Optimal Power Allocation for Type II H–ARQ via Geometric Programming

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Abstract —

In mobile wireless data communications, it is very important to reduce the average energy consumption while maintaining a target frame-error-rate (FER) and frame latency. These goals can be achieved by means of Hybrid–ARQ (re)transmission power control. By allocating different symbol energy for each (re)transmission, the average energy consumption can be optimized. For a type-II Hybrid–ARQ scheme over an i.i.d. Rayleigh block fading channel, we show that the above optimization can be formulated and solved as a geometric programming problem. For the special case of two (re)transmissions, the optimal power allocation can also be analytically derived. For a Rate Compatible Punctured Convolutional (RCPC) code, our simulation results show that a gain up to 4dB can be achieved at a target FER of 1E-4 with the optimized power allocation scheme.

I. Introduction

As a simple yet powerful error control scheme, the type II Hybrid–ARQ scheme with Incremental Redundancy (IR–HARQ) has been used in many wireless communication systems. As most wireless systems have only limited power supply, it is especially important to reduce the average energy consumption while maintaining a target frame-error-rate (FER) and frame latency.

For the traditional Hybrid–ARQ scheme, usually the same bit energy is allocated for all (re)transmissions in one ARQ round. This non-adaptive power allocation scheme is not power efficient. During one ARQ round, as the number of (re)transmissions increases, the amount of energy needed to achieve the target FER also changes. Therefore, if different bit or symbol energy is allocated for different (re)transmissions, we can reduce the power consumption and keep the same target FER and frame latency.

For the Hybrid–ARQ scheme, the average energy consumption can be expressed as the weighted sum of the bit energy for each (re)transmission in one ARQ round. For a target FER and frame latency, with different bit or symbol energy allocated for different (re)transmissions, the average energy consumption can be minimized. The optimization problem that minimizes the average energy consumption for an AWGN channel has been analyzed in our previous work [1]. In this paper, we extend this work to an i.i.d. Rayleigh block fading channel. We find that for this type of channel, the optimization problem can be formulated and solved as a geometric programming problem. For the special case of two (re)transmissions, we also derive a closed form solution, which we validate through simulation.

This paper is organized as follows: In section II, we analyze the FER over an i.i.d Rayleigh block fading channel with different symbol energy for each (re)transmission. In section III, the optimization problem is formulated and solved as a geometric programming problem. In section IV, the simulation results for the optimal power allocation strategy are discussed and we conclude with some future directions.

II. FER for independent Rayleigh Block Fading Channel

For a linear code, the union bound to the FER, P_e, is the sum of the pairwise error probabilities between one codeword and all the other codewords. Since all the codewords have the same error property for a linear code, without loss of generality, we assume the all-zero codeword $\mathbf{0}$ is sent. The probability that a non-zero codeword is received can be written as

$$P_e \leq \sum_{i=1}^{M-1} P_2(\mathbf{c_i}, \mathbf{0}),$$

where M is the total number of codewords in a codebook and $P_2(\mathbf{c_i}, \mathbf{0})$ are the pairwise error probabilities between the all-zero codeword and non-zero codewords.

A block fading model [2] is used to model a fading channel that has a high correlation within a decoding block. The correlation comes from non-ideal interleaving in a system that has a decoding delay requirement. Assume the system employs coherent detection and has the perfect side information about the channel at the receiver. The block fading model representation of the channel is given by

$$r(t) = c(t)s(t) + n(t)$$

where $r(t)$ is the received signal; $c(t)$ is the gain of the fading channel that remains constant for L symbol periods; $s(t)$ is the transmitter signal and $n(t)$ is the additive white Gaussian noise. We also assume the fading gain is an i.i.d. Rayleigh random variable.

In the following discussion, we will first derive the pairwise word error probability $P_2(\mathbf{c_i}, \mathbf{0})$ for a block fading channel with different symbol energy across the fading blocks. Using that, we can obtain an approximation for the FER at high signal-to-noise ratios (SNR) for a type II Hybrid–ARQ scheme.

