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Sum Capacity and TSC Bounds in Collaborative Multibase Wireless Systems

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***Abstract*—We consider a wireless system with base stations which collaborate, and derive bounds on sum capacity and total squared correlation for uniform channels between users and bases. The correspondence also investigates structural properties which must be satisfied by user transmit covariance matrices at the optimal sum capacity/total squared correlation (TSC) point, and shows that for multibase systems, maximizing sum capacity and minimizing TSC are, in general, not equivalent problems.**

***Index Terms*—Collaboration, multiple-input multiple-output (MIMO), multibase systems, sum capacity, total squared correlation.**

I. Introduction

We consider a multibase wireless system consisting of multiple users and base stations distributed over a given geographical area. The available spectrum is shared by all users and bases as would be the case in unlicensed bands. Furthermore, the stations are allowed to collaborate and we here make no particular assignment of users to base stations. Thus, transmitted signals from all users are observed at all bases, unlike the usual cellular setup which assumes no cooperation among bases, and in which users are observed only at the bases with which they are associated. As such, the ensemble of bases is logically a "super receiver" with multiple distributed antennas. The overabundance of optical fiber deployed in the past but not yet used (so called "dark fiber") makes such widespread collaboration plausible, and as a specific example, imagine an abstraction of WiFi/802.11 access points which can share information at high speed over a fiber backbone to do joint decoding. Such a collaborative scenario has been considered in previous work dealing with systems with multiple transmitters and receivers [2], [3], [10] and provides upper bounds on various measures of interest since one can do no better than to jointly decode.

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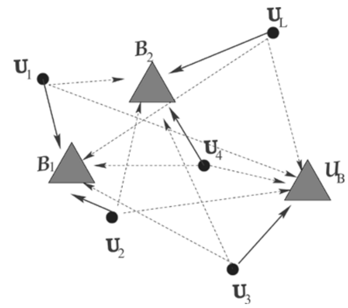

Fig. 1. A multibase system with L users and B base stations. Triangles denote receivers/bases and circles denote transmitters/users.

We consider two global criteria, sum capacity and total squared correlation (TSC) which have been extensively used to characterize the performance of single-base systems [1], [5], [9], [11], [12]. In this correspondence, we derive analytic bounds on these metrics for multibase collaborative systems in a general signal space framework, and investigate structural properties that must be satisfied by user transmit covariance matrices for the special case where the gains between any given user and any given base are identical over all time–bandwidth signal space dimensions used by the system—what we call a "uniform channel" assumption. Such uniform channel models are appropriate when a single path between any pair of transmitters and receivers is dominant, as well as for subchannels within a given coherence bandwidth.

The correspondence is organized as follows: in Section II, we introduce our model for multibase collaborative systems, and show that the uniform channel assumption implies a special structure on the received covariance matrix. In Sections III and IV, we present the main results of the correspondence: bounds on sum capacity and TSC implied by the special structure of the received covariance matrix. In Section V, we extend the results to consider carrier phase offsets.

II. System Description

We consider a system with B base stations and L users distributed over a given area (described schematically in Fig. 1) for which we assume a common signal space representation of dimension N for all users/bases implied by finite bandwidth and finite signaling interval constraints [6]. Unlike the usual cellular scenario, we assume no particular assignment of users to bases, and transmissions from all users are received by all bases.

The received signal during an arbitrary symbol interval at base station j is written as

$$\boldsymbol{r}_j = \sum_{\ell=1}^{L} \boldsymbol{G}_{\ell j}\boldsymbol{x}_\ell + \boldsymbol{w}_j, \qquad \forall j = 1, \dots, B \tag{1}$$

where $\boldsymbol{x}_\ell$ is the N-dimensional codeword transmitted by user ℓ, $\boldsymbol{r}_j$ is the N-dimensional received vector at base station j containing the additive Gaussian noise vector $\boldsymbol{w}_j$. $\boldsymbol{G}_{\ell j}$ is the $N \times N$ gain matrix that characterizes the vector channel between user ℓ and base station j. For simplicity of notation, we assume general gain matrices now and will later specialize the gain matrices appropriately for what we call "uniform channels."

We assume a (temporal) sequence of $\{\boldsymbol{x}_\ell\}$ is transmitted by user ℓ during successive symbol intervals and is decoded at the receiver. Since we will assume Gaussian channels, we assume a Gaussian codebook for the $\boldsymbol{x}_\ell$ which allows us to worry only about the covariance matrix $\boldsymbol{X}_\ell = E\left[\boldsymbol{x}_\ell \boldsymbol{x}_\ell^\top\right]$ when considering issues of capacity.

