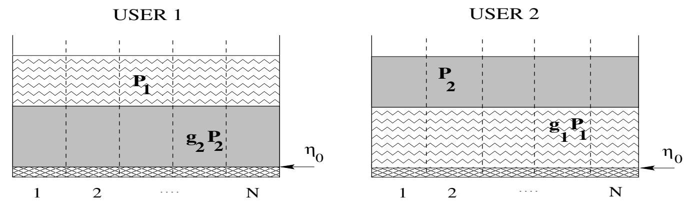

Fig. 4. Complete overlap between users.

the transmit covariance matrices of the two users expressed using the spectral factorization theorem as

$$\begin{array}{l} \mathbf{X}_1 = \mathbf{S}_1\mathbf{S}_1^\top = \sum_{i=1}^{N} \alpha_i\psi_i\psi_i^\top = \mathbf{\Psi}\mathbf{A}\mathbf{\Psi}^\top \\ \mathbf{X}_2 = \mathbf{S}_2\mathbf{S}_2^\top = \sum_{i=1}^{N} \beta_i\phi_i\phi_i^\top = \mathbf{\Phi}\mathbf{B}\mathbf{\Phi}^\top \end{array} \tag{3}$$

where $\mathbf{\Psi}$ and $\mathbf{\Phi}$ are the matrices of eigenvectors $\{\psi_i\}_{i=1}^N$ and $\{\phi_i\}_{i=1}^N$ for $\mathbf{X}_1$ and $\mathbf{X}_2$, respectively. $\mathbf{A}$ and $\mathbf{B}$ are the corresponding diagonal eigenvalue matrices, with eigenvalues $\alpha_1 \geq \ldots \geq \alpha_N$ and $\beta_1 \leq \ldots \leq \beta_N$, in decreasing and increasing order, respectively. We note that eigenvalues are real and non-negative since the $\mathbf{X}_i$ are covariance matrices.

From the perspective of a given user, the other user acts as interference, and in order to achieve minimum mutual interference the eigenvectors of its transmit covariance matrix must align with the eigenvectors of the interference covariance matrix [7], [9], [10]. In addit ion, the eigenvalues of the transmit covariance matrix must satisfy a water filling distribution over those eigenvectors with minimum eigenvalues. From a codeword perspective, this implies that user codewords must reside in the subspace spanned by the eigenvectors of the other user's covariance matrix with smallest eigenvalues. Three signal space configurations are possible at a fixed point.

A. Complete Overlap

Suppose both user codeword matrices span the signal space. Water filling requires that matrices $\mathbf{R}_1$ and $\mathbf{R}_2$ be scaled identity matrices

$$\begin{array}{l} \mathbf{R}_1 = \mathbf{X}_1 + g_2\mathbf{X}_2 + \eta_0\mathbf{I}_N = c_1\mathbf{I}_N \\ \mathbf{R}_2 = \mathbf{X}_2 + g_1\mathbf{X}_1 + \eta_0\mathbf{I}_N = c_2\mathbf{I}_N \end{array} \tag{4}$$

where the "water levels" c_1 and c_2 are obtained using the previously mentioned trace constraints as $c_1 = 1 + g_2 + \eta_0$ and $c_2 = 1 + g_1 + \eta_0$.

The solution to this system of equations is $\mathbf{X}_1 = \mathbf{X}_2 = \mathbf{I}_N$, and either of the eigenvector matrices $\mathbf{\Psi}$ or $\mathbf{\Phi}$ could be used for signal space representation[2] by the users. Since eigenvalues of the transmit covariance matrices for both users are in this case equal to unity, the water filling structure is unique and is given in Figure 4. Each user has a unique "water level" over the whole signal space.

[2] In fact, any orthogonal basis could be used.

B. Incomplete Overlap

Suppose both user codeword matrices span only subspaces of the signal space. We assume that these subspaces are of dimension k_i, with overlap in k_1+k_2-N dimensions between them. From user 1's perspective, the first $N-k_2$ dimensions are not shared, which implies that in matrix $\mathbf{R}_1$ the eigenvalues corresponding to these dimensions must be equal. A similar condition must be valid in these dimensions for matrix $\mathbf{R}_2$. Similarly, because user 2 does not share the last $N-k_1$ dimensions, eigenvalues of $\mathbf{R}_1$ and $\mathbf{R}_2$ in these dimensions must also be equal.