Assume a codeword is binary-PSK modulated and transmitted through F fading blocks. We further divide the codeword into F length-L sub-codewords. Let E_i be the symbol energy of the i-th sub-codeword and α_i be the fading gain of

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the i-th sub-codeword. We assume the normalized random variables α_i have unit second moment, i.e., $\mathbf{E}[\alpha_i^2] = 1$, where $\mathbf{E}[\cdot]$ denotes the expectation of a random variable. Define the weight of the i-th sub-codeword as d_i. Then the squared Euclidean distance between the i-th sub-codeword and the all-zero sub-codeword can be written as $\frac{2E_i}{N_0} d_i \alpha_i^2$. Therefore, the squared Euclidean distance between a codeword $\mathbf{c}$ and $\mathbf{0}$ becomes

$$d^2(\mathbf{c}, \mathbf{0}) = \sum_{i=1}^{d_H^F} \frac{2E_i}{N_0} d_i \alpha_i^2,$$

where d_H^F is the number of non-zero d_i.

For an i.i.d. Rayleigh block fading channel, following the same procedure as in [3], we can bound $P_2(\mathbf{c}, \mathbf{0})$ as follows:

$$\begin{aligned} P_2(\mathbf{c}, \mathbf{0}) &= \int Q(\sqrt{z}) p(z) dz \\ &\leq \frac{1}{2} \int p(z) e^{-\frac{1}{2}z} dz \\ &= \frac{1}{2} \phi_z(-\frac{1}{2}) \\ &= \frac{1}{2} \prod_{i=1}^{d_H^F} \frac{1}{1 + d_i \frac{E_i}{N_0}} \end{aligned} \quad (1)$$

where $z = d^2(\mathbf{c}, \mathbf{0}) = \sum_{i=1}^{d_H^F} \frac{2E_i}{N_0} d_i \alpha_i^2$ and $\phi_z(\mathbf{s})$ is the characteristic function of z defined by

$$\phi_z(\mathbf{s}) = \int p(z) e^{\mathbf{s}z} dz = \prod_{i=1}^{d_H^F} \frac{1}{1 - \mathbf{s} d_i \frac{2E_i}{N_0}}. \quad (2)$$

The pairwise error probability for non-equal symbol power allocation as shown in equation (1) is consistent with the known result for an ideal Rayleigh fading channel, which is a special case of equation (1). An ideal Rayleigh fading channel means a fully interleaved i.i.d. Rayleigh fading channel with equal symbol power allocation, i.e., $E_i = E_s, d_i = 1$ with d_H^F as the Hamming weight of the codeword $\mathbf{c}$. In this case, the above bound can be written as (see [4], [5]):

$$P_2(\mathbf{c}, \mathbf{0}) \leq \frac{1}{2} (1 + \frac{E_s}{N_0})^{-d_H^F}.$$

As in equation (1), it can be further bounded as

$$P_2(\mathbf{c}, \mathbf{0}) \leq \frac{1}{2} \prod_{i=1}^{d_H^F} \frac{1}{d_i E_i / N_0} = C \prod_{i=1}^{d_H^F} \frac{1}{E_i / N_0} \quad (3)$$

with $C = \frac{1}{2} \prod_{i=1}^{d_H^F} \frac{1}{d_i}$. As the SNR increases, the above bound is asymptotically tight. Therefore, at high SNR, we can approximate the pairwise error probability and the union bound to the FER as

$$\begin{aligned} P_2(\mathbf{c}, \mathbf{0}) &\approx C \prod_{i=1}^{d_H^F} \frac{1}{E_i / N_0} \\ P_e \leq \sum_{j=1}^{M-1} P_2(\mathbf{c_j}, \mathbf{0}) &\approx \sum_{j=1}^{M-1} C_j \prod_{i=1}^{(d_H^F)_j} \frac{1}{(E_i)_j / N_0} \end{aligned} \quad (4)$$

For a block fading channel with a large block length, we assume all sub-codewords have non-zero weights, i.e., $(d_H^F)_j = F$. Therefore, the union bound to the FER at high SNR can be written as

$$P_e \leq \sum_{j=1}^{M-1} C_j \prod_{i=1}^{F} \frac{1}{E_i / N_0} \quad (5)$$

The FER is also lower bounded by the largest pairwise error probability, i.e.,

$$P_e \geq P_2^{\max}(\mathbf{c}, \mathbf{0}) \approx C_{\max} \prod_{i=1}^{F} \frac{1}{E_i / N_0}, \quad (6)$$

with $C_{\max} = \max\{C_j, j = 1, \cdots, M-1\}$. Combining the equations (5) and (6), we can write

$$C_{\max} \prod_{i=1}^{F} \frac{1}{E_i / N_0} \leq P_e \leq \sum_{j=1}^{M-1} C_j \prod_{i=1}^{F} \frac{1}{E_i / N_0}.$$

Therefore, we can approximate the FER for an i.i.d. block fading channel as

$$P_e \approx A \prod_{i=1}^{F} \frac{1}{E_i / N_0} \quad (7)$$

with $C_{\max} \leq A \leq \sum_{j=1}^{M-1} C_j$.