We consider two potential sources of noise: 1) independent thermal noise at the receiver $\boldsymbol{n}_j$ with covariance matrix $\boldsymbol{\mathcal{V}}_j = E\left[\boldsymbol{n}_j \boldsymbol{n}_j^\top\right]$, and 2) noise from random emitters (possibly associated with other systems) from μ discrete geographic locations $\boldsymbol{e}_n$, $n = 1, \dots, \mu$, with covariances $\boldsymbol{\mathcal{W}}_n = E\left[\boldsymbol{e}_n \boldsymbol{e}_n^\top\right]$, $n = 1, 2, \dots, \mu$. Thus, we can write

$$\boldsymbol{w}_j = \boldsymbol{n}_j + \sum_{n=1}^{\mu} \boldsymbol{H}_{nj} \boldsymbol{e}_n \tag{2}$$

with $\boldsymbol{H}_{nj}$ representing the gain from emitter n to base station j, and the noise covariance matrix at base station j is written as

$$\boldsymbol{W}_j = E\left[\boldsymbol{w}_j \boldsymbol{w}_j^\top\right] = \boldsymbol{\mathcal{V}}_j + \sum_{n=1}^{\mu} \boldsymbol{H}_{nj} \boldsymbol{\mathcal{W}}_n \boldsymbol{H}_{nj}^\top. \tag{3}$$

Our goal is to derive bounds on global performance measures for this system, like information-theoretic capacity which characterizes achievable data rates for reliable transmission, or total squared correlation which characterizes the total interference in the system. In order to do this, we assume a collaborative scenario in which received signals at all bases are collected and used for joint decoding since one can do no better than to jointly decode. Assuming collaboration, a BN-dimensional received vector is constructed by gathering all received vectors from all bases

$$\underbrace{\begin{bmatrix} \boldsymbol{r}_1 \\ \vdots \\ \boldsymbol{r}_B \end{bmatrix}}_{\boldsymbol{r}} = \sum_{\ell=1}^{L} \underbrace{\begin{bmatrix} \boldsymbol{G}_{\ell 1} \\ \vdots \\ \boldsymbol{G}_{\ell B} \end{bmatrix}}_{\boldsymbol{G}_\ell} \boldsymbol{x}_\ell + \underbrace{\begin{bmatrix} \boldsymbol{w}_1 \\ \vdots \\ \boldsymbol{w}_B \end{bmatrix}}_{\boldsymbol{w}} \tag{4}$$

with correlation matrix

$$\boldsymbol{R} = E[\boldsymbol{r}\boldsymbol{r}^\top] = \sum_{\ell=1}^{L} \boldsymbol{R}(\ell) + \boldsymbol{W} \tag{5}$$

where matrix $\boldsymbol{R}(\ell)$ represents the user ℓ contribution to $\boldsymbol{R}$ and is expressed in terms of its transmit covariance matrix and corresponding gain matrices as

$$\boldsymbol{R}(\ell) = \boldsymbol{G}_\ell \boldsymbol{X}_\ell \boldsymbol{G}_\ell^\top \tag{6}$$

and $\boldsymbol{W}$ is the covariance matrix of the resulting noise vector $\boldsymbol{w}$. Due to the structure of the noise at each base station, the resulting noise vector is written as

$$\underbrace{\begin{bmatrix} \boldsymbol{w}_1 \\ \vdots \\ \boldsymbol{w}_B \end{bmatrix}}_{\boldsymbol{w}} = \underbrace{\begin{bmatrix} \boldsymbol{n}_1 \\ \vdots \\ \boldsymbol{n}_B \end{bmatrix}}_{\boldsymbol{n}} + \sum_{n=1}^{\mu} \underbrace{\begin{bmatrix} \boldsymbol{H}_{n1} \\ \vdots \\ \boldsymbol{H}_{nB} \end{bmatrix}}_{\boldsymbol{H}_n} \boldsymbol{e}_n \tag{7}$$

and its covariance matrix can be written as

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{\mathcal{V}}_1 & & & \\ & \boldsymbol{\mathcal{V}}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{\mathcal{V}}_B \end{bmatrix} + \sum_{n=1}^{\mu} \boldsymbol{H}_n \boldsymbol{\mathcal{W}}_n \boldsymbol{H}_n^\top. \tag{8}$$

Thus, we have

$$\boldsymbol{R} = \sum_{\ell=1}^{L} \boldsymbol{G}_\ell \boldsymbol{X}_\ell \boldsymbol{G}_\ell^\top + \boldsymbol{W}. \tag{9}$$

We note that the received signal in (4) which characterizes the collaborative scenario corresponds to a base with multiple antennas and single-antenna users—a type of multiple-input multiple-output (MIMO) system. Joint decoding occurs through use of a backbone network as would be the case for bases distributed like 802.11 access points connected to the Internet. Since decoding is collaborative at what amounts to an aggregate "super-receiver," we can use global performance measures such as information-theoretic sum capacity or TSC as a measure of system performance.

Sum capacity characterizes the sum of achievable rates for reliable transmission by all users, and under Gaussian signaling and noise assumptions is expressed as [3], [14]

$$C_{\text{sum}} = \frac{1}{2}\log|\boldsymbol{R}| - \frac{1}{2}\log|\boldsymbol{W}|. \tag{10}$$

The TSC characterizes the total interference in the system and is computed at the "super-receiver" which collects received signals from all bases and forms $\boldsymbol{r}$ in (4) as the sum of squared correlations between any two user received signals, that is,

$$\begin{aligned} \text{TSC} &= \sum_{\ell=1}^{L}\sum_{m=1}^{L} E\left\{\left[(\boldsymbol{G}_\ell \boldsymbol{x}_\ell)^\top (\boldsymbol{G}_m \boldsymbol{x}_m)\right]^2\right\} \\ &= \text{Trace}\left[\left(\sum_{\ell=1}^{L} \boldsymbol{G}_\ell \boldsymbol{X}_\ell \boldsymbol{G}_\ell^\top\right)^2\right]. \end{aligned} \tag{11}$$

