Strong Interference and Spectrum Warfare

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***Abstract*— We consider a wireless system with multiple user-base pairs randomly distributed in some region, in which users try to greedily optimize their performance without any exchange of information between bases, and for which a fixed point is reached. This is a Nash equilibrium point for the system and corresponds to a simultaneous water filling solution. In this paper we focus on systems with strong interference for which the simultaneous water filling solution implies not an unique fixed point but a set of fixed points, among which the information theoretic capacities of users vary widely. We propose a dynamic game to move the system from a possible suboptimal point to a better simultaneous water filling fixed point, eventually to the optimal point.**

I. Introduction

In a wireless information network multiple users (transmitters) and base stations (receivers) are distributed over some geographical region, and users are assigned to particular bases as in a typical cellular setup. However, in spite of user assignment to particular bases, the shared nature of the spectrum that characterizes wireless systems implies that all users will interfere with each other leading to an interference channel scenario which is mostly an open research topic (see [3, p. 382] and references therein).

Recent research [7] approaches the Gaussian interference channel from a non-cooperative game theoretic perspective in which users compete for data rates, and each user's objective is greedy performance maximization regardless of other users in the system. From this perspective it is shown [7] that for a system with two users and bases as depicted in Figure 1

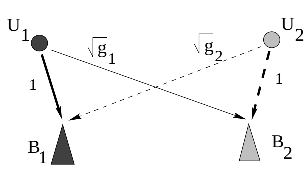
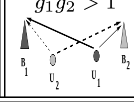
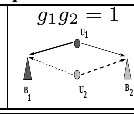
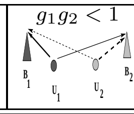


Fig. 1. A system with two users and two base stations. g_1 is the gain corresponding to user 1's signal at base 2, g_2 is the gain corresponding to user 2's signal at base 1. Gains to the own base are normalized to 1.

performance of both users is optimized by a "simultaneous water filling" distribution of user powers, which represents a Nash equilibrium solution for the system.

A detailed analysis of the water filling solution set relating geographic distribution of users and bases (as characterized by user-base gains) with the set of potential Nash equilibria for the system is presented in [6]. Using a signal space approach reference [6] shows that three signal space configurations are possible at a fixed point: complete overlap between users in all the dimensions, partial overlap between users (when only some dimensions of the signal space are shared), and no overlap between users (when users reside in orthogonal subspaces). Table 1 summarizes the results in [6] and relates the number of potential Nash equilibria and overlap scenarios to the relative gains of users to bases.

TABLE I

Overlap	**Equilibrium Points**		
	$g_1g_2>1$ 	$g_1g_2=1$ 	$g_1g_2<1$ 
Complete	*unique*	*unique*	*unique*
Incomplete	*many*	*many*	-
None	*many*	*unique*	-

Reference [6] defines also the *simultaneous water filling region* as the set of all possible fixed points achievable with simultaneous water filling, for given user power, gains and signal space dimension.

In this paper we focus on the case corresponding to $g_1g_2 \geq 1$ which is the "strong interference" case and for which the simultaneous water filling solution corresponds to multiple Nash equilibria. We note that the case $g_1g_2 < 1$ which corresponds to "weak interference" [2] has an unique Nash equilibrium point [6], [7] and is considered in more detail in [5]. We analyze the simultaneous water filling region in the case of strong interference and present a procedure that moves the system from any possible low performance point to a better point, eventually to the optimum point. We use as global performance measure the "collective capacity" [6] to show that the most important improvement in performance is achieved by moving the system on the border of the simultaneous water filling region, corresponding to orthogonal users in signal space. In terms of the collective capacity all points on the border offer comparable performance.

II. Strong Interference and Simultaneous Water Filling

The case $g_1g_2 \geq 1$ corresponds to a physical situation in which the distance from a given user to the base to which it is associated is larger than the distance to the other base for which the given user's transmission is not intended and creates interference. This situation corresponds to a "strong interference" case since usually, with equal user transmitted power, the interfering signal at any base station is stronger than the actual desired signal, and is illustrated in Figure 2.

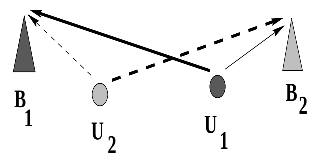


Fig. 2. The "strong interference" case. A user is closer to the base to which it produces interference, than to the base to which its transmission is intended.