The incomplete overlap between the two users implies that $\mathbf{X}_1$ is orthogonal to the last $N-k_1$ eigenvectors of $\mathbf{X}_2$

$$\mathbf{X}_1 \perp \{\phi_i\} \qquad i = k_1+1, \ldots, N \tag{5}$$

Let us denote $\bar{\mathbf{\Phi}} = [\phi_1 \phi_2 \ldots \phi_{k_1}]$ the k_1 eigenvectors of $\mathbf{X}_2$ with smallest eigenvalues, which span the user 1 codeword matrix. Then, there is an $\bar{\mathbf{S}_1}$ such that $\mathbf{S}_1 = \bar{\mathbf{\Phi}}\bar{\mathbf{S}}_1$ and we can write the covariance matrix at base 1 as

$$\mathbf{R}_1 = \bar{\mathbf{\Phi}}\bar{\mathbf{S}}_1\bar{\mathbf{S}}_1^\top\bar{\mathbf{\Phi}}^\top + g_2\sum_{i=1}^{N}\beta_i\phi_i\phi_i^\top + \eta_0\mathbf{I}_N \tag{6}$$

A projection onto the subspace generated by $\bar{\mathbf{\Phi}}$ leads to

$$\bar{\mathbf{\Phi}}^\top\mathbf{R}_1\bar{\mathbf{\Phi}} = \bar{\mathbf{S}}_1\bar{\mathbf{S}}_1^\top + g_2\begin{bmatrix}\beta_1 & & 0\\ & \ddots & \\ 0 & & \beta_{k_1}\end{bmatrix} + \eta_0\mathbf{I}_{k_1} \tag{7}$$

The water filling solution implies a constant level over the first k_1 dimensions (see $\mathbf{R}_1$ perspective), thus the above expression is further equal to $\mathcal{L}_1\mathbf{I}_{k_1}$, where $\mathcal{L}_1$ is the water level in user 1's subspace and is determined by the constant traces:

$$\mathcal{L}_1 = \left(\frac{\mathrm{Tr}[\mathbf{S}_1\mathbf{S}_1^\top] + g_2\sum_{i=1}^{k_1}\beta_i}{k_1} + \eta_0\right) = \left(\frac{N + g_2\sum_{i=1}^{k_1}\beta_i}{k_1} + \eta_0\right) \tag{8}$$

Thus, we have

$$\bar{\mathbf{S}_1}\bar{\mathbf{S}_1}^\top = \mathcal{L}_1\mathbf{I}_{k_1} - g_2\begin{bmatrix}\beta_1 & & 0\\ & \ddots & \\ 0 & & \beta_{k_1}\end{bmatrix} + \eta_0\mathbf{I}_{k_1} \tag{9}$$

Reconstructing the original codeword covariance matrix, $\mathbf{X}_1 = \mathbf{S}_1\mathbf{S}_1^\top = \bar{\mathbf{\Phi}}\bar{\mathbf{S}_1}\bar{\mathbf{S}_1}^\top\bar{\mathbf{\Phi}}^\top$, we get

$$\mathbf{X}_1 = \sum_{i=1}^{k_1}(\mathcal{L}_1 - g_2\beta_i - \eta_0)\phi_i\phi_i^\top \tag{10}$$

which implies we can represent both $\mathbf{S}_i$ in terms of the $\{\phi_i\}$.

Using a similar line of reasoning, user 2's transmit covariance matrix can be written as

$$\mathbf{X}_2 = \mathbf{S}_2\mathbf{S}_2^\top = \sum_{j=N-k_2+1}^{N}(\mathcal{L}_2 - g_1\alpha_j - \eta_0)\phi_j\phi_j^\top \tag{11}$$

where the level from this perspective is given by

$$\mathcal{L}_2 = \left(\frac{N + g_1\sum_{i=N-k_2+1}^{N}\alpha_i}{k_2} + \eta_0\right) \tag{12}$$

We now consider two cases.