This measure can be regarded as an extension of the TSC used for single-base systems [1], [5], [9], [11] to our collaborative multibase system. By treating the noise vector as coming from a virtual interferer and adding terms corresponding to the squared correlations between all users and this additional interferer (including the correlation of the noise vector with itself) we define the generalized TSC (GTSC) as the trace of the squared received signal correlation matrix

$$\begin{aligned} \text{GTSC} &= \text{Trace}\left[\left(\sum_{\ell=1}^{L} \boldsymbol{G}_\ell \boldsymbol{X}_\ell \boldsymbol{G}_\ell^\top + \boldsymbol{W}\right)^2\right] \\ &= \text{Trace}\left[\boldsymbol{R}^2\right]. \end{aligned} \tag{12}$$

We note that sum capacity is concave in $\boldsymbol{R}$ and GTSC is convex in $\boldsymbol{R}$ [5], [8], [9], [13].

We assume that channels between users and bases are *uniform*, and characterized by identical gains across all signal space dimensions, that is, $\boldsymbol{G}_{\ell j} = g_{\ell j}\boldsymbol{I}_N$, $\forall \ell = 1, \dots, L$, $j = 1, \dots, B$ with $\boldsymbol{I}_N$ the identity matrix of order N. Such uniform channels could arise in a variety of settings where a single path (not necessarily line of sight) predominates between transmitters and receivers, or for narrow-band subchannels within some coherence bandwidth. Thus, user gain matrices are expressed as

$$\boldsymbol{G}_\ell = \begin{bmatrix} g_{\ell 1}\boldsymbol{I}_N \\ \vdots \\ g_{\ell B}\boldsymbol{I}_N \end{bmatrix} \tag{13}$$

which implies that

$$\boldsymbol{R}(\ell) = \begin{bmatrix} g_{\ell 1}^2 \boldsymbol{X}_\ell & \cdots & g_{\ell 1} g_{\ell B} \boldsymbol{X}_\ell \\ \vdots & \ddots & \vdots \\ g_{\ell B} g_{\ell 1} \boldsymbol{X}_\ell & \cdots & g_{\ell B}^2 \boldsymbol{X}_\ell \end{bmatrix} \tag{14}$$

and we assume that these gains are stable for sufficiently long sequences of $\boldsymbol{x}_\ell$ transmissions.

We then note that the uniform channel assumption implies that $\boldsymbol{R}$ has the following special structure.

1) $\boldsymbol{R}$ is composed of $N \times N$ subblocks.
2) Each subblock $\boldsymbol{R}_{ij}$ is trace constrained. That is,

$$\begin{aligned} \mathrm{Trace}\,[\boldsymbol{R}_{ij}] &= E_{ij} \\ &= \sum_{\ell=1}^{L} g_{\ell i} g_{\ell j} \underbrace{\mathrm{Trace}\,[\boldsymbol{X}_\ell]}_{P_\ell} + \underbrace{\mathrm{Trace}\,[\boldsymbol{W}_{ij}]}_{\omega_{ij}} \\ &= \sum_{\ell=1}^{L} g_{\ell i} g_{\ell j} P_\ell + \omega_{ij} = E_{ji} \end{aligned} \tag{15}$$

with P_ℓ being the power corresponding to user ℓ.

We will show that this special structure implies a particular form for the $\boldsymbol{R}$ which maximizes (10) (or minimizes (11)). We will also show that maximizing sum capacity and minimizing GTSC are, in general, not equivalent problems, as it was the case for single-base systems.

III. Bounds on Sum Capacity

Here we identify properties of the $\boldsymbol{R}$ which maximize sum capacity under our uniform gain matrix assumptions. Note that we do not attempt to solve the problem of maximizing sum capacity in (10), and/or propose an algorithm that does this—we refer readers to the paper by Yu *et al.* [14] which solved the problem of maximizing sum capacity for a general multiaccess vector channel as a spectral optimization problem

$$\max_{\boldsymbol{X}_\ell} C_{\mathrm{sum}} \text{ subject to } \mathrm{Trace}\,[\boldsymbol{X}_\ell] = P_\ell, \qquad \ell = 1, \dots, L \tag{16}$$

and proposes an iterative algorithm for finding optimal user transmit covariance matrices $\boldsymbol{X}_\ell$ which maximize sum capacity. Rather, we investigate structural properties of matrix $\boldsymbol{R}$ at the maximum sum capacity point.

