Performance of Reed-Solomon Codes for Hybrid-ARQ over Rayleigh Fading Channels under Imperfect Interleaving

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Abstract

We present a new method to analyze the performance of Reed-Solomon codes for hybrid-ARQ over Rayleigh fading channels with imperfect interleaving. Using the asymptotic results of level crossings of the faded envelope, we characterize the packet error probability as a function of the Doppler frequency and mean signal-to-noise ratio (SNR). We then use the notion of equivalent Doppler frequency to include the effect of imperfect interleaving on performance. This approach allows us to directly evaluate the throughput of hybrid-ARQ schemes and also specify minimum interleaving depths for achieving optimum throughput on correlated Rayleigh channels.

Keywords: Rayleigh Fading, Reed-Solomon Codes, Imperfect interleaving

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1 Introduction

Signal fading due to multipath propagation is one of the major impairments in mobile communications. In narrowband channels, fades are generally characterized by considering the envelope of the received signal to be a correlated Rayleigh signal. Due to correlation or memory of the fading process, fades usually cause bursts of errors. Most of the work involving channel coding is based on mathematically simpler memoryless channel models. The error dependence is generally assumed to be removed by perfect interleaving/deinterleaving or simply ignored. However, the interleaving in real systems may be limited due to the delay constraints in certain applications. If the interleaving is not perfect, the channel coding performance, delay, and throughput are likely to be affected. On the other hand, even if the delay is more tolerable in some non-real-time data communications, application specific design constraints may restrict the interleaving depth to a certain level. Thus it is of interest to investigate the performance of imperfectly interleaved coding schemes and compare their performance relative to that of perfect interleaving schemes.

In previous works, Reed-Solomon (RS) codes with perfect interleaving were analyzed in hybrid-ARQ schemes for fading channels [1, 2]. Markov models have been used to evaluate the packet error rates of error-correcting block codes in fading channels [3–6]. Recently, a new analytical method has been proposed to analyze non-interleaved RS codes in Rayleigh fading channels [7]. This method uses the fade and interfade statistics in Rayleigh fading channels [8–10] to estimate the performance of non-interleaved RS codes. It provides an alternative to traditional Markov model based approaches. This new approach is attractive in that packet error performance can be evaluated directly as a function of the Doppler frequency and the mean SNR value.

In this paper, by considering the equivalence between a block interleaved system and a non-interleaved system, we extend our analysis on non-interleaved RS codes to the more general case of imperfectly interleaved RS codes for hybrid-ARQ. We first derive statistics of fade duration and arrivals using asymptotic results from level crossings of Rayleigh processes. This allows us to characterize packet error probabilities as a function of the Doppler frequency, the mean SNR value, and required SNR value. We then use the notion of equivalent Doppler frequency to include the effect of imperfect interleaving on packet error performance. Using this methodology, we estimate the required minimum interleaving depth to achieve optimum performance under different fading rates.

This paper is organized as follows. In Section 2, the fading statistics of Rayleigh channels are presented and some assumptions are justified. In Section 3, we point out the equivalence between block interleaved system and non-interleaved system with higher Doppler frequency. In Section 4, the error performance of imperfectly interleaved RS codes in Rayleigh fading channels is analyzed. We present the numerical results in Section 5 and conclude in Section 6.

2 Fading Statistics of Rayleigh Fading Channels

In land-mobile radio communications, for the situations of non-line-of-sight, the Rayleigh distribution is commonly used to describe the rapid fluctuations of the envelope of the received signal [11]. Considering non-coherent symbol detection or coherent detection with perfect phase recovery, only the amplitude of the fading process is relevant for channel symbol detection. Adopting the two states from the Gilbert channel model [3], we divide the channel into fade and interfade states. The usual first order 2-state Markov model [3–6] assumes that the distributions of the duration of fade and interfade are geometric. In this paper, we use a similar 2-state model but we derive the distributions of fade and interfade durations using the asymptotic level crossings of the Rayleigh signal envelope [8]. The justification for using asymptotic results is that signal outage can be typically considered to be a rare event that occurs when the signal drops below a specified threshold target value. This also corresponds to deep fades.

If R_{req} denotes the specified signal target, then the level crossing rate N_R, defined as the expected rate at which the envelope crosses the signal level $r = R_{req}$ in the positive (or negative) direction, is given as [11]

$$N_R = \sqrt{2\pi} f_D \rho e^{-\rho^2}, \tag{1}$$

where f_D is the maximum Doppler frequency of the Rayleigh fading signal and $\rho = R_{req}/R_{rms}$, with R_{rms} being the root mean square value of the Rayleigh fading signal. It is also shown in [11] that the mean fade duration $\bar{\tau}_f$ of Rayleigh fading signal is given as

$$\bar{\tau}_f = \frac{P(r \le R_{req})}{N_R} = \frac{e^{\rho^2} - 1}{\sqrt{2\pi} f_D \rho}. \tag{2}$$

While the above results only describe the average fading rate and duration, Rice [8] has derived an asymptotic result for the distribution of fade durations for deep fades by a Rayleigh signal $r(t)$. Specifically it is shown in [8] that the complementary cumulative distribution function (CDF) of fade duration (τ_f, as shown in Figure 1) is given as

$$P_{\tau_f}(\tau_f > \tau) = \frac{2}{u} I_1(\frac{2}{\pi u^2}) exp(-\frac{2}{\pi u^2}) \tag{3}$$

where u is the normalized fade duration with respect to mean fade duration $\bar{\tau}_f$, defined as $u = \tau/\bar{\tau}_f$, and $I_1(\cdot)$ is the modified Bessel function of the first kind (page 375 in [12]).

From the function $P_{\tau_f}(\tau_f > \tau)$, the probability density function (pdf) $f_{\tau_f}(\tau)$ can be obtained as

$$\begin{aligned} f_{\tau_f}(\tau) &= -\frac{dP_{\tau_f}(\tau_f > \tau)}{d\tau} = -\frac{1}{\bar{\tau}_f} \frac{dP_{\tau_f}(\tau_f > \tau)}{du} \\ &= -\frac{1}{\bar{\tau}_f} \Bigg[-\frac{2}{u^2} I_1(\frac{2}{\pi u^2}) exp(-\frac{2}{\pi u^2}) + \frac{8}{\pi u^4} I_1(\frac{2}{\pi u^2}) exp(-\frac{2}{\pi u^2}) \\ &\quad -\frac{4}{\pi u^4}(I_1(\frac{2}{\pi u^2}) + I_2(\frac{2}{\pi u^2})) exp(-\frac{2}{\pi u^2}) \Bigg]. \end{aligned} \tag{4}$$

Note that the pdf in the above equation which characterizes fade duration is different from the usual assumption of 2-state Markov models for fade characterization that assumes this pdf is exponential in the continuous model or geometric in the discrete model.