For a type II Hybrid–HARQ with Incremental Redundancy, a sub-frame of a high rate FEC code is sent in the first transmission. If this sub-frame cannot be decoded, more redundancy bits are transmitted in the next sub-frame to soft-combine with previous transmissions and form a lower rate FEC code. As a result, at the receiver side, one codeword is composed of one or more sub-frames that are received from different (re)transmissions. In our proposed scheme, we assume equal symbol energy within one sub-frame and different symbol energy across sub-frames.

Let $(E_s)_j$ be the symbol energy for the j-th sub-frame. Define D_j as the number of E_i in equation (7) that are equal to $(E_s)_j$. Equation (7) can then be written as

$$P_e \approx A_n \prod_{j=1}^{n} \left(\frac{(E_s)_j}{N_0} \right)^{-D_j}, \quad (8)$$

where A_n is defined as the parameter A in equation (7) for a codeword with n sub-frames. After the i-th (re)transmission, the receiver will decode a codeword with i sub-frames. Then the residual FER becomes

$$f_i = (P_e)_i \approx A_i \prod_{j=1}^{i} \left(\frac{(E_s)_j}{N_0} \right)^{-D_j}. \quad (9)$$

In the remainder of the paper, we will use the above expression for the FER in deriving optimal power allocation strategies for the type II Hybrid–ARQ scheme over an i.i.d. Rayleigh block fading channel.

III. Optimal power allocation on an i.i.d. Rayleigh block fading channel

In this section we will analyze the optimal power allocation scheme for an i.i.d block fading channel. Define n_i as the number of symbols sent in the i-th sub-frame. Let f_i be the residual FER after receiving all i sub-frames as defined in equation (9). The average power consumption of the Hybrid–ARQ system is the sum of the average power consumption in each (re)transmission. For a maximum of N (re)transmissions, the optimal power allocation problem that minimizes the average power consumption can be formulated as

$$\min_{(E_s)_1,(E_s)_2,\cdots,(E_s)_N} \left\{ n_1(E_s)_1 + \sum_{i=2}^{N} n_i(E_s)_i f_{i-1} \right\} \quad (10a)$$

$$\text{subject to} \quad f_N \leq P_e^{\max} \quad (10b)$$

$$(E_s)_i > 0,\ \forall i = 1, \cdots, N$$

After using equation (9), defining $x_i = (E_s)_i/N_0$, $(i = 1, \cdots, N)$, $a_i = A_i n_{i+1}/n_1$, $(i = 1, \cdots, N-1)$, $a_N = A_N$ and rearranging the constraints in (10b), we can rewrite above optimization problem as:

$$\begin{aligned} \min_{x_1,x_2,\cdots,x_N} \{ & f(\mathbf{x}) = x_1 + a_1 x_2 x_1^{-D_1} + a_2 x_3 x_1^{-D_1} x_2^{-D_2} + \\ & \cdots + a_{N-1} x_N \prod_{i=1}^{N-1} x_i^{-D_i} \} \\ \text{subject to} \quad & a_N \prod_{i=1}^{N} x_i = P_e^{\max} \\ & x_i > 0,\ \forall i = 1, \cdots, N \end{aligned} \quad (11)$$

The constraint in the above problem becomes an equality constraint due to the monotonicity of f_N.

This is a geometric programming problem with a zero degree of difficulty [6]. According to the property of the geometric programming, this problem has a unique solution. The degree of difficulty is defined as $\mathcal{N} - \mathcal{M} - 1$, where $\mathcal{N}$ denotes the total number of terms in all the polynomials of both the objective function and all the constraints. $\mathcal{M}$ denotes the number of design variables. Therefore, in the problem defined in equation (11), $\mathcal{N} = N + 1$, $\mathcal{M} = N$ and the degree of difficulty is $\mathcal{N} - \mathcal{M} - 1 = 0$.