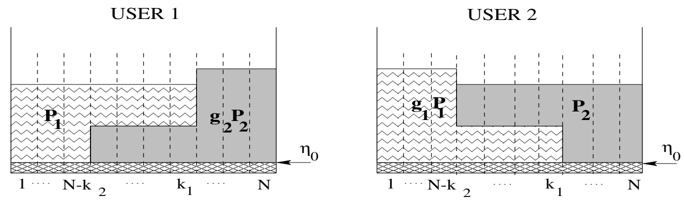

Fig. 5. Incomplete overlap between users

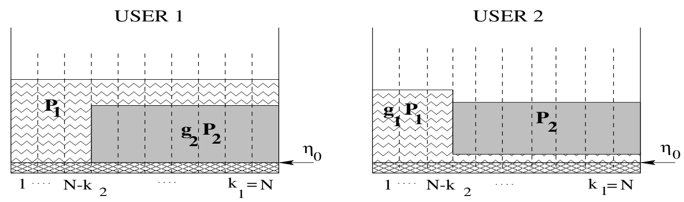

Fig. 6. The "nesting" case

1) k_1, $k_2 < N$: We note that the water level from the perspective of user 1/user 2 is imposed by its largest eigenvalue, α_1/ β_N, and the structure imposes the following constraints:

$$\begin{aligned}&\alpha_1 = \ldots = \alpha_{N-k_2} =\\ &= \alpha_{N-k_2+1} + g_2\beta_{N-k_2+1} = \ldots = \alpha_{k_1} + g_2\beta_{k_1}\\ &\beta_N = \ldots = \beta_{k_1+1} =\\ &= g_1\alpha_{k_1} + \beta_{k_1} = \ldots = g_1\alpha_{N-k_2+1} + \beta_{N-k_2+1}\end{aligned} \tag{13}$$

All the other eigenvalues can be written in terms of the largest eigenvalues, α_1 and β_N,

$$\begin{cases}\alpha_1 = \alpha_2 = \ldots = \alpha_{N-k_2}\\ \alpha_{N-k_2+1} = \ldots = \alpha_{k_1} = \dfrac{g_2\beta_N - \alpha_1}{g_1g_2 - 1}\end{cases} \tag{14}$$

and

$$\begin{cases}\beta_N = \beta_{N-1} = \ldots = \beta_{k_1+1}\\ \beta_{N-k_2+1} = \ldots = \beta_{k_1} = \dfrac{g_1\alpha_1 - \beta_N}{g_1g_2 - 1}\end{cases} \tag{15}$$

Thus, the eigenvalues of each user transmit covariance matrix can have only two values: one for the unshared portion of the signal space and another for the shared portion of the signal space, as depicted in Figure 5.

We then note that the competitive stability of mutual water filling with incomplete overlap implies the following two conditions for the largest eigenvalues of the two users

$$\begin{aligned}\alpha_1 &\le g_2\beta_N\\ \beta_N &\le g_1\alpha_1\end{aligned} \tag{16}$$

which further implies that in order to have incomplete overlap the gains must satisfy $g_1g_2 \ge 1$.

2) $k_1 = N$, $k_2 < N$: With no loss of generality we let user 1's codewords span the signal space and have user 2's codewords occupy a proper subspace. That is, user 2's subspace is completely contained (nested) in user 1's space, Figure 6. The previous derivation still holds with minor changes resulting in

$$g_2\beta_N + \alpha_N = \alpha_1 \tag{17}$$

Overlap	Scenarios		
	$g_1 g_2 > 1$	$g_1 g_2 = 1$	$g_1 g_2 < 1$
Complete	*unique*	*unique*	*unique*
Incomplete	*many*	*many*	-
None	*many*	*unique*	-

TABLE I

FIXED POINT POSSIBILITIES AS A FUNCTION OF USER PLACEMENT.

and

$$\beta_N + g_1 \alpha_N \leq g_1 \alpha_1 \tag{18}$$

which again implies $g_1 g_2 \geq 1$

We note that the signal space dimension pair (k_1, k_2) for incomplete overlap is not unique.