In order to characterize the fade arrival process, we rely on the following results from [9]. The study of level-crossings of χ^2 processes by Aronowich and Adler [9] proved that the asymptotic low-level down-crossings of a χ^2 process form a Poisson process. Since for a Rayleigh signal $r(t)$, $r^2(t)$ is a χ^2 process with 2 degrees of freedom, the asymptotic down-crossings of a Rayleigh signal (denoted as the fade arrival in Figure 1) is also a Poisson process. Thus the CDF of the fade arrival interval τ_s can be expressed as

$$P_{\tau_s}(\tau_s \le \tau) = 1 - \exp(-\frac{\tau}{\bar{\tau}_s}) \tag{5}$$

where the mean value of the fade arrival interval $\bar{\tau}_s$ is given by

$$\bar{\tau}_s = \frac{1}{N_R} = \frac{1}{\sqrt{2\pi} f_D \rho e^{-\rho^2}} \tag{6}$$

The probability of i fade arrivals during an interval of L seconds (such as a packet duration) is given as

$$P\{N(L) = i\} = \frac{(L/\bar{\tau}_s)^i e^{-L/\bar{\tau}_s}}{i!} \tag{7}$$

Hence the number of fade arrivals depends on the channel only through the mean fade arrival interval $\bar{\tau}_s$. Notice that this equation is generally true for arbitrary L (even for small L). The fade arrival interval τ_s is the sum of the fade duration τ_f and the interfade duration τ_i ($\tau_s = \tau_f + \tau_i$). Since we are considering the case of deep fades, the interfade durations (τ_i) are generally much larger than the fade durations (τ_f). This means that the fade arrival intervals (τ_s) are mainly decided by the interfade durations (τ_i). Thus the mean fade arrival interval $\bar{\tau}_s$ is more or less independent of the fade durations. Hence the number of fade arrivals during an interval is assumed to be independent of the fade durations. For this assumption to be valid, the only condition is that the interfade durations (τ_i) should be generally much larger than the fade durations (τ_f), which is the case for deep fades. Also, since the fade arrival intervals are generally much longer than the effective range of the autocorrelation of the fading signal in deep fades, it is reasonable to assume that the fade durations are mutually independent. These assumptions will be applied in the derivation of the probability of packet error in section 4.

The description of the fading process in this section is characterized purely by the Doppler frequency and signal strength. In a practical packet communication system, there is channel coding as well as interleaving. Thus durations of fades may not directly map to bit or packet errors. Further, interleaving also undoes the error correlation among sequences of transmitted bits in a packet. We describe in the next section how interleaving affects the correlation and fading statistics of the received signal.

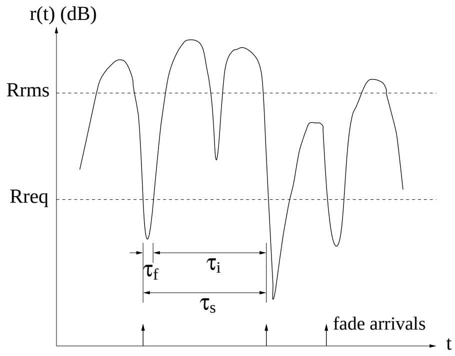

Figure 1: The illustration of fade arrivals, fade arrival interval τ_s, fade duration τ_f, interfade duration τ_i.

3 Interleaving and Equivalent Doppler Frequency

Interleaving is commonly used to reduce the correlation of errors in fading channels. If the interleaving is perfect or sufficiently deep, then adjacent symbols appear to be independent at the decoder. However, the perfect interleaving is not always feasible due to delay constraints and hardware memory limits especially in high data rates. A more realistic case is imperfect interleaving, where adjacent received symbols are not completely independent. As pointed out in [13,14], a slowly fading system with interleaving is equivalent to a faster fading system without interleaving. Let us consider that a block code (n, k) is used in conjunction with a commonly used $D \times N$ block interleaver shown in Figure 2 and assume that N is an integer multiple of n to avoid the "wraparound" effect [15]. The parameter D is the interleaving depth and N is the interleaving length. The encoded symbols are written into a $D \times N$ matrix by rows. The sequence for transmission is by columns. At the receiving end, symbols are stored into a $D \times N$ matrix by columns. Decoding then commences in row-by-row fashion. After deinterleaving, two adjacent code symbols at the receiving end are separated by $D - 1$ code symbols during transmission. For a fading channel with normalized Doppler frequency of $f_D T_s$ with T_s being the symbol duration, after interleaving/deinterleaving, the equivalent normalized Doppler frequency becomes $f_D(DT_s)$. Thus the interleaved system with interleaving depth D and Doppler frequency of f_D is equivalent to a non-interleaved system ($D = 1$) with Doppler frequency of Df_D [13]. We call Df_D the equivalent Doppler

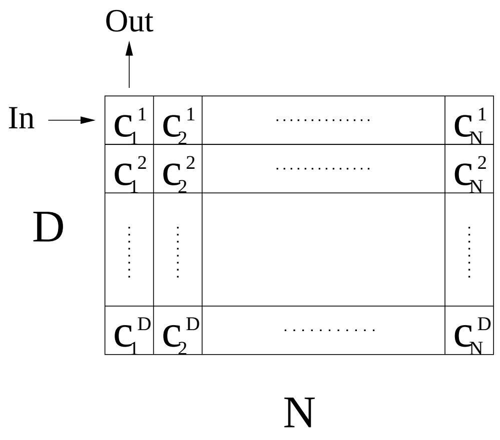

Figure 2: Block interleaver.

frequency of the interleaved system. Thus, in the case of common block interleaver, we could estimate the performance of imperfectly interleaved RS codes using the fading statistics of the Rayleigh fading channel with the equivalent Doppler frequency.

4 Performance Analysis

Using the results from previous sections, we will present an approach that allows us to analyze the packet error performance of imperfectly interleaved RS codes in Rayleigh fading channels.