In the following, we outline the solution to this problem. Define

$$\begin{aligned} x_0 &= f(\mathbf{x}) \\ x_i &= e^{w_i} \\ \Delta_i &= \frac{a_{i-1} x_i \prod_{j=1}^{i-1} x_j^{-D_j}}{f(\mathbf{x})}, i = 1, ..., N. \end{aligned} \quad (12)$$

It follows that

$$\begin{aligned} \sum_{i=1}^{N} \Delta_i &= 1 \\ \ln \frac{\Delta_i}{a_{i-1}} &= -w_0 + w_i - \sum_{j=1}^{i-1} D_j w_j \\ \ln(P_e^{\max}/a_N) &= -\sum_{j=1}^{N} D_j w_j. \end{aligned} \quad (13)$$

The problem in equation (11) is then equivalent to minimizing x_0 with constraints as shown in equation (13). The Lagrange function corresponding to the optimization problem becomes

$$\begin{aligned} L(\mathbf{w}, \boldsymbol{\Delta}, \lambda) = & w_0 + \lambda_0 (\sum_{i=1}^{N} \Delta_i - 1) \\ & + \sum_{i=1}^{N} \lambda_i (-w_0 + w_i - \sum_{j=1}^{i-1} D_j w_j - \ln \frac{\Delta_i}{a_{i-1}}) \\ & + \lambda_{N+1} (-\sum_{j=1}^{N} D_j w_j - \ln \frac{P_e^{\max}}{a_N}) \end{aligned} \quad (14)$$

The optimal solution should satisfy the following equations:

$$\frac{\partial L}{\partial w_0} = 1 - \sum_{i=1}^{N} \lambda_i = 0 \quad (15a)$$

$$\frac{\partial L}{\partial w_i} = \lambda_i - \sum_{j=i+1}^{N} \lambda_j D_i - \lambda_{N+1} D_i, i = 1, ..., N \quad (15b)$$

$$\frac{\partial L}{\partial \lambda_0} = \sum_{i=1}^{N} \Delta_i - 1 = 0 \quad (15c)$$

$$\frac{\partial L}{\partial \lambda_i} = -w_0 + w_i - \sum_{j=1}^{i-1} D_j w_j - \ln \frac{\Delta_i}{a_{i-1}} = 0, i = 1, ..., N \quad (15d)$$

$$\frac{\partial L}{\partial \lambda_{N+1}} = -\sum_{j=1}^{N} D_j w_j - \ln \frac{P_e^{\max}}{a_N} = 0 \quad (15e)$$

$$\frac{\partial L}{\partial \Delta_i} = \lambda_0 - \frac{\lambda_i}{\Delta_i} = 0 \quad (15f)$$

From equation (15a), equation (15c) and equation (15f), the following equations can be derived:

$$\Delta_i = \frac{\lambda_i}{\lambda_0} \quad (16a)$$

$$\sum_{i=1}^{N} \Delta_i = \frac{\sum_{i=1}^{N} \lambda_i}{\lambda_0} = 1 \quad (16b)$$

$$\sum_{i=1}^{N} \lambda_i = \lambda_0 = 1 \quad (16c)$$

$$\Delta_i = \lambda_i \quad (16d)$$

$\{\lambda_i\}_{i=1}^{N+1}$ can be solved using equations (15a) and (15b). We can then use these optimal values and solve for $\{w_i\}_{i=0}^{N}$ using equations (16d), (15d) and (15e). In both cases, there are an equal number of unknown variables and linearly independent equations. Therefore, $\{\lambda_i^*\}_{i=0}^{N+1}$ are uniquely defined and $\{w_i\}_{i=0}^{N}$ are solved uniquely. The optimal power consumption value $f(\mathbf{x}) = e^{w_0}$ and the corresponding power allocation for each transmission $x_i = e^{w_i}, i = 0, \cdots, N$ are also uniquely defined.

For the special case of $N = 2$, a closed form solution to the optimal SNR values for the first and the second (re)transmissions can be derived analytically. For $N = 2$, we will solve the optimization problem:

$$\begin{aligned} \min_{x_1,x_2} \{ & f(\mathbf{x}) = x_1 + a_1 x_2 x_1^{-D_1} \} \\ \text{subject to} & \\ & a_2 x_1^{-D_1} x_2^{-D_2} = P_e^{\max} \\ & x_i > 0,\ i = 1, 2 \end{aligned} \quad (17)$$

From equation (15a) and (15b), $\lambda_1^*, \lambda_2^*, \lambda_3^*$ should satisfy the equations:

$$\begin{aligned} \lambda_1 + \lambda_2 &= 1 \\ \lambda_1 - (\lambda_2 + \lambda_3) D_1 &= 0 \\ \lambda_2 - \lambda_3 D_2 &= 0 \end{aligned} \quad (18)$$