Let P_i be the transmitted power for user i and η_i be the spectral height of the additive white Gaussian noise that corrupts the signal received at base station i. If the two users were not interfering then they can transmit at rates

$$R_i \leq C_i = \frac{1}{2}\log\left(1+\frac{P_i}{\eta_i}\right) \qquad i=1,2 \tag{1}$$

The capacity region will be the rectangle defined by the origin $(0,0)$ and points $(0,C_1)$, $(C_2,0)$, and (C_1,C_2), and any rate pair (R_1,R_2) in this region is achievable. Furthermore, Carleial has shown [1] that if interference is strong enough, then it can be subtracted and the capacity region of the considered two-user system with strong interference is identical to the capacity region of the system without interference. This is true when gains satisfy [1]

$$g_i \geq \frac{P_j+\eta_j}{\eta_i} \qquad i,j=1,2 \qquad i\neq j \tag{2}$$

A simultaneous water filling solution for the considered system is satisfied when each user distributes its power according to a traditional water filling allocation [3, p. 253] regarding interference from other users as Gaussian noise. From a game theoretic perspective a simultaneous water filling solution corresponds to a Nash equilibrium for the system and is unique only in the case of "weak interference" which corresponds to $g_1g_2 < 1$ [5]–[7]. In the case of "strong interference", when $g_1g_2 \geq 1$, the simultaneous water filling solution corresponds to multiple Nash equilibria for the system, and the *simultaneous water filling region* was defined [6] as the set of all possible user capacities achievable with simultaneous water filling distribution, for given user power, gains and signal space dimension (as implied by fixed communication bandwidth and signaling interval constraints).

Figure 3 presents a typical simultaneous water filling region for a symmetric system with equal user gains $g_1 = g_2 = g$, signal space dimension $N = 100$, user power $P = 100$ and background noise with $\eta_0 = 10$ at each base. From Figure 3 one can see that in the case of weak interference the water filling region consists of a single point (upper left plot), while in the case of strong interference the region consists of multiple points among which user capacities vary widely. More insight into the water filling structure that corresponds to these points can be found in [6].

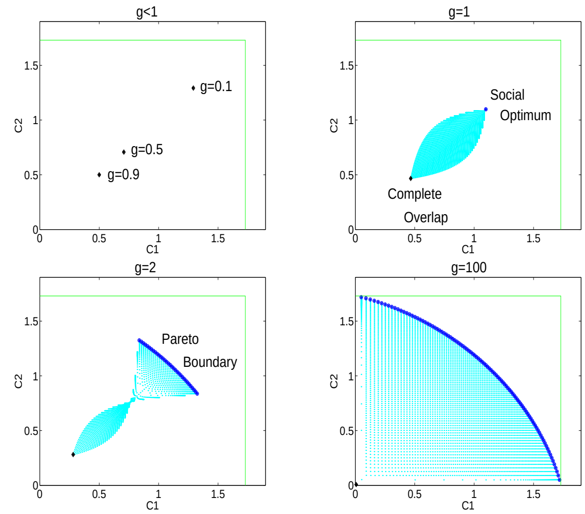


Fig. 3. Simultaneous water filling region for a symmetric system.

We note that the water filling region is inside the rectangular capacity region with no interference, and that in the case of "strong interference", when $g_1g_2 \geq 1$, the simultaneous water filling region consists of multiple points among which information theoretic capacities of the two users vary widely. The most interior point of the simultaneous water filling region in this case corresponds to a complete user overlap in signal space, the points inside the region correspond to incomplete user overlap in signal space, and points on the outer border correspond to user separation in signal space. For $g = 1$ the outer border of the water filling region reduces to a single point, and as g increases, the outer border expands and more points corresponding to user separation in signal space are possible. Furthermore, the point corresponding to complete user overlap moves closer to the origin as g increase. The points inside the water filling region are suboptimal with respect to achievable capacity since the largest user capacity is achieved by points on the outer border, and the worst situation corresponds to the complete user overlap in signal space case for which users achieve the lowest capacities.