Since for stationary noise the noise covariance matrix is fixed, then maximum sum capacity in (10) implies that the determinant of the system covariance matrix $\boldsymbol{R}$ is maximized. We note that in general, maximizing $|\boldsymbol{R}|$ is subject to a trace constraint on $\boldsymbol{R}$ implied by the total energy in the system. However, in our case, maximization of $|\boldsymbol{R}|$ is subject to additional constraints on traces of subblocks of $\boldsymbol{R}$ as specified by (15). We also note that while in general, a positive-definite $K \times K$ matrix $\boldsymbol{A}$ with a trace constraint has maximum determinant when it is a scaled identity matrix [4], that is,

$$\max |\boldsymbol{A}| \iff \boldsymbol{A} = \frac{\mathrm{Trace}\,[\boldsymbol{A}]}{K} \boldsymbol{I}_K \tag{17}$$

this is not the case with our received covariance matrix $\boldsymbol{R}$, and unless the off-diagonal blocks of $\boldsymbol{R}$ have zero trace, the covariance matrix $\boldsymbol{R}$ *cannot* be a scaled identity matrix. Thus, we must seek the structure of $\boldsymbol{R}$ which maximizes $|\boldsymbol{R}|$ subject to the imposed trace constraints on individual $N \times N$ subblocks in (15).

The following mathematical result, proven in [7], is useful as we seek to maximize the determinant of our subblock trace-constrained matrix $\boldsymbol{R}$.

Theorem 1: Let $\mathcal{Q}_{NJ}$ be the class of symmetric positive-definite matrices $\boldsymbol{Q}$ of the following form:

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} & \cdots & \boldsymbol{Q}_{1J} \\ \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ \boldsymbol{Q}_{J1} & \cdots & \cdots & \boldsymbol{Q}_{JJ} \end{bmatrix}$$

where the subscript N denotes the size of the square subblocks and J denotes the number of vertical and horizontal subblocks $\boldsymbol{Q}_{ij} = \boldsymbol{Q}_{ji}^\top$ with trace constraint $\mathrm{Trace}\,[\boldsymbol{Q}_{ij}] = E_{ij}$. Then the determinant of $\boldsymbol{Q}$ is maximized when

$$\boldsymbol{Q}_{ij} = \frac{E_{ij}}{N} \boldsymbol{I}_N, \qquad 1 \le i, j \le J \tag{18}$$

and its maximum value is equal to

$$|\boldsymbol{Q}| = \frac{1}{N^{NJ}} |\boldsymbol{E}|^N \tag{19}$$

where $\boldsymbol{E}$ is a $J \times J$ symmetric matrix with elements $\{E_{ij}\}$. □

If we assume independent white noise/interference, then each block of $\boldsymbol{W}$ is of the form

$$\boldsymbol{W}_{ij} = \frac{\omega_{ij}}{N} \boldsymbol{I}_N, \qquad i, j = 1, \dots, B \tag{20}$$

and direct application of Theorem 1 implies that

$$\sum_{\ell=1}^{L} g_{\ell i} g_{\ell j} \boldsymbol{X}_\ell = \frac{E_{ij} - \omega_{ij}}{N} \boldsymbol{I}_N \tag{21}$$

which is always satisfied if user covariance matrices $\boldsymbol{X}_\ell$ are scaled identity matrices (although there may be other solutions as well). Thus, for white noise/interference with subblock traces ω_{ij}, sum capacity is maximized when all user transmit covariances are f

$$\boldsymbol{X}_\ell = \frac{P_\ell}{N} \boldsymbol{I}_N \tag{22}$$

with implicit Gaussian codebooks for the $\{\boldsymbol{x}_\ell\}$. The corresponding maximum sum capacity value is

$$C_{\max} = \frac{N}{2} \left(\log |\boldsymbol{E}| - \log |\boldsymbol{\Omega}| \right) \tag{23}$$

where $\boldsymbol{E}$ is the $B \times B$ symmetric matrix with elements $\boldsymbol{E}_{ij}$ in (15), and $\boldsymbol{\Omega}$ is the $B \times B$ matrix with elements $\omega_{ij} = \mathrm{Trace}\,[\boldsymbol{W}_{ij}]$.

For colored noise, the trace constraints are identical to those in (21) and application of Theorem 1 requires that

$$\sum_{\ell=1}^{L} g_{ik} g_{jk} \boldsymbol{X}_\ell + \boldsymbol{W}_{ij} = \frac{E_{ij}}{N} \boldsymbol{I}_N, \qquad 1 \le i, j \le B \tag{24}$$

which, due to the symmetry of matrix $\boldsymbol{R}$, can be regarded as a system of $B(B+1)/2$ matrix equations with L unknown covariances $\boldsymbol{X}_\ell$, $\ell = 1, \dots, L$. In this case, one needs to answer the following questions: a) does there exist a realizable/feasible set of user transmit covariances $\{\boldsymbol{X}_\ell\}$, $\ell = 1, \dots, L$ which satisfies the system of matrix equations (24), and b) if no such set $\{\boldsymbol{X}_\ell\}$ exists, what is the actual sum capacity maximizing set.

We note that, when a feasible solution to the system of matrix equations in (24) exists, then (24) provides necessary and sufficient conditions for user transmit covariance matrices at the maximum sum capacity point. That is, maximum sum capacity is achieved if and only if user transmit covariance matrices satisfy the system of matrix equations (24), provided it has a solution, and maximum sum capacity value is

$$C_{\max} = \frac{1}{2}\left(N\log|\boldsymbol{E}| - NB\log N - \log|\boldsymbol{W}|\right). \tag{25}$$

If no set $\{\boldsymbol{X}_\ell\}$ that satisfies the system of matrix equations (24) exists,[1] then the value in (25) is an upper bound. Of course, the sum capacity maximizing set of user transmit covariance matrices can always be obtained numerically through water-filling schemes [14]. But more careful characterization of when the bound can be achieved and providing tighter analytic bounds when it cannot would be useful and could be the subject of future work.