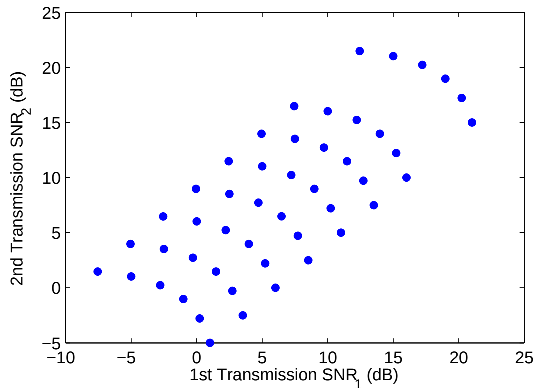


Figure 1: Simulation power allocation schemes for the first and the second transmissions (Each dot corresponds to one set of simulation settings that we obtain the FER values.)

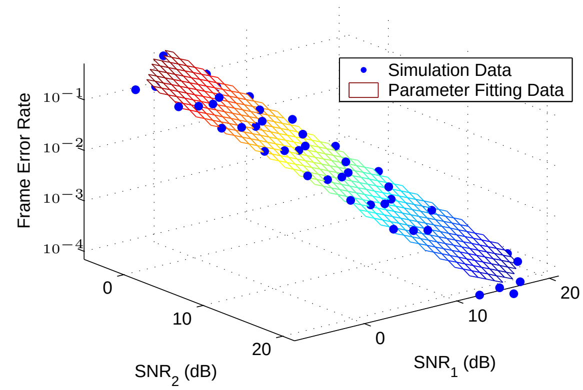


Figure 2: FER as a function of the SNRs of the first and the second transmissions (Each dot corresponds to the FER from one simulation run; the mesh surface corresponds to the FER values from the parameter fitting.)

The optimal $\lambda_i^*, i = 1, 2, 3$ are solved as

$$\begin{aligned} \lambda_1^* &= \frac{D_1(D_2+1)}{D_1+D_2+D_1D_2} \\ \lambda_2^* &= \frac{D_2}{D_1+D_2+D_1D_2} \\ \lambda_3^* &= \frac{1}{D_1+D_2+D_1D_2} \end{aligned} \tag{19}$$

Applying $\Delta_i = \lambda_i$ to equation (15e) and (15f), we find the optimal solution w_0^*, w_1^*, w_2^* satisfies the equations:

$$\begin{aligned} -w_0 + w_1 - \ln\lambda_1^* &= 0 \\ -w_0 + w_2 - D_1 w_1 - \ln\frac{\lambda_2^*}{a_1} &= 0 \\ -D_1 w_1 - D_2 w_2 - \ln\frac{P_e^{\max}}{a_2} &= 0 \end{aligned} \tag{20}$$

The optimal solution can be written as:

$$\begin{aligned} w_0^* &= \lambda_1^*\ln\lambda_1^* - \lambda_2^*\ln\frac{\lambda_2^*}{a_1} - \lambda_3^*\ln\frac{P_e^{\max}}{a_2} \\ w_1^* &= \lambda_3^*(D_2\ln\frac{\lambda_1^*}{\lambda_2^*}a_1 - \ln\frac{P_e^{\max}}{a_2}) \\ w_2^* &= -\lambda_3^*(D_1\ln\frac{\lambda_1^*}{\lambda_2^*}a_1 - (1+D_1)\ln\frac{P_e^{\max}}{a_2}) \end{aligned} \tag{21}$$

And the optimal power allocated for the first and the second (re)transmissions are written as

$$\begin{aligned} x_1^* &= \frac{E_1}{N_0} = \left(\frac{a_1\lambda_1^*}{\lambda_2^*}\right)^{D_2\lambda_3^*}\left(\frac{a_2}{P_e^{\max}}\right)^{\lambda_3^*} \\ x_2^* &= \frac{E_2}{N_0} = \left(\frac{a_1\lambda_1^*}{\lambda_2^*}\right)^{-D_1\lambda_3^*}\left(\frac{a_2}{P_e^{\max}}\right)^{(1+D_1)\lambda_3^*}, \end{aligned} \tag{22}$$

which follows from the expressions $x_i^* = e^{w_i^*}, i = 1, 2, 3$.