The simultaneous water filling region has an ellipsoidal border, and the coordinates of a fixed point on the border are given by the user capacities. Since points on the border correspond to user separation in signal space user capacities can be expressed as

$$C_i = \frac{1}{2}k_i\log\left(1+\frac{P_i}{k_i\eta_0}\right) \qquad i=1,2 \quad k_1+k_2=N \tag{3}$$

where k_i is the dimension of the signal subspace occupied by user i. The distance between the origin and a given point on the border is

$$r=\sqrt{C_1^2+C_2^2} \tag{4}$$

and varies as the point moves along the border. For a symmetric system (equal user powers, gains, and background noise) this distance is maximum when users reside in orthogonal subspaces of equal dimension $k_1 = k_2 = N/2$. The point

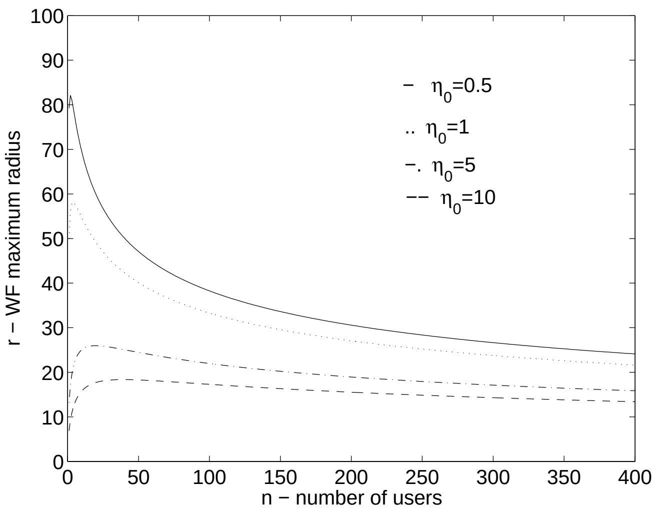


Fig. 4. The variation of $r_{\max}$ in equation (7) as a function of number of users and noise level for $P = N = 100$. Peak value for $\eta_0 = 0.5, 1, 5, 10$ corresponds to $n = 2, 4, 20, 39$.

with maximum distance from the origin corresponds also to the socially optimal solution for the system defined in terms of the collective capacity, which is a global measure used to characterize the overall system performance and was defined in [5], [6] as the sum of user capacities

$$\mathcal{C} = C_1 + C_2 \tag{5}$$

For the symmetric system considered this corresponds to that fixed point on the border with equal user capacities $C_1 = C_2$.

This analysis can be extended to an arbitrary number of user-base pairs n for which users are separated in signal space. User capacities will be expressed as in equation (3) with $i = 1, \ldots, n$ and $k_1 + \ldots + k_n = N$, and the distance from the origin to a point on the ellipsoidal border of the simultaneous water filling region is given in this case by

$$r = \sqrt{\sum_{i=1}^{n} C_i^2} \tag{6}$$

Maximum r is achieved also for equipartition of signal space among users, that is for $k_i = N/n$, and is written as

$$r_{\max} = \frac{N}{2\sqrt{n}} \log\left(1 + \frac{nP}{\eta_0 N}\right) \tag{7}$$

Figure 4 shows the variation of $r_{\max}$ for different background noise levels η_0 and number of user-base pairs n. In all cases, the plot illustrates a peak value after which $r_{\max}$ decreases slowly as the number of user-base pairs increases. We note that as the number of users increases the border of the simultaneous water filling region moves closer to the origin, and the simultaneous water filling region dwindles.

We conclude our analysis of the simultaneous water filling region for the "strong interference" case by noting that a socially optimal solution implies separation of users in signal space and corresponds to a simultaneous water filling distribution with no user overlap. Such a solution can be obtained using various algorithms which yield simultaneous water filling distributions [5], [7], [8]. However, regardless of what algorithm is used, it is not known a priori to what type of simultaneous water filling solution these algorithms may converge, and suboptimal simultaneous water filling distributions in which users overlap (partially or totally) in signal space are also possible in this case. Thus, in order to reach a socially optimal solution in the "strong interference" case we may need to augment the algorithm which drives the system to a simultaneous water filling solution with an extra procedure designed to move the system toward the Pareto boundary of the simultaneous water filling region.

III. Spectrum Warfare: A Dynamic Game for Performance Improvement

From the perspective of maximizing individual user capacity, it is desirable for a given user to reside alone in as large a signal subspace as possible. Thus, in the case of a simultaneous water filling distribution in which users overlap in signal space a given user might employ aggressive strategies which seek to drive other users out of one or more signal space dimensions in order to gain sole residence in those dimensions. This can be achieved through a dynamic game (or competition) that moves the system from interior points of the simultaneous water filling region toward the Pareto border which contains the socially optimal solution, and we propose such a game in the sequel.