IV. Bounds on GTSC

In this section, we identify the properties that are satisfied by the received signal covariance matrix $\boldsymbol{R}$ at the minimum GTSC point. We provide bounds for GTSC and show that under certain circumstances, minimizing the GTSC in uniform channel multibase systems is equivalent to maximizing sum capacity.

The following mathematical result, also proven in [7], is useful since the matrix multiplications implied by GTSC and the special structure of $\boldsymbol{R}$ lead to off-diagonal subblock products whose $\mathrm{Trace}[\cdot]$ extrema are not obvious.

Theorem 2: Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two square matrices, such that $\boldsymbol{A}$ is positive definite, $\mathrm{Trace}[\boldsymbol{A}] = E_A$, and $\mathrm{Trace}[\boldsymbol{B}] = E_B$. Then

$$\min_{\boldsymbol{B}} \mathrm{Trace}\left[\boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}^\top\right] = \frac{E_{\boldsymbol{B}}^2}{E_{\boldsymbol{A}}} \text{ for } \boldsymbol{B} = \frac{E_{\boldsymbol{B}}}{E_{\boldsymbol{A}}}\boldsymbol{A}. \tag{26}$$

In order to derive bounds on GTSC we first note that

$$\begin{aligned}\mathrm{GTSC} &= \mathrm{Trace}\left[\boldsymbol{R}^2\right] \\ &= \sum_{i=1}^{B}\sum_{j=1}^{B}\mathrm{Trace}\left[\boldsymbol{R}_{ij}\boldsymbol{R}_{ji}\right] \\ &= \sum_{i=1}^{B}\sum_{j=1}^{B}\mathrm{Trace}\left[\boldsymbol{R}_{ij}^\top\boldsymbol{R}_{ij}\right]\end{aligned} \tag{27}$$

and then using Theorem 2 with $\boldsymbol{A} = \boldsymbol{I}$ we obtain that the GTSC is minimized when each of the subblocks of $\boldsymbol{R}$ is a scaled identity matrix

$$\boldsymbol{R}_{ij} = \frac{E_{ij}}{N}\boldsymbol{I}, \quad 1 \le i, j \le B \tag{28}$$

and the minimum GTSC is obtained as

$$\mathrm{GTSC}_{min} = \sum_{i=1}^{B}\sum_{j=1}^{B}\frac{E_{ij}^2}{N} = \frac{1}{N}\mathrm{Trace}\left[\boldsymbol{E}^2\right] \tag{29}$$

with $\boldsymbol{E}$ the same matrix as in (23).

When white noise/interference is assumed, each block of $\boldsymbol{W}$ is an identity matrix as in (20), minimum GTSC can be obtained when all user transmit covariance matrices $\boldsymbol{X}_\ell$ are scaled identity matrices as in (22). For colored noise, (28) implies that the same system of matrix equations as in (24) must be satisfied by user transmit covariance matrices in order to minimize the GTSC. As before, we note here that there may exist no feasible set of user transmit covariance matrices that satisfy the conditions in (24), in which case (29) serves only as a lower bound for GTSC.

[1] Most likely when $B(B+1)/2 > L$; otherwise, a system with the number of equations less than or equal to the number of unknowns has in general at least one solution.

We also note here that as long as a feasible solution to the system of matrix (24) exists, then the same set of user transmit covariance matrices $\{\boldsymbol{X}_\ell\}$ will both maximize sum capacity in (10) and minimize GTSC in (12). However, this is not true in general, as can be seen from the following example in which a single user is active in the multibase system. In this case, there is no need for the user index ℓ, and the correlation matrix of the composite received signal is written as

$$\boldsymbol{R} = \boldsymbol{G}\boldsymbol{X}\boldsymbol{G}^\top + \boldsymbol{W} \tag{30}$$

where

$$\boldsymbol{G} = \begin{bmatrix} g_1\boldsymbol{I}_N \\ \vdots \\ g_B\boldsymbol{I}_N \end{bmatrix} = \underbrace{\begin{bmatrix} g_1 \\ \vdots \\ g_B \end{bmatrix}}_{\boldsymbol{g}} \otimes \boldsymbol{I}_N = \boldsymbol{g}\otimes\boldsymbol{I}_N \tag{31}$$

is the gain matrix that corresponds to the user, and can also be expressed as a Kronecker product [4] between the B-dimensional vector containing the uniform gain values to all bases $\boldsymbol{g}$ and the identity matrix of order N. We can then define

$$\boldsymbol{U} = \frac{1}{\|\boldsymbol{g}\|}\boldsymbol{G} \tag{32}$$

which is a matrix with orthonormal columns, that is, $\boldsymbol{U}^\top\boldsymbol{U} = \boldsymbol{I}_N$. We also define the orthonormal complement of $\boldsymbol{U}$ as $\bar{\boldsymbol{U}}$ and note that

$$[\boldsymbol{U} \quad \bar{\boldsymbol{U}}]\begin{bmatrix}\boldsymbol{U}^\top \\ \bar{\boldsymbol{U}}^\top\end{bmatrix} = \boldsymbol{I}_{BN}. \tag{33}$$