Note that for a geometric programming problem with zero degree of difficulty, the minimum of the primal problem can be obtained by maximizing the corresponding dual function. In this paper, we do not explore the dual problem approach but it is a topic of interest for future study.

IV. Simulation validation

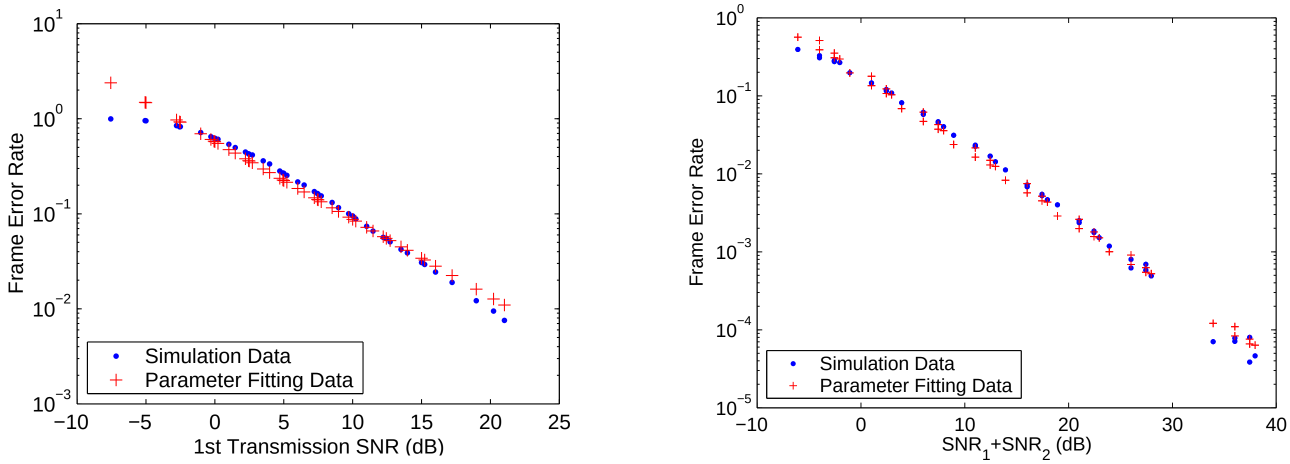
In this section, we will apply the optimal power allocation scheme in a simulated Hybrid–ARQ system. The system parameters and the simulation results will be discussed next.

The simulated Hybrid–ARQ system allows up to two (re)transmissions, i.e. $N = 2$. Assume the data source is encoded with a Rate Compatible Punctured Convolutional (RCPC) [7] code based on a rate 1/4 convolutional code with constraint length $K = 9$. The generating polynomials are (765 671 513 473) in octal format. The puncturing pattern for the first transmission is simply {"0101"}, which means puncturing every other bit to generate a 1/2 rate code. The codeword bits removed by puncturing at the first transmission are sent in the second transmission if the first transmission has a decoding error.

The uncoded source frame has a length of 1000 bits. After the RCPC encoding and puncturing, for the first transmission, the sub-frame has a length of 2000 bits. The sub-frame sent in the second transmission has the same length. We assume that the fading gain is constant within one transmission and changes independently over different (re)transmissions.

To obtain expressions for f_1 and f_2, i.e., the parameters A_1, A_2, D_1 and D_2, we set up simulations for varying ratios of powers allocated to the first and second (re)transmission. The simulation settings for these power allocation schemes are shown in Figure 1. Each dot corresponds to one pair of SNRs allocated to the first and the second transmissions. For each pair of SNRs, one FER value is obtained from the simulation. With all the FER values we get from the simulation, we fit the parameters of the FER model in equation (9). Both the fitted FER and the FER from the simulation are shown in Figure 2. The simulation data is marked with dots and the fitted data is plotted as a mesh surface.

An alternate comparison of the simulation results and the parameter fitting results are shown in Figure 3(a) and Figure 3(b). Figure 3(a) shows the FER v.s. SNR for the first transmission. In Figure 3(b), the y-axis represents the FER value and the x-axis is the sum of the SNRs from the first and the second transmissions. From these figures, we observed very slight difference between the simulation results and the parameter fitting results. Therefore, we can apply the FER approximation model and the optimization scheme to the Hybrid–ARQ system.



(a) FER after the first transmission

(b) FER after both transmissions

Figure 3: Comparison of simulation results and data fitting results