For this game the time is quantized by equal-duration "epochs" in which one user acts as leader and the other as follower. During one epoch the leader and follower make only one move each, which can be either aggressive (attack) or passive (retreat). Each user's payoff is the capacity achieved at the end of the epoch.

One epoch starts with an aggressive move (attack) of the leader for one or more signal dimensions, and requires that the leader deploy some energy in the targeted dimensions. The amount of energy used and the size of the targeted subspace are both part of the leader's (attacker's) strategy. In response to the leader's move, the follower reacts in a rational way by applying a water filling procedure: estimates the interference covariance and adjusts its transmit covariance accordingly. Thus, unlike the leader's move, the follower's response is completely predictable given the leader's move.

We note that in general the leader's move results in a spectral distribution of energy which is *not* water filling. Thus, the leader will suffer an immediate decrease in information theoretic capacity after its move. However, the assumption is that the follower will redistribute its energy in a rational way and retreat from the signal dimension(s) targeted by the leader, which will result in higher capacity for the leader.

In the following epoch users take turns and the leader of the previous epoch becomes follower and viceversa.

Figure 5 depicts an epoch of the game in which user 1 is the leader and attacks signal dimension k which is shared with user 2. For clarity, the figure illustrates the energy

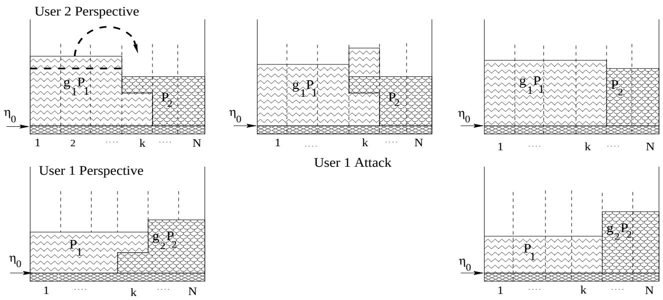


Fig. 5. User 1's attack for dimension k shared with user 2.

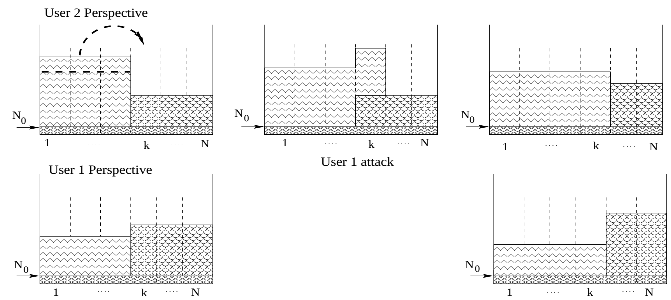


Fig. 6. User 1's attack for dimension k occupied by user 2 only.

distribution from both user 1 and user 2 perspectives. The attack performed by user 1 is better seen from the perspective of user 2: depending on the amount of energy user 1 deploys in dimension k user 2 is forced to retreat from that dimension either partially or totally, and in Figure 5 we assumed that enough energy is used in the attack to force user 2 to vacate dimension k.

Figure 6 depicts an epoch of the game in which user 1 is the leader and attacks signal dimension k which is occupied only by user 2. We assumed again that enough energy is used in the attack to force user 2 to vacate dimension k.

In a similar way, the leader could attack multiple signal space dimensions using the same basic method – "taxing" energy from certain dimensions and placing that energy in the targeted signal dimensions. The maximum number of signal space dimensions that the leader can attempt to occupy alone can be computed, based on its power and on the interference that the follower creates. For example, for equal user power and gains ($P_1 = P_2 = P$ and $g_1 = g_2 = 1$) a given user acting as leader cannot hope to command more than half the signal space. Trying to command a larger portion will enable the other user acting as follower, to water fill over some part of the leader's signal space during its retreat, thus resulting in lower capacity for the leader at the end of the epoch.