C. No Overlap

Finally, suppose user codeword matrices span orthogonal subspaces of the signal space at a fixed point, as in Figure 7. We assume that user 1 resides in a subspace of dimension k, and that user 2 occupies the remaining $N - k$ dimensions of the signal space. Using the trace constraints we obtain

$$\begin{cases} \alpha_1 = \alpha_2 = \ldots = \alpha_k = \dfrac{N}{k} \\ \alpha_1 \leq g_2 \beta_N \end{cases} \tag{19}$$

and

$$\begin{cases} \beta_{k+1} = \ldots = \beta_{N-1} = \beta_N = \dfrac{N}{N-k} \\ g_1 \alpha_1 \geq \beta_N \end{cases} \tag{20}$$

which implies $g_1 g_2 \geq 1$ and in addition

$$\frac{1}{g_2 + 1} \leq \frac{k}{N} \leq \frac{g_1}{g_1 + 1} \tag{21}$$

The solution for k in general is not unique and as g increases the range of k expands. For the particular case of $g_1 g_2 = 1$ there might be no solution for k in the discrete representation considered, but as N approaches infinity there will always exist a stable separated solution.

D. The Water-Filling Region

The disposition of the inequality $g_1 g_2 \geq 1$ is an indicator for the possibility of any type of overlap between the user signal spaces as summarized in Table I.

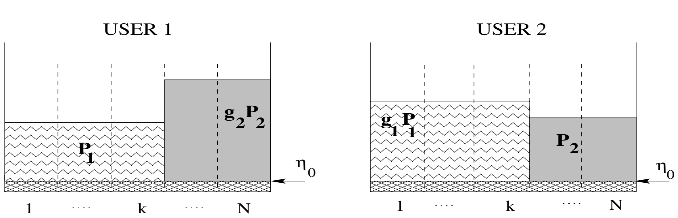

Fig. 7. Complete separation between users

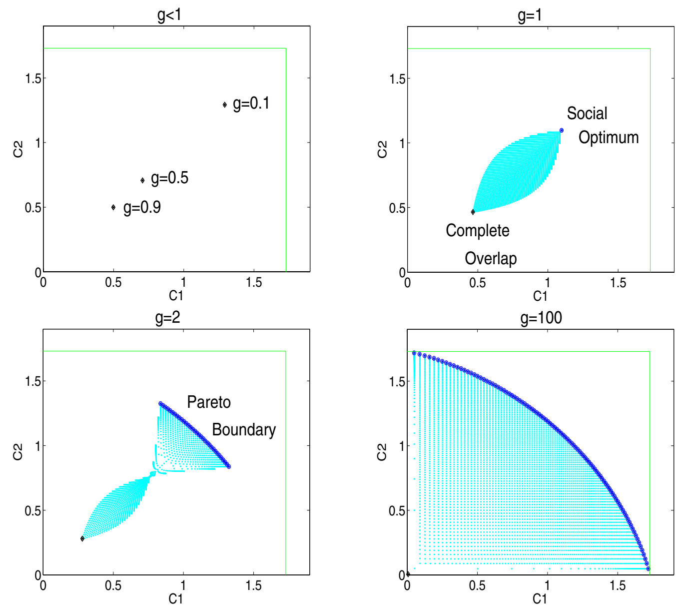

Fig. 8. Water-filling region varies as a function of gain

We define the *water-filling region* as the set of all possible fixed points achievable with simultaneous water-filling, for given user power, gains and signal space dimension. If for each particular fixed point both users compute their capacities (2), the set of all possible pairs (C_1, C_2) describe the water-filling region. Different points in the water-filling region correspond to different sizes of the subspaces in which users reside.

User capacity is a function of the gains (see (2)), and the possible fixed points are function of the gains (see Table 1). To illustrate gain dependence we consider a particular example of a symmetric system with $g_1 = g_2 = g$, $N = 100$, user power $P = N = 100$ and $\eta_0 = 10$. Figure 8 shows the variation of the water-filling region as a function of g. In these figures, for $g \geq 1$, the most interior point corresponds to complete overlap, the interior region corresponds to incomplete overlap, and the outer border corresponds to complete separation. As g increases, the point corresponding to complete overlap moves toward $(0, 0)$ and the border corresponding to complete separation expands with more possible fixed point partitions of signal space between the users.