Applying the following similarity transformation to $\boldsymbol{R}$:

$$\begin{aligned}\begin{bmatrix}\boldsymbol{U}^\top \\ \bar{\boldsymbol{U}}^\top\end{bmatrix}\boldsymbol{R}[\boldsymbol{U} \quad \bar{\boldsymbol{U}}] &= \boldsymbol{R}' \\ &= \begin{bmatrix}\|\boldsymbol{g}\|^2\boldsymbol{X} + \boldsymbol{W}'_{11} & \boldsymbol{W}'_{12} \\ \boldsymbol{W}'_{21} & \boldsymbol{W}'_{22}\end{bmatrix}\end{aligned} \tag{34}$$

where

$$\begin{aligned}\boldsymbol{W}'_{11} &= \boldsymbol{U}^\top\boldsymbol{W}U & \boldsymbol{W}'_{12} &= \boldsymbol{U}^\top\boldsymbol{W}\bar{\boldsymbol{U}} \\ \boldsymbol{W}'_{21} &= \bar{\boldsymbol{U}}^\top\boldsymbol{W}U & \boldsymbol{W}'_{22} &= \bar{\boldsymbol{U}}^\top\boldsymbol{W}\bar{\boldsymbol{U}}.\end{aligned} \tag{35}$$

We can now write $|\boldsymbol{R}'|$ using the Schur factorization [4]

$$|\boldsymbol{R}'| = |\boldsymbol{W}'_{22}|\,\left|\|\boldsymbol{g}\|^2\boldsymbol{X} + \boldsymbol{W}'_{11} - \boldsymbol{W}'_{12}(\boldsymbol{W}'_{22})^{-1}\boldsymbol{W}'_{21}\right| \tag{36}$$

and

$$\begin{aligned}&\mathrm{Trace}\left[(\boldsymbol{R}')^2\right] = \mathrm{Trace}\left[(\|\boldsymbol{g}\|^2\boldsymbol{X} + \boldsymbol{W}'_{11})^2\right] \\ &+\mathrm{Trace}\left[\boldsymbol{W}'_{12}\boldsymbol{W}'_{21}\right] + \mathrm{Trace}\left[\boldsymbol{W}'_{21}\boldsymbol{W}'_{12}\right] + \mathrm{Trace}\left[(\boldsymbol{W}'_{22})^2\right].\end{aligned} \tag{37}$$

When maximizing $|\boldsymbol{R}'|$ in (36), the eigenvalues of the user covariance matrix $\boldsymbol{X}$ will be determined by the eigenvalues of matrix

$$\boldsymbol{W}'_{11} - \boldsymbol{W}'_{12}(\boldsymbol{W}'_{22})^{-1}\boldsymbol{W}'_{21}.$$

In contrast, when minimizing $\mathrm{Trace}\left[(\boldsymbol{R}')^2\right]$ in (37), only the eigenvalues of $\boldsymbol{W}'_{11}$ matter. Therefore, unless $\boldsymbol{W}'_{12}(\boldsymbol{W}'_{22})^{-1}\boldsymbol{W}'_{21} = 0$, optimizing (36) and (37) in $\boldsymbol{X}$ may lead to different results.

This example shows that unlike single-base systems for which sum capacity maximization and TSC minimization are equivalent problems [8], [9], for multibase systems they are in the most general case not equivalent, and may result in different solutions. However, so long as $\boldsymbol{R}$ can be realized with scaled identity subblocks, then maximizing sum capacity for multibase systems will be equivalent to minimizing the GTSC.

V. Incorporating Carrier Phase Delays

So far we have assumed complete synchronization at all receivers between all users, which may be justified in baseband by assuming sufficiently long signaling intervals relative to the communication bandwidth allotted. However, for carrier-modulated signals, simple propagation delay can cause signals modulated on the in-phase rail to appear on the quadrature rail at the receiver, and *vice versa*. We also note that relative phases can be compensated for a single user, but that compensation for multiple users with different delays to the same receivers is not possible in general for omnidirectional transmission. In this section, we show that the structural results derived for synchronized systems also apply to uniform gain systems where propagation delay is considered explicitly.

We start by assuming a set of baseband orthonormal waveforms that span the signal space of interest, which is implied by the allotted bandwidth and duration of signaling interval [6]. To be consistent with the signal space representation used in the previous sections we assume N even, and let $\{\phi_i(t)\}$, $i = 1, 2, \dots, N/2$ be the corresponding baseband orthonormal functions set. Modulation with both $\cos 2\pi f_c t$ and $\sin 2\pi f_c t$ provides N passband orthonormal basis functions which can be used to represent user ℓ transmitted waveform

$$x_\ell(t) = \begin{bmatrix} x_\ell^{(1)}\phi_1(t)\cos 2\pi f_c t \\ x_\ell^{(2)}\phi_1(t)\sin 2\pi f_c t \\ \vdots \\ x_\ell^{(2n-1)}\phi_n(t)\cos 2\pi f_c t \\ x_\ell^{(2n)}\phi_n(t)\sin 2\pi f_c t \\ \vdots \\ x_\ell^{(N-1)}\phi_{\frac{N}{2}}(t)\cos 2\pi f_c t \\ x_\ell^{(N)}\phi_{\frac{N}{2}}(t)\sin 2\pi f_c t \end{bmatrix} \tag{38}$$

using the N-dimensional vector

$$\boldsymbol{x}_\ell = \begin{bmatrix} x_\ell^{(1)} \\ x_\ell^{(2)} \\ \vdots \\ x_\ell^{(2n-1)} \\ x_\ell^{(2n)} \\ \vdots \\ x_\ell^{(N-1)} \\ x_\ell^{(N)} \end{bmatrix}. \tag{39}$$