Assuming equal power budgets, for given gains g_1 and g_2 (with $g_1 g_2 \geq 1$) and signal space fraction $x = k/N$ occupied solely by user 1, a separated but mutually water filled configuration is stable if

$$\frac{1}{g_2 + 1} \leq x \leq \frac{g_1}{1 + g_1} \tag{8}$$

For $g_1 g_2 = 1$ the space partitioning for complete separation is given by

$$x = \frac{g_1}{1 + g_1} = \frac{1}{g_2 + 1} \tag{9}$$

(which corroborates the specific $g_1 = g_2 = 1$ example). This implies a single possible stable point. Thus, user 1 can target, attain and stably hold at most fraction $\frac{g_1}{1+g_1}$ of the signal space. It is important to note that this unique partition of the signal space represents also the fixed point that maximizes both user capacities, and consequently the collective capacity, thus corresponds to the social optimum for the system. For symmetric systems this point corresponds to equal partition of the signal space. Thus, for systems with $g_1 g_2 = 1$, the dynamic game moves the system from any interior suboptimal fixed point of the simultaneous water filling region (with users overlapping over some region of the signal space) to the socially optimal point.

For $g_1 g_2 > 1$ the fraction of the signal space in which user 1 can reside with complete separation, x in equation (8), does not have a unique solution but a set of solutions. For example, for $g_1 = g_2 = g = 2$ each user can command between one third and two thirds of the signal space. The larger the gain value g the larger the solution interval. As opposed to the case in which $g_1 g_2 = 1$, in this case there is no fixed point on the simultaneous water filling region border where both user capacities are maximized. For a symmetric system the social optimum is still the point on the border with $C_1 = C_2$, but none of the user capacities are maximized at this point. Thus, for $g_1 g_2 > 1$ if a water filling procedure ends up in an interior point of the simultaneous water filling region (with users overlapping in some signal subspace) the dynamic game described will move the system on the border of the simultaneous water filling region, separating the users in signal space, in only one epoch. However, the point on the border where the system is moved is not necessary the socially optimal point. As a consequence, different dynamic games can be designed based on the number of signal space dimensions that the leading user targets to gain during one epoch of the game. We describe here a greedy procedure with users trying to maximize their performances for which the dynamic game is designed such that at each epoch, the leading user targets the maximum number of signal space dimensions from which a full retreat by the follower can be enforced without leaving some other portion of their signal space open to water filling procedure performed by the follower. We illustrate such a competition in Figure 7 for a symmetric system with equal gains $g_1 = g_2 = g$ and user powers $P_1 = P_2 = P$. For $g = 2$ one user can occupy between one third and two thirds of the signal space at a simultaneous water filling point on the Pareto border of the simultaneous water filling region, and when the user acts as leader it will target maximum occupancy of two thirds of the signal space and will drive the follower to reside in only one third of the signal space. However, when users change turns, the follower will act as leader and target maximum occupancy of two thirds of the signal space. Thus, owing to the perfect symmetry, the competition enters a limit cycle with users alternating between occupancy of