Without interference ($g = 0$), the capacity region of the system is the rectangle [11] with $C_i = \frac{1}{2}\log(1 + \frac{P_i}{\eta_0})$ (vertices $(0, 0)$ and $(1.73, 1.73)$ in this example). The water-filling region is included in this rectangle. In [11] Carleial showed that for strong interference (which here means $g \geq 1 + P/\eta_0$ or $g \geq 11$) the capacity region of the interference channel coincides with the rectangle. Thus, for strong interference, the maximum of the capacity region is given by $C_1 = C_2 = 1.73$, or a *collective capacity* $C = C_1 + C_2 = 3.46$. A useful point of reference for the water-filling regions is the symmetric capacity point, defined as that point on the outer boundary where $C_1 = C_2$. It corresponds to separation of the users in equal and complementary subspaces, and maximizes the

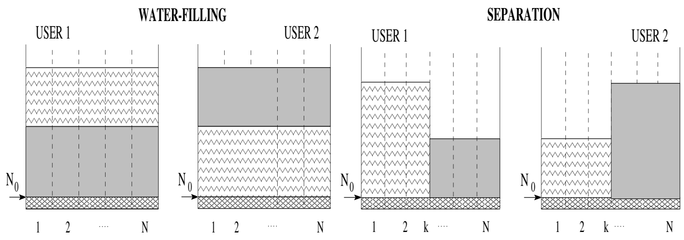

Fig. 9. The case of $g_1 g_2 < 1$

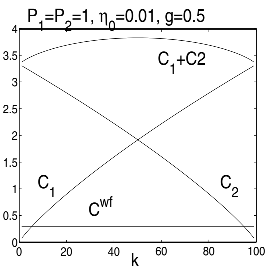

Fig. 10. Capacity variations as a function of user subspace width

collective capacity, $C = 2.196$ ($g > 1$). We define this point as the *social optimum* for $g \geq 1$.

Figure 8 shows three cases of interest. The simplest is for $g = 1$ (or in general $g_1 g_2 = 1$) where there is a single outermost point at which both users maximize their capacities by complete separation. At this point both individual and collective interest are satisfied, since the collective capacity is also maximized, and we define it as the *social optimum* for this case.

For $g_1 g_2 > 1$ the water-filling region displays an outer *pareto* boundary. From a collective point of view the capacity $C = C_1 + C_2$ does not vary radically on the boundary. For the symmetric case considered we define the social optimum for $g_1 g_2 > 1$ as the point on the border with $C_1 = C_2$, which provides identical capacities for each user.

The most natural case is probably $g_1 g_2 < 1$ where users are attached to the closest base and the interference is significantly smaller than the signal power (at least for one of the bases). This case is also more problematic since the mutual water filling solution is a unique and strongly suboptimal point as compared to complete separation. To be more specific, consider Figure 9 which depicts the energy distribution for two cases: 1) a water-filling solution and 2) an equipartitioned signal space in a symmetric system with equal gains $g_i = g$ and user powers $P_i = P$. In Figure 10, k is the width of the subspace occupied by user 1, and varies from 1 to 99 (for $N = 100$). User 1's capacity, C_1, increases monotonically in k whereas user 2's capacity, C_2, decreases monotonically. In the figure C^{wf} represents the capacity of a single user for the fixed point water-filling solution and it can be seen that a large range of partitions gives both users much higher rates. Also, the collective capacity is uniformly high and does not vary more than 12% from it's maximum (the equipartition social optimum) for any possible partition.

IV. Discussion and Conclusion

Using interference gains as surrogates for relative user and base positions, we have analyzed a simple two users/bases system and have identified the spectral morphology of simultaneous water-filling solutions: *completely overlapped* where each user's spectrum is white (Figure 4), *partially overlapped* where each user is white in the overlap region and also white in regions where there is no overlap but not white overall (Figures 5 and 6), and *completely separated* with each user white in their respective subspaces (Figure 7).

We have also found that simultaneous water-filling "may not good neighbors make." That is, though in many cases ($g_1 g_2 \geq 1$) simultaneous waterfilling can result in socially optimal (or at least pareto optimal) resource sharing, without external guidance toward the pareto boundary many strongly suboptimal solutions are possible. Furthermore, in many cases of interest ($g_1 g_2 < 1$) the simultaneous water-filling solution does not even include the pareto boundary defined by complete separation of users in the signal space.

Thus, we are currently investigating dynamic strategies such as "spectrum warfare" where users seek to dispossess one another from portions of the signal space and "tit for tat" [12] where users reward socially acceptable behavior and punish bad actors. Empirically these result in much better performance and will be the subject of future work.

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