We also assume that baseband basis functions are not affected by the delay, that is, $\phi_i(t) \approx \phi_i(t-\tau)$ for $i = 1, 2, \dots, N/2$, which is reasonable for typical propagation delays τ. However, the carrier will be affected and we can write

$$\begin{aligned} \cos 2\pi f_c(t-\tau) &= \cos 2\pi f_c t \cos 2\pi f_c \tau + \sin 2\pi f_c t \sin 2\pi f_c \tau \\ \sin 2\pi f_c(t-\tau) &= \sin 2\pi f_c t \cos 2\pi f_c \tau - \sin 2\pi f_c \tau \cos 2\pi f_c t \end{aligned} \tag{40}$$

which implies that for any pair of passband basis functions $\phi_n(t)\cos 2\pi f_c t$ and $\phi_n(t)\sin 2\pi f_c t$ we can write

$$\begin{bmatrix} \phi_n(t)\cos 2\pi f_c(t-\tau) \\ \phi_n(t)\sin 2\pi f_c(t-\tau) \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{\boldsymbol{O}(\theta)} \begin{bmatrix} \phi_n(t)\cos 2\pi f_c t \\ \phi_n(t)\cos 2\pi f_c t \end{bmatrix} \tag{41}$$

where $\theta = -2\pi f_c \tau$, and $\boldsymbol{O}(\theta)$ is a standard rotation matrix which satisfies $\boldsymbol{O}^{-1}(\theta) = \boldsymbol{O}(\theta)^\top = \boldsymbol{O}(-\theta)$. Thus, after propagation delay, the signal vector that corresponds to the transmitted signal in (39) can be written as

$$\begin{bmatrix} \boldsymbol{O}(\theta) & & & \\ & \boldsymbol{O}(\theta) & & \\ & & \ddots & \\ & & & \boldsymbol{O}(\theta) \end{bmatrix} \boldsymbol{x}_\ell = \boldsymbol{\Theta}(\theta)\boldsymbol{x}_\ell \tag{42}$$

which shows that the effect of propagation delay is a pairwise rotation of signal components.

We now denote the carrier phase rotation that corresponds to the received signal from user ℓ at base j as $\theta_{\ell j}$ with the corresponding $N \times N$ rotation matrix $\boldsymbol{\Theta}_{\ell j} = \boldsymbol{\Theta}(\theta_{\ell j})$. We can then write the covariance matrix of user ℓ signal received at base j as $\boldsymbol{\Theta}_{\ell j}\boldsymbol{X}\boldsymbol{\Theta}_{\ell j}$ which implies that each $N \times N$ block $\boldsymbol{R}_{ij}$ in the covariance matrix $\boldsymbol{R}$ can be written as

$$\boldsymbol{R}_{ij} = \sum_{\ell=1}^{L} g_{\ell i} g_{\ell j} \boldsymbol{\Theta}_{\ell i} \boldsymbol{X}_\ell \boldsymbol{\Theta}_{\ell j}^\top + \boldsymbol{W}_{ij}. \tag{43}$$

The following properties of rotation matrices, which can be easily verified

$$\begin{aligned} \mathrm{Trace}\,[\boldsymbol{A}\boldsymbol{O}(\theta)] &= \mathrm{Trace}\,[\boldsymbol{O}(\theta)\boldsymbol{A}] \\ &= \mathrm{Trace}\,[\boldsymbol{A}]\cos\theta, \qquad \forall \boldsymbol{A} \in \mathbb{R}^2 \end{aligned} \tag{44}$$

$$\boldsymbol{O}(\theta_i)^\top \boldsymbol{O}(\theta_j) = \boldsymbol{O}(\theta_i - \theta_j) \tag{45}$$

extend to matrix $\boldsymbol{\Theta}(\theta)$ in (42) as

$$\begin{aligned} \mathrm{Trace}\,[\boldsymbol{A}\boldsymbol{\Theta}(\theta)] &= \mathrm{Trace}\,[\boldsymbol{\Theta}(\theta)\boldsymbol{A}], \qquad \forall \boldsymbol{A} \in \mathbb{R}^N \\ &= \mathrm{Trace}[\boldsymbol{A}]\cos\theta, \qquad N \text{ even} \end{aligned} \tag{46}$$

$$\boldsymbol{\Theta}(\theta_i)^\top \boldsymbol{\Theta}(\theta_j) = \boldsymbol{\Theta}(\theta_i - \theta_j). \tag{47}$$