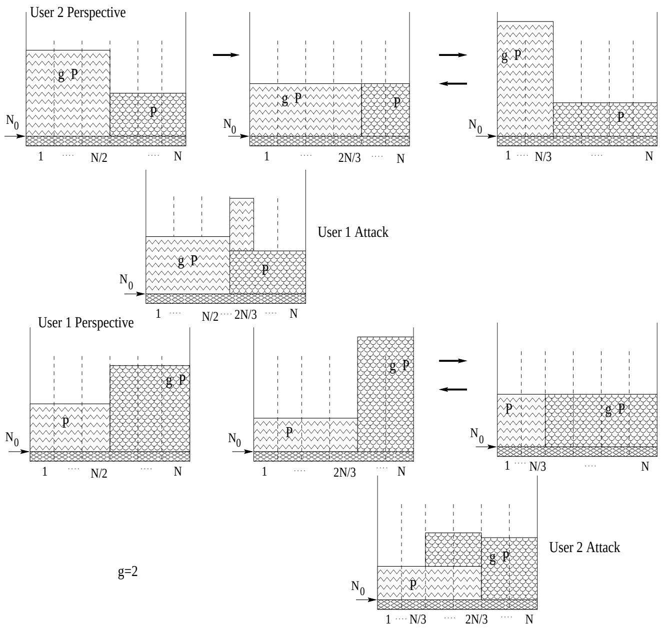


Fig. 7. Competing for maximum signal space occupancy

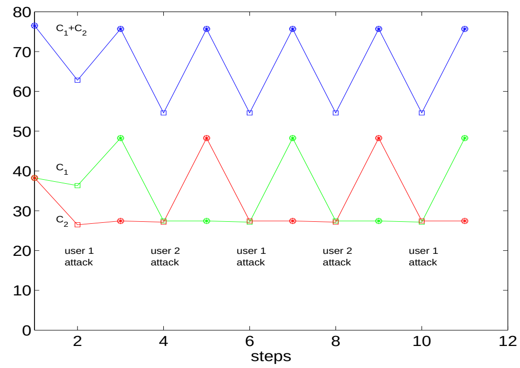


Fig. 8. Capacity variations during one epoch of the dynamic game for $g = 2$, $P = N = 100$, $\eta_0 = 1$.

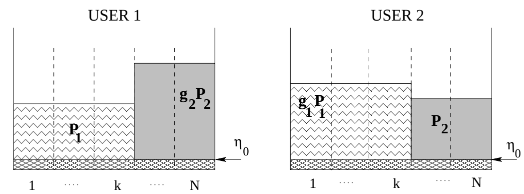


Fig. 9. Signal space partition for two orthogonal users.

one third and two thirds of the signal space. We note that the average performance is improved by moving the system from any interior point to a point on the Pareto boundary, and this is accomplished in only one epoch for a two-user system, and that the cyclic behavior does not bring any more improvements.

Figure 8 shows capacity variations for the game played in the particular case presented in Figure 7. Circles indicate the capacity values corresponding to the simultaneous water filling fixed points, and the squares indicate capacity values during one epoch of the game, after the leader's move and before the follower's response. The initial point on the graph corresponds to the socially optimum point, with each user residing in half of the signal space, and we included it as a reference point. Each user's capacity variation shows that the leader's capacity decreases after its aggressive move, but the decrease in capacity for the leader is not significant. However, the decrease in capacity suffered by the follower is significant. After the follower's water filling move the capacity of the leader will be significantly increased.

Figure 8 illustrates also the variation of the collective capacity during the game. We note that variations of the collective capacity among different points on the Pareto boundary of the simultaneous water filling region are insignificant, and the collective capacity at the socially optimal point where users occupy one half of the signal space each is almost identical to the collective capacity at the extreme points where one user occupies one third of the signal space and the other user occupies two thirds of the signal space. We also note that the aggressive move of the follower during one epoch has a significant negative impact on the collective capacity, but this is compensated by the follower's water filling move which restores the collective capacity value.

From a collective point of view we can conclude that, once a point on the Pareto boundary of the simultaneous water filling region has been reached, continuing the game will not bring improvement anymore, but has a negative impact due to the large variations in capacity experienced by users. Thus, the dynamic game may be designed such that the competition is stopped as soon as capacity oscillations occur. In order to ensure fairness in resource allocation some refined procedures may be necessary to lead the system to the socially optimum point. These procedures may be based on changing the rational behavior of users such that, once limit cycles occur users begin targeting less signal dimensions for occupancy. Since such behavior could be exploited by greedy users in the long run, the game can be augmented with "tit for tat" strategies [4] which reward socially acceptable behavior, and punish greedy users.

IV. The Bouncing Region

We saw in the previous section that competing only for maximum rate users lead the system into limit cycles, in which each user is bouncing between maximum and minimum signal space occupancy. We will call this region "the bouncing region", and for the two user symmetric case considered in the previous section this represents between one third and two thirds of the signal space. In this section we look more closely to this region and to the parameters that may affect it.

We note that a point on the border of the simultaneous water filling region is characterized by orthogonal users in signal space, as shown in Figure 9. Let P_i be the power of user i. The simultaneous water filling conditions

$$g_1 \frac{P_1}{k} \geq \frac{P_2}{N-k} \quad \text{and} \quad \frac{P_1}{k} \leq g_2 \frac{P_2}{N-k} \tag{10}$$

imply that the bouncing region for user 1 is given by

$$\frac{P_1}{P_1 + g_2 P_2} \leq \frac{k}{N} \leq \frac{g_1 P_1}{P_2 + g_1 P_1} \tag{11}$$

and the bouncing region for user 2 is given by

$$\frac{P_2}{P_2 + g_1 P_1} \leq \frac{N-k}{N} \leq \frac{g_2 P_2}{P_1 + g_2 P_2} \tag{12}$$

Because of the symmetry of the two expressions we can work with no loss of generality with the expressions for user 1.