4.1 Probability of packet error

We will first evaluate the probability of packet error for a t-error-correcting (n, k) RS code with code symbols from GF(2^m) in Rayleigh fading channels. Each code symbol is composed of m channel bits (also m channel symbols when using binary modulation). As presented in [1], RS code with code symbol interleaving offers significantly better performance than the same code with channel symbol interleaving. Thus we only consider code symbol interleaved RS codes in this paper. As shown in Figure 2, the interleaved unit is the code symbol composed of m channel bits. It is assumed that the channel fading is frequency nonselective and is slow with respect to the rate of code symbol transmission so that the signal envelope is constant during a code symbol interval. Thus each entire code symbol is either in the

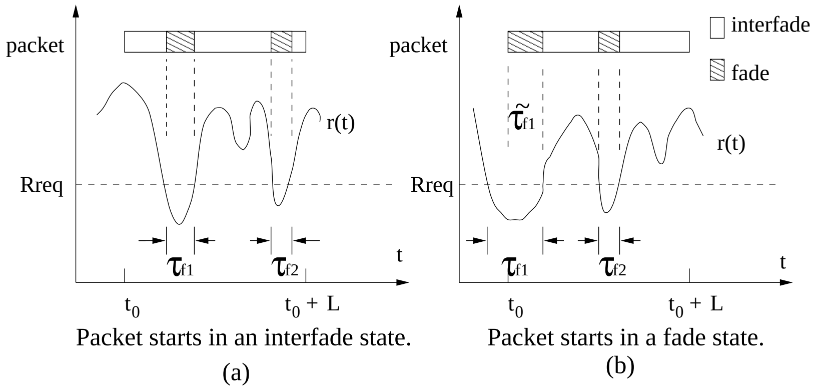

Figure 3: The packet starts in (a) an interfade; (b) a fade state.

fade or in the interfade state. Such a piecewise constant assumption is widely used in several previous works to simplify the analysis [1,4,5,16,17]. The difference in each of those works is the relative time interval over which such a piecewise constant assumption is made. It could be bit duration as in [4,17], code symbol (multiple bits) as in [1,5] and this paper, or even the entire packet as in [16]. In our examples later, code symbol duration (T_s) is chosen to be 0.3 ms. For a Doppler frequency of 80Hz, $f_D T_s$ is $0.024 \ll 1$. It is reasonable to assume that the signal is constant over a code symbol interval.

The code symbol errors caused by fades in a codeword/packet may come in bursts due to the correlation of the equivalent channel after imperfect interleaving as illustrated in Figure 3. The channel is in fade when the signal-to-noise ratio (SNR) is less than SNR_{req} (corresponding to R_{req}) and in interfade when the SNR is greater or equal to SNR_{req}. The selection of SNR_{req} and its effect on the probability of packet error will be discussed later in section 5. Since the signal envelope is assumed to be constant during a code symbol, the conditional probability of code symbol error in interfade (P_{si}) and in fade (P_{sf}) can be calculated as

$$\begin{aligned} P_{si} &= \frac{\int_{SNR_{req}}^{\infty} [1-(1-P_b(x))^m] p(x)dx}{\int_{SNR_{req}}^{\infty} p(x)dx} \\ P_{sf} &= \frac{\int_{0}^{SNR_{req}} [1-(1-P_b(x))^m] p(x)dx}{\int_{0}^{SNR_{req}} p(x)dx} \end{aligned} \tag{8}$$

where $P_b(x)$ is the probability of channel symbol error and $p(x)$ is the probability density function (pdf) of SNR. For BPSK modulation, P_b is given as

$$P_b(\frac{E_s}{N_0}) = Q(\sqrt{\frac{2E_s}{N_0}}) \tag{9}$$

where $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt$. For a Rayleigh fading channel, the pdf $p(x)$ is given as

$$p(x) = \frac{1}{\Gamma} \exp(-\frac{x}{\Gamma}), \;\; x \geq 0 \tag{10}$$

with Γ being the mean SNR. Note that the SNR here is for the channel symbol, i.e., E_s/N_0. It is related to E_b/N_0 as

$$\frac{E_s}{N_0} = \frac{E_b}{N_0} R \tag{11}$$

where $R = k/n$ is the code rate.

For the most commonly used hard-decision bounded distance decoding techniques for RS codes, such as Berlekamp-Massey algorithm [18, 19], the packet is regarded in error when the total code symbol errors (e) in a packet exceed the error-correcting capability (t) of the code (where erasure is not considered). Thus the minimum number of code symbol errors to cause a packet error is $t+1$. Since P_{si} is much less than P_{sf}, we could ignore the code symbol errors in interfade (i.e., assume $P_{si} = 0$) to simplify the analysis. When the number of code symbols in fade in a packet is equal to n_f ($t + 1 \leq n_f \leq n$), the conditional probability of packet error can be calculated as

$$\begin{aligned} P_E|n_f &= P\{e \geq t+1 | n_f\} \\ &= \sum_{i=t+1}^{n_f} \binom{n_f}{i} P_{sf}^i (1 - P_{sf})^{n_f - i}. \end{aligned} \tag{12}$$

The unconditional probability of packet error for an n-code-symbol packet is given as

$$\begin{aligned} P_E &= P\{e \geq t+1\} \\ &= \sum_{n_f=t+1}^{n} P(n_f, n) \sum_{i=t+1}^{n_f} \binom{n_f}{i} P_{sf}^i (1 - P_{sf})^{n_f - i}. \end{aligned} \tag{13}$$

The problem still remains of evaluating $P(n_f, n)$, which is the probability that an n-code-symbol packet contains n_f code symbols in fade. Notice that the code symbols in fade may come from one or more fades and these fades may be partially located inside the packet duration. All these make the estimation of $P(n_f, n)$ nontrivial. We start with the probability that an n-code-symbol packet contains $n_f \geq n_{f0}$ code symbols in fade

$$P_{n_f} \equiv P(n_f \geq n_{f0}, n) = P\{\sum_i \tau_{fi} \geq n_{f0} T_s \equiv \tau_m\} \tag{14}$$

where $\sum_i \tau_{fi}$ represents the sum of fade durations over a codeword epoch and T_s is the code symbol duration. Note that τ_m denotes the minimum fade duration (combined over all fades within a codeword/packet) to cause $n_f \geq n_{f0}$ code symbols in fade. The subscript m in τ_m should not be confused with the number of channel bits per code symbol. P_{n_f} can be derived under the condition that the envelope of the faded signal at the beginning of a packet is either in the interfade or the fade state, as illustrated in Figure 3. The results are summarized below while the details of derivations are given in the Appendix.