Thus,

$$\begin{aligned} \mathrm{Trace}\left[\boldsymbol{\Theta}_{\ell i}\boldsymbol{X}_\ell\boldsymbol{\Theta}_{\ell j}^\top\right] &= \mathrm{Trace}\left[\boldsymbol{X}_\ell\boldsymbol{\Theta}_{\ell j}^\top\boldsymbol{\Theta}_{\ell i}\right] \\ &= \mathrm{Trace}\,[\boldsymbol{X}_\ell\boldsymbol{\Theta}(\theta_{\ell i} - \theta_{\ell j})] \\ &= P_\ell\cos(\theta_{\ell i} - \theta_{\ell j}) \end{aligned} \tag{48}$$

which implies that

$$\begin{aligned} \mathrm{Trace}\,[\boldsymbol{R}_{ij}] &= E_{ij} \\ &= \sum_{\ell=1}^{L} g_{\ell i} g_{\ell j} P_\ell \cos(\theta_{\ell i} - \theta_{\ell j}) + \mathrm{Trace}\,[\boldsymbol{W}_{ij}]. \end{aligned} \tag{49}$$

This shows that when carrier phase delays are taken into account, the traces of all subblocks $\boldsymbol{R}_{ij}$ in $\boldsymbol{R}$ will continue to be trace constrained as it was when no phase delays were considered, and $\mathrm{Trace}\,[\boldsymbol{R}_{ij}]$ will also depend on the phase delay through

$$\theta_{\ell i} - \theta_{\ell j} = 2\pi f_c(\tau_{\ell j} - \tau_{\ell i}). \tag{50}$$

The fact that the $N \times N$ subblocks of $\boldsymbol{R}$ which have the expression in (43) are trace constrained as shown by (49) ensures that the general necessary and sufficient conditions in (24) derived in the previous sections also hold when carrier phase delays are considered—that is, $|\boldsymbol{R}|$ is maximized when

$$\sum_{\ell=1}^{L} g_{\ell i} g_{\ell j} \boldsymbol{\Theta}_{\ell i} \boldsymbol{X}_{\ell} \boldsymbol{\Theta}_{\ell j}^{\top} + \boldsymbol{W}_{ij} = \frac{E_{ij}}{N} \boldsymbol{I}_N, \qquad 1 \leq i, j \leq B. \quad (51)$$

And as before, it is possible that no such set of $\boldsymbol{X}_\ell$ exists, in which case, our results provide an upper/lower bound on sum capacity/TSC.

VI. Conclusion

The overall structure of collaborative but geographically dispersed bases is interesting in light of the proliferation of consumer wireless systems like 802.11 and the amount of dark fiber available from past fiber (over)deployments. In this correspondence, we considered an abstraction of such systems as multiple collaborating base stations and uniform channels between users and bases and derived bounds on sum capacity and TSC via structural properties of the received covariance matrix. We also showed that as compared to single-base systems, where maximizing sum capacity and minimizing TSC are equivalent problems, in multibase systems TSC and sum capacity optimization can lead to different results.

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Frequency Hopping Sequences With Optimal Partial Autocorrelation Properties

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***Abstract*—We classify some p^k-ary (p prime, k integer) generalized m-sequences and generalized Gordon–Mills–Welch (GMW) sequences of period $p^{2k} - 1$ over a residue class ring $R = \mathrm{GF}(p)[\xi]/(\xi^k)$ having optimal partial Hamming autocorrelation properties. In frequency hopping (FH) spread-spectrum systems, these sequences are useful for synchronizing process. Suppose, for example, that a transmitting p^k-ary FH patterns of period $p^{2k} - 1$ are correlated at a receiver. Usually, the length of a correlation window, denoted by L, is shorter than the pattern's overall period. In that case, the maximum value of the out-of-phase Hamming autocorrelation is lower-bounded by $\lceil \frac{L}{p^k+1} \rceil$ but the classified sequences achieve this bound with equality for any positive integer L.**

***Index Terms*—Finite rings, frequency hopping, generalized Gordon–Mills–Welch (GGMW) sequences, Hamming correlation, partial autocorrelation.**

I. Introduction

In frequency hopping multiple-access (FHMA) spread-spectrum systems employing orthogonal modulation, we have to use a set of frequency hopping patterns to minimize the maximum of Hamming out-of-phase autocorrelation and cross correlation to effectively discriminate between their own signals and reduce multiple-access interference (MAI). Specific methods to generate such sets originate from the properties of m-sequences, Reed–Solomon codes, or combinatorial methods used in the ring of integers $\bmod p$ for appropriate prime p [1], [2]. For example, an optimal family of frequency hopping (FH) sequences having p^k (p is a prime and k is a positive integer) symbols can be easily constructed from m-sequence over a Galois field $\mathrm{GF}(p)$ [3] or from a generalized m-sequence (GM) or a generalized Gordon–Mills–Welch (GGMW) sequence over a polynomial residue class ring [4], [5]. Such sequences have optimal periodic autocorrelation functions. However, usually the length of a correlation window is shorter than the period of the chosen FH sequence due to the limited synchronization time or hardware complexity. Moreover, the window length may vary from time to time depending on the channel conditions. In that case, the partial Hamming autocorrelation,

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