The bouncing region defined by the inequalities in equation (11) can be rewritten in terms of user power ratio $\rho = P_1/P_2$ as

$$\frac{\rho}{\rho + g_2} \leq \frac{k}{N} \leq \frac{g_1 \rho}{1 + g_1 \rho} \tag{13}$$

This implies that the region varies as a function of user gains and power ratio. As gain g_i and/or power P_i increases the area that user i can control extends. The width of the bouncing region is given by

$$\Delta \frac{k}{N} = \frac{g_1 \rho}{1 + g_1 \rho} - \frac{\rho}{\rho + g_2} \tag{14}$$

and is maximized when $\rho = \sqrt{g_2/g_1}$. Thus, for a symmetric system with equal user powers and equal gains, the bouncing region is maximum. We note that the larger the bouncing region, the larger the user capacity variation along the border of the simultaneous water filling region. From this perspective, the symmetric case illustrated before is the least favorable one for the dynamic game proposed. We also note that a narrower bouncing region is preferable since it makes capacity variations less significant which in turn implies that the exact location of the point on border becomes less important.

For a two user system with equal user powers the bouncing region shrinks to a single value for systems with $g_1 g_2 = 1$, and implies

$$\frac{k}{N} = \frac{g_1}{1+g_1} = \frac{1}{1+g_2} \text{ and } \frac{N-k}{N} = \frac{1}{1+g_1} = \frac{g_2}{1+g_2} \tag{15}$$

This point corresponds to the social optimum in this case. In the symmetric case corresponds to $g_1 = g_2 = g = 1$ as illustrated in Figure 3.

The results can be extended for systems with more than two users-base pairs. Simultaneous water filling conditions for a system with M user-base pairs, with P_i the power of user i and g_{ij} the gain from user i to base j imply the bouncing region for user i to be

$$\frac{P_i}{\sum_{j=1}^{M} g_{ji} P_j} \leq \frac{k_i}{N} \leq \frac{P_i}{\sum_{j=1}^{M} \frac{1}{g_{ij}} P_j} \tag{16}$$

After some algebra it can be shown that for systems with equal user powers the bouncing region of each user reduces again to a single point if all cross-products $g_{ij} g_{ji} = 1$, such that

$$\frac{k_i}{N} = \frac{1}{\sum_{j=1}^{M} g_{ji}} \tag{17}$$

This corresponds to the socially optimum point for the multi-user system. In the case of a symmetric system with multiple user-base pairs, for which all user powers are equal, gains to the own base $g_{ii} = 1$, and gains to other bases $g_{ij} = g$, $i \neq j$, the width of the bouncing region is given by

$$\begin{aligned} \Delta \frac{k}{N} &= \frac{1}{1+\frac{M-1}{g}} - \frac{1}{1+(M-1)g} \\ &= \frac{(M-1)\left(g - \frac{1}{g}\right)}{[1+(M-1)g]\left[1+(M-1)\frac{1}{g}\right]} \end{aligned} \tag{18}$$

and decreases as the number of user-base pairs M increases.

The conclusion of this bouncing region analysis is that the symmetric case with two user-base pairs has the largest bouncing region. This implies a worst-case scenario in which the competition for maximum signal space occupancy between users leads to limit cycles. However, we have seen that the performance on any point on the border of the simultaneous water filling region is very close to optimum. Thus, it is not crucial for the system to reach the socially optimum point once the border has been reached, and most improvement in performance is achieved by moving the system from interior points of the simultaneous water filling region to a point on the border. Depending on the particular water filling structure corresponding to the interior point, this can usually be accomplished in a few steps (only one step for a two-user system). After the system has reached the border, the effort (in terms of energy and time consumed) in moving it along the border may not be worth the extra gain in collective capacity.

V. Conclusion

In this paper we analyzed the simultaneous water filling solution for a system with two user-base pairs with "strong interference" corresponding to relative gains $g_1 g_2 \geq 1$. Multiple simultaneous water filling fixed points are possible for this system for which individual and collective performances vary widely. Thus, simultaneous water filling alone does not necessary imply optimum resource sharing in this case. In order to improve system performance we propose a dynamic game that moves the system to better simultaneous water filling fixed points, eventually to the socially optimum point.

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