Case 1: The packet starts in an interfade state (Figure 3(a)).

Consider first, the situation where the faded envelope $r(t)$ is above the threshold R_{req} at the beginning of a packet epoch. The probability that the packet starts in an interfade state can be obtained for the Rayleigh fading signal as

$$P^{(1)} = P(r \geq R_{req}) = e^{-1/F} \tag{15}$$

where r is the fading envelope at the beginning of the packet, and $F = R_{rms}^2/R_{req}^2$ is the fade margin in absolute value. P_{n_f} due to i fades under case 1, denoted as $P_{n_f}^{(1)}(i)$, can be expressed as follows.

(a) $i = 1$,

$$P_{n_f}^{(1)}(1) = N_R(L - \tau_m)e^{-N_R L} P_{\tau_f}(\tau_{f1} \geq \tau_m) \tag{16}$$

where L is the packet length, N_R is the mean downward level crossing rate, and $P_{\tau_f}(\tau_{f1} \geq \tau_m)$ can be calculated using equation (3).

(b) $i = 2$,

$$\begin{aligned} P_{n_f}^{(1)}(2) &= \int_0^{\tau_m} \frac{1}{2!}(N_R(L - \tau_m + \tau_{f1}))^2 e^{-N_R L} P_{\tau_f}(\tau_{f2} \geq \tau_m - \tau_{f1}) f_{\tau_f}(\tau_{f1}) d\tau_{f1} \\ &+ \frac{1}{2!}(N_R L)^2 e^{-N_R L} \int_{\tau_m}^{L} f_{\tau_f}(\tau_{f1}) d\tau_{f1} \end{aligned} \tag{17}$$

where $f_{\tau_f}(\cdot)$ is the probability density function of the fade duration given by equation (4).

(c) $3 \leq i \leq t$,

$$\begin{aligned} P_{n_f}^{(1)}(i) &= \int_0^{\tau_m} \frac{1}{i!}(N_R(L-\tau_m+\tau_{1,i-1}))^i e^{-N_R L} P_{\tau_f}(\tau_{fi} \geq \tau_m - \tau_{1,i-1}) f_{\tau_{1,i-1}}(\tau_{1,i-1}) d\tau_{1,i-1} \\ &+ \frac{1}{i!}(N_R L)^i e^{-N_R L} \int_{\tau_m}^{L} f_{\tau_{1,i-1}}(\tau_{1,i-1}) d\tau_{1,i-1} \end{aligned} \tag{18}$$

where

$$\tau_{1,i-1} = \tau_{f1} + \tau_{f2} + ... + \tau_{fi-1} \tag{19}$$

$$f_{\tau_{1,i-1}}(\tau) = f_{\tau_{f1}}(\tau) * f_{\tau_{f2}}(\tau) * ... * f_{\tau_{fi-1}}(\tau). \tag{20}$$

with * denoting the convolution operation.

(d) $i > t$,

$$P_{n_f}^{(1)}(i) = \frac{1}{i!}(N_R L)^i e^{-N_R L}. \tag{21}$$

Case 2. The packet starts in a fade state (Figure 3(b)).

Consider next, the situation where the faded envelope $r(t)$ is below the threshold R_{req} at the beginning of a packet epoch. The probability that the packet starts in a fade state can now be obtained for the Rayleigh fading channel as

$$P^{(2)} = P(r < R_{req}) = 1 - e^{-1/F}. \tag{22}$$

The probability of packet error due to i fades under case 2 can be expressed as follows.

(a) $i = 1$,

$$P_{n_f}^{(2)}(1) = e^{-N_R L} P_{\tilde{\tau}_{f1}}(\tilde{\tau}_{f1} \geq \tau_m) \tag{23}$$

where $\tilde{\tau}_{f1}$ is the portion of the first fade duration τ_{f1} located in the packet duration as explained in Appendix.

(b) $i=2$,

$$\begin{aligned}P_{n_f}^{(2)}(2) &= \int_0^{\tau_m} N_R(L-\tau_m+\tilde{\tau}_{f1})e^{-N_RL}P_{\tau_f}(\tau_{f2}\geq\tau_m-\tilde{\tau}_{f1})f_{\tilde{\tau}_{f1}}(\tilde{\tau}_{f1})d\tilde{\tau}_{f1}\\ &+ N_RLe^{-N_RL}\int_{\tau_m}^{L} f_{\tilde{\tau}_{f1}}(\tilde{\tau}_{f1})d\tilde{\tau}_{f1}. \end{aligned} \tag{24}$$

(c) $3\leq i\leq t$,

$$\begin{aligned}P_{n_f}^{(2)}(i) &= \int_0^{\tau_m} \frac{1}{(i-1)!}(N_R(L-\tau_m+\tilde{\tau}_{1,i-1}))^{i-1}e^{-N_RL}P_{\tau_f}(\tau_{fi}\geq\tau_m-\tilde{\tau}_{1,i-1})f_{\tilde{\tau}_{1,i-1}}(\tilde{\tau}_{1,i-1})d\tilde{\tau}_{1,i-1}\\ &+ \frac{1}{(i-1)!}(N_RL)^{i-1}e^{-N_RL}\int_{\tau_m}^{L} f_{\tilde{\tau}_{1,i-1}}(\tilde{\tau}_{1,i-1})d\tilde{\tau}_{1,i-1} \end{aligned} \tag{25}$$

where

$$\tilde{\tau}_{1,i-1} = \tilde{\tau}_{f1}+\tau_{f2}+...+\tau_{fi-1} \tag{26}$$

$$f_{\tilde{\tau}_{1,i-1}}(\tau) = f_{\tilde{\tau}_{f1}}(\tau)*f_{\tau_{f2}}(\tau)*...*f_{\tau_{fi-1}}(\tau). \tag{27}$$

(d) $i>t$

$$P_{n_f}^{(2)}(i)=\frac{1}{(i-1)!}(N_RL)^{i-1}e^{-N_RL}. \tag{28}$$

Combining the above two cases, P_{n_f} can be calculated as

$$P_{n_f}=P^{(1)}\sum_i P_{n_f}^{(1)}(i)+P^{(2)}\sum_i P_{n_f}^{(2)}(i). \tag{29}$$

Thus we have given the general expressions to calculate P_{n_f}, which leads to the calculation of $P(n_f,n)$ and the probability of packet error P_E (equation (13)). Although the value of i in the above equations can be arbitrarily large, it cannot be greater than the total number of code symbols in the packet (n). Actually, when i increases, the packet error rate due to i fades decreases very quickly. Hence only the items with significant contribution to total P_E need to be included in the numerical computation.

4.2 Reed-Solomon codes for hybrid-ARQ

As shown in [1], a single pair of RS encoder/decoder can be used in hybrid-ARQ for both error correction and error detection (without extra CRC error detection). For commonly used bounded distance decoding for RS codes, such as the Berlekamp-Massey algorithm [18, 19], within the error-correction capability of the code, the symbol errors can be corrected. If the received codeword does not lie in the decoding sphere of any codeword, the decoder will indicate an error detected, i.e., the error detection case. Thus, an automatic-repeat-request (ARQ) is sent back to the transmitter to request retransmission. If the received codeword falls within a decoding sphere of some other codeword, the decoder will not detect the errors and will decode the received codeword as some other codeword, i.e., the incorrect decoding case. The incorrect decoding errors are undetectable and cannot be corrected by retransmissions in general. Hence, in the hybrid-ARQ protocol, we need to distinguish the probability of incorrect decoding (P_{ICD}) and the probability of error detection (P_{ED}).

The probability of incorrect decoding is dependent on the Hamming distances between the transmitted codeword and all other codewords. For any linear code, we could assume that the all-zero codeword is sent without loss of generality. Thus P_{ICD} is dependent on the weight distribution of the code, which is the Hamming distances from the all-zero codeword to all other codewords. Since the RS code is a linear and maximum-distance-separable (MDS) code, the weight distribution of a (n, k) RS code over GF(2^m) is known exactly to be [18]

$$A_j = \binom{n}{j}(2^m - 1)\sum_{i=0}^{j-d_{min}}(-1)^i\binom{j-1}{i}2^{m(j-i-d_{min})} \tag{30}$$

where the minimum distance $d_{min} = n - k + 1$ and A_j represents the number of codewords of weight j.

To calculate P_{ICD} and P_{ED}, we also need the probability of j errors ($P_j = P\{e = j\}, j = 1, 2, ...n$) in an n-code-symbol codeword. The values of P_j can be evaluated easily from the equation of $P_E = P\{e \geq t+1\}$ (equation (13)) by assigning t to different values and differentiating it. Since we assume that the all-zero codeword is transmitted, P_j is the probability of weight-j errors. Notice that P_j is due to all possible weight-j error patterns.

When transmitting one code symbol, which is composed of m channel symbols, the number of possible error patterns is $2^m - 1$. This can be seen clearly from Figure 4. When transmitting j code symbols, if every code symbol is in error, the total number of possible error patterns is $(2^m - 1)^j$. When transmitting an n-symbol packet, the number of possible

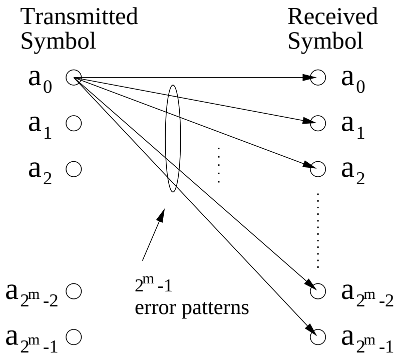

Figure 4: Error patterns for each code symbol transmission.

combinations of j code symbols is $\binom{n}{j}$. Hence the total number of possible weight-j error patterns when transmitting an n-symbol packet, denoted as K_j, is

$$K_j = \binom{n}{j}(2^m - 1)^j \tag{31}$$

The probability of weight-j errors is the summation of probabilities of all possible weight-j error patterns, given as

$$P_j = \sum_{i=1}^{K_j} P_j(i) \tag{32}$$

where $P_j(i)$ represents the probability of the i-th weight-j error pattern. If we assume that all possible weight-j error patterns occur with equal probability, that is

$$P_j(i) = P_j{}', \quad \forall i, \tag{33}$$

then we have

$$P_j{}' = \frac{P_j}{K_j} = \frac{P_j}{\binom{n}{j}(2^m - 1)^j}. \tag{34}$$

Precisely speaking, the probabilities of different error patterns with the same weight are different, no matter non-interleaved codes, imperfect interleaved codes, or perfect interleaved codes are used. The assumption of equal probability for different error patterns with the same weight is somewhat unrealistic. But in spite of this it is still generally assumed to make the analytical estimation of P_{ICD} tractable [1, 19]. Under such assumption, the probability of incorrect decoding P_{ICD} is given as [19]

$$
\begin{aligned}
P_{ICD} &= \sum_{h=d_{min}}^{n} P_{ICD}(h) \\
&= \sum_{h=d_{min}}^{n} \sum_{s=0}^{t} \sum_{j=h-s}^{h+s} \sum_{r=r_1}^{r_2} \binom{h}{h-s+r} \binom{s-r}{j-h+s-2r} \\
&\quad \times \binom{n-h}{r} (2^m-2)^{j-h+s-2r} (2^m-1)^r {P_j}' A_h \qquad (35)
\end{aligned}
$$

where $r_1 = max\{0, j-h\}$, $r_2 = \lfloor \frac{j-h+s}{2} \rfloor$, and ${P_j}'$ is given as equation (34).

The probability of error detection P_{ED} can then be evaluated as

$$P_{ED} = P_E - P_{ICD}. \qquad (36)$$

The throughput for RS coded hybrid-ARQ is a function of the retransmission scheme. For a selective-repeat type-I hybrid-ARQ system without code combining, the throughput is given as [1, 20]

$$\eta = R(1 - P_R) = R(1 - P_{ED}) \qquad (37)$$

where $R = k/n$ is the code rate and P_R is the probability of retransmission which equals to P_{ED} in this case.

5 Numerical Results

In this section numerical performance results are presented for imperfectly interleaved RS codes in a Rayleigh fading channel. The influence of Doppler frequency (f_D), interleaving depth (including non-interleaving, $D = 1$), mean SNR, and error-correcting capabilities on the performance is studied. We illustrate the results using (63,k) RS codes over GF(2^6) ($k = 47, 53, 57$). Notice that the non-interleaved (63,47) RS code is also used in Cellular Digital Packet Data (CDPD) [21]. Using a $D \times N$ block interleaver, the interleaving length N is chosen to be equal to the code word length n, which is 63 in this example. For all the analytical and simulation results, the packet length is $N_c = 378$ bits. With channel

$SNR_{req}(dB)$	P_{si}	P_{sf}	P_E
2	5.658×10^{-3}	0.5247	3.477×10^{-2}
3	3.582×10^{-3}	0.4524	3.808×10^{-2}
4	2.018×10^{-3}	0.3819	3.952×10^{-2}
5	9.852×10^{-4}	0.3167	3.899×10^{-2}
6	4.029×10^{-4}	0.2593	3.675×10^{-2}
7	1.323×10^{-4}	0.2109	3.307×10^{-2}

Table 1: Sensitivity of P_E to the value of SNR_{req} (RS(47,63) code, mean SNR = 15 dB, $f_D = 40$ Hz, non-interleaving).

symbol rate $R_c = 19.2$ kbps, the packet length $L = N_c/R_c$ is approximately 20 ms. In the simulations, the RS encoder/decoder functions are from MATLAB Communications Toolbox. In all the results presented, we assume coherent BPSK demodulation.

As mentioned in Section 4, the parameter SNR_{req} is used to define the fade and the interfade state of the channel. Similar to the case of the Gillbert-Elliott model, the selection of the threshold SNR_{req} is somewhat "ad hoc" as described in [6]. It can be dependent on the channel requirement and possibly the performance criteria for reliable communication. For instance, if we define the channel symbol error probability less than 10^{-2} to be a good channel requirement, we could select $SNR_{req} = 5$ dB for BPSK. The important fact is that the channel symbols or code symbols located in the interfade duration are received in error with much lower probability than that in the fade duration. It is desirable that the probability of packet error is not changed significantly by a minor variation of the selection of parameter SNR_{req}. Table 1 shows the sensitivity of probability of packet error to the parameter SNR_{req}. For different values of SNR_{req}, the resulting symbol error probabilities in fade (P_{sf}) and in interfade (P_{si}) may change dramatically, however the probability of packet error (P_E) is relatively stable. A similar phenomenon is also observed in [6].

Next we will show the validation of analysis of imperfectly interleaved RS codes using the concept of equivalent Doppler frequency. Figure 5 shows the comparison of simulation and analytical results of the probability of packet error with different interleaving depths. Notice

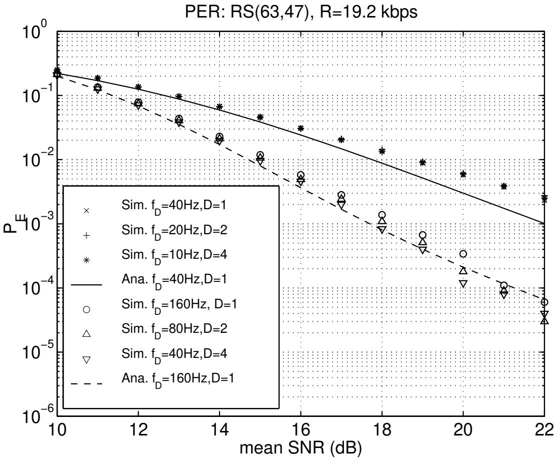

Figure 5: Simulation and analytical results of probability of packet error with different interleaving depth. BPSK, RS(63,47), channel rate $R_c = 19.2$ kbps, equivalent Doppler frequency $D * f_D$=40, 160 Hz.

that the analysis of imperfectly interleaved RS codes using equivalent Doppler frequency as described above works quite well. For instance, the probability of packet error (P_E) for a non-interleaved code at $f_D = 160$ Hz from our analysis matches the P_E from simulations for D =1, 2, and 4 interleaved codes at f_D =160, 80 and 40 Hz, respectively. Thus by applying our analytical method described in Section 4, we could directly compute the performance of imperfectly interleaved RS codes in Rayleigh fading channels.

Figure 6 shows P_E as a function of mean SNR for RS codes with the same packet length and different error-correcting capabilities. Specifically, we have plotted results for RS codes of length 63 with error correcting capabilities of $t = 3, 5, 8$. As expected, we note that P_E is lower for the higher error-correcting code. However this does not mean that the higher error-correcting code is always preferred since the corresponding lower code rate will affect the information throughput. As shown in Figure 7, the lower error-correcting code has higher information throughput in the higher SNR region. Such analysis on information throughput is useful in adaptive coding study in which different codes can be switched according to some SNR threshold in order to keep the possible largest information throughput.

Another key performance criterion is the probability of incorrect decoding (P_{ICD}). Since

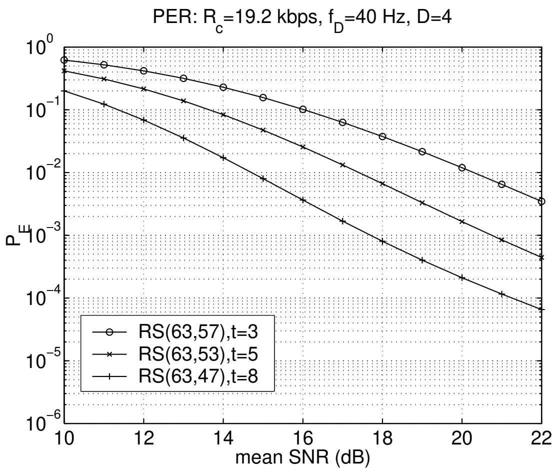

Figure 6: Probability of packet error (analysis) versus mean SNR for codes with different error-correcting capabilities.

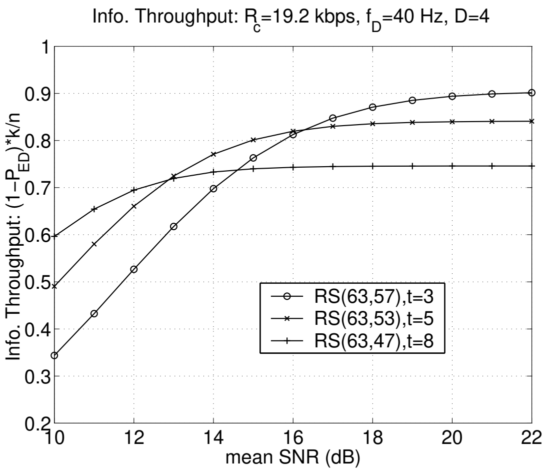

Figure 7: Information throughput (analysis) versus mean SNR for different error-correcting codes. $Df_D = 160$ Hz.

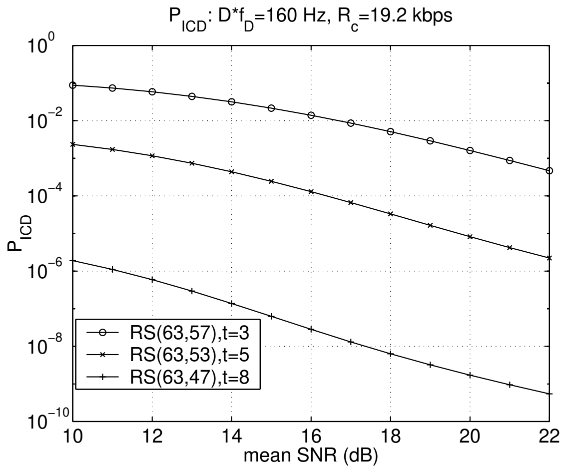

Figure 8: Probability of incorrect decoding (analysis) versus mean SNR for different error-correcting codes. $Df_D = 160$ Hz.

such errors cannot be recovered by retransmission, the selection of the coding scheme has to ensure that P_{ICD} is smaller than a given value, say 10^{-5}. Under the same conditions of Figure 6, the corresponding P_{ICD} is plotted as a function of mean SNR for different codes in Figure 8. Notice in the low SNR region, P_{ICD} might be not small enough for a low error-correcting code (e.g. RS(63,57)). Thus extra CRC bits might be needed to satisfy the P_{ICD} requirement. Comparing Figures 6 and 8, it is noted that P_{ICD} is much smaller than P_E in the case of the RS(63,47) code. In this case the probability of error detection (P_{ED}) is approximately equal to P_E.

Figure 9 shows the effect of interleaving on throughput at a fixed value of mean SNR. The perfect interleaving case (shown as $D = Inf$) is also shown for comparison. Note that at this relatively good mean SNR value, the interleaving decreases P_E and increases the throughput. As the product of interleaving depth D and Doppler frequency f_D increases (by increasing D and/or f_D), the throughput approaches the limit with perfect interleaving which is constant for all Doppler frequencies. Using such an analysis, we could estimate the required minimum depth of interleaving at a particular fade rate to achieve near optimum performance (for example, 99% of the throughput limit with perfect interleaving) for a particular code. When the users can be separated into different groups according to their speeds, it is possible that

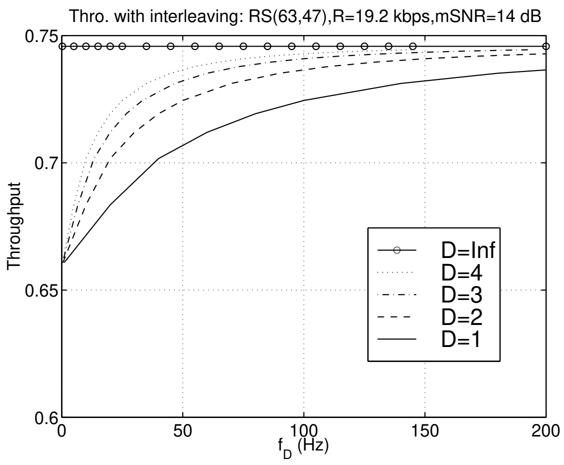

Figure 9: Information throughput (analysis) with different interleaving depth. BPSK, RS(63,47), channel rate R=19.2 kbps, mean SNR=14 dB.

the appropriate interleaving depths can be chosen for different groups to reduce the delay and satisfy the error requirement at the same time. Figure 10 shows the required minimum interleaving depth as a function of Doppler frequency (f_D). As shown in the figure, for a given code at fixed mean SNR, the required minimum interleaving depth increases as f_D decreases. For a very low Doppler frequency, it may not be possible to increase the interleaving depth beyond the constraints imposed due to delay requirements of the applications. In this case, either the performance will degrade due to imperfect interleaving, or some other scheme has to be used to compensate for slow fading, such as slow frequency hopping as used in GSM systems [22].

6 Conclusions

In this paper we have presented a new method to analyze the performance of Reed-Solomon codes for hybrid-ARQ over Rayleigh fading channels with imperfect interleaving. Our approach was based on the equivalent level crossing statistics of the faded envelope during a packet epoch after imperfect interleaving/de-interleaving. It can be used to calculate the performance of RS codes over Rayleigh fading channels directly from the system parame-

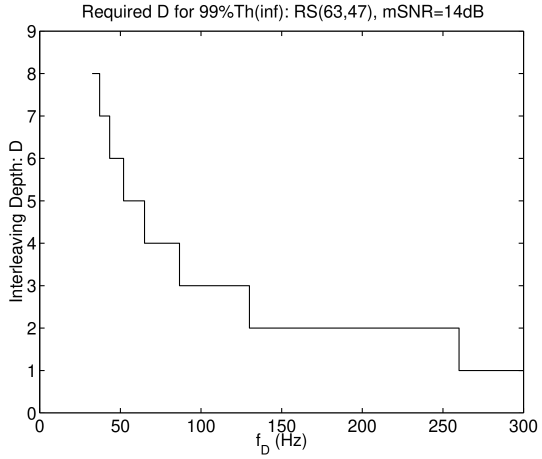

Figure 10: The required minimum interleaving depth to achieve 99% of the throughput limit with perfect interleaving. BPSK, RS(63,47), channel rate R=19.2 kbps, mean SNR=14 dB.

ters, such as mean SNR, Doppler frequency, and interleaving depth. The advantage is that this approach does not require fitting of models to simulation or experimental data for every pair of mean SNR and Doppler frequency as generally required in Markov model based methods. This method can be used to estimate the required minimum depth of interleaving as a function of the Doppler frequency to achieve optimum performance for a given code. The simulation results show very good agreement with the analysis. This method can also be applied to analyze performance of other block codes under imperfect interleaving over Rayleigh fading channels.

Appendix

We will derive analytical expressions of P_{n_f} for a t-error-correcting RS code due to i fades in a packet as presented in Section 4.

Case 1. The packet starts in an interfade state (Figure 3(a)).

Let the faded envelope $r(t)$ be above the threshold R_{req} at the beginning of a packet epoch. The packet length is assumed to be L seconds. The distribution of fade duration τ_f is given by equation (3). As described in Section 2, the fade arrival process $N(t)$ is assumed to be a Poisson counting process with parameter N_R, which is the mean downward level crossing rate given in equation (1). Further, the number of fade arrivals during a packet duration is assumed independent of the fade durations. Under these assumptions, $P_{n_f}^{(1)}(i)$ can be expressed as follows.

(a) When $i = 1$,

$$\begin{aligned} P_{n_f}^{(1)}(1) &= P\{N(0)=0, N(L-\tau_m)=1, N(L)=1\}P(\tau_{f1} > \tau_m) \\ &= P\{N(L-\tau_m)-N(0)=1, N(L)-N(L-\tau_m)=0\}P(\tau_{f1} > \tau_m) \\ &= N_R(L-\tau_m)e^{-N_R(L-\tau_m)}e^{-N_R\tau_m}P_{\tau_f}(\tau_{f1} > \tau_m) \\ &= N_R(L-\tau_m)e^{-N_R L}P_{\tau_f}(\tau_{f1} > \tau_m). \end{aligned} \tag{38}$$

The expression $(N(L-\tau_m)=1)$ ensures that the portion of τ_{f1} located in the packet duration is at least τ_m to cause $n_f \geq n_{f0}$ code symbols in fade. Similarly, when dealing with i fades, the first $i-1$ fades are all located in the packet duration. However the last fade may be only partially located in the packet duration. Thus they need to be treated differently.

(b) When $i = 2$, the conditional probability $P_{n_f}^{(1)}(2|\tau_{f1})$ can be expressed as follows

$$P_{n_f}^{(1)}(2|\tau_{f1}) = \begin{cases} P\{N(0)=0, N(L-\tau_m+\tau_{f1})=2, N(L)=2\} & \\ \quad \times P_{\tau_f}(\tau_{f2} \geq \tau_m - \tau_{f1}) & \text{, if } 0 \leq \tau_{f1} < \tau_m \\ P\{N(0)=0, N(L)=2\} & \text{, if } \tau_m \leq \tau_{f1} \leq L \\ 0 & \text{, otherwise.} \end{cases} \tag{39}$$

The above expressions follow from identifying two cases for the range of τ_{f1}, namely $0 \leq \tau_{f1} < \tau_m$ and $\tau_m \leq \tau_{f1} \leq L$. Therefore the unconditional probability $P_{n_f}^{(1)}(2)$ is given by averaging over τ_{f1}

$$\begin{aligned} P_{n_f}^{(1)}(2) &= \int_0^{\tau_m} \frac{1}{2!}(N_R(L-\tau_m+\tau_{f1}))^2 e^{-N_R L} P_{\tau_f}(\tau_{f2} \geq \tau_m - \tau_{f1}) f_{\tau_f}(\tau_{f1}) d\tau_{f1} \\ &+ \frac{1}{2!}(N_R L)^2 e^{-N_R L} \int_{\tau_m}^{L} f_{\tau_f}(\tau_{f1}) d\tau_{f1}. \end{aligned} \tag{40}$$

(c) For $3 \le i \le t$, first define

$$\tau_{1,i-1} = \tau_{f1} + \tau_{f2} + ... + \tau_{fi-1}. \tag{41}$$

Then the probability density function of $\tau_{1,i-1}$ can be obtained as

$$f_{\tau_{1,i-1}}(\tau) = f_{\tau_{f1}}(\tau) * f_{\tau_{f2}}(\tau) * ... * f_{\tau_{fi-1}}(\tau) \tag{42}$$

by convolution since the fade durations are mutually independent. Now we can treat the i fades case similar to the case of two fades with $\tau_{1,i-1}$ and τ_i as the respective fade durations. Thus we obtain

$$\begin{aligned} P_{nf}^{(1)}(i) &= \int_0^{\tau_m} \frac{1}{i!}(N_R(L-\tau_m+\tau_{1,i-1}))^i e^{-N_R L} P_{\tau_f}(\tau_{fi} \ge \tau_m - \tau_{1,i-1}) f_{\tau_{1,i-1}}(\tau_{1,i-1}) d\tau_{1,i-1} \\ &+ \frac{1}{i!}(N_R L)^i e^{-N_R L} \int_{\tau_m}^{L} f_{\tau_{1,i-1}}(\tau_{1,i-1}) d\tau_{1,i-1}. \end{aligned} \tag{43}$$

(d) When $i > t$, there are more than t fade arrivals in the packet. Since we assume that each code symbol is entirely in the fade or the interfade state as described at the beginning of Section 4, each fade arrival located in the packet will cause at least one code symbol in fade within the codeword. It is assumed that the last fade arrival will also cause at least one code symbol in fade within the codeword even if the last fade may be partially located within the packet. Therefore, there must be more than t code symbols in fade within the packet. Thus for $i > t$, $P_{nf}^{(1)}(i)$ is equal to the probability of i fade arrivals in the packet (equation (7)) and is given as

$$P_{nf}^{(1)}(i) = \frac{1}{i!}(N_R L)^i e^{-N_R L}. \tag{44}$$

Case 2. The packet starts in a fade state (Figure 3(b)).

Let the faded envelope $r(t)$ be below the threshold R_{req} at the beginning of a packet epoch. The first fade (τ_{f1}) has portion ($\tilde{\tau}_{f1}$) located inside the packet duration. We define a random variable α such that the first burst $\tilde{\tau}_{f1} = \alpha\tau_{f1}$. Since the starting point of the packet is arbitrary relative to the starting point of the first fade, we claim that the random variable

α is uniformly distributed over $(0, 1]$. The probability density function of α is given as

$$f_\alpha(\alpha) = \begin{cases} 1 & , 0 < \alpha \leq 1 \\ 0 & , \text{otherwise} \end{cases} \tag{45}$$

Then the distribution function of $\tilde{\tau}_{f1}$ can be obtained as

$$\begin{aligned} P_{\tilde{\tau}_{f1}}(\tilde{\tau}_{f1} > \tau) &= \int_0^1 P_{\tau_f}(\alpha\tau_{f1} > \tau|\alpha) f_\alpha(\alpha) d\alpha \\ &= \int_0^1 P_{\tau_f}(\tau_{f1} > \frac{\tau}{\alpha}|\alpha) f_\alpha(\alpha) d\alpha. \end{aligned} \tag{46}$$

The probability density function of $\tilde{\tau}_{f1}$ can be obtained as

$$f_{\tilde{\tau}_{f1}}(\tau) = -\frac{dP_{\tilde{\tau}_{f1}}(\tilde{\tau}_{f1} > \tau)}{d\tau}. \tag{47}$$

Once we get the distribution function of the first burst in the packet, the rest of the derivation is similar to case 1. The final results are given in Section 4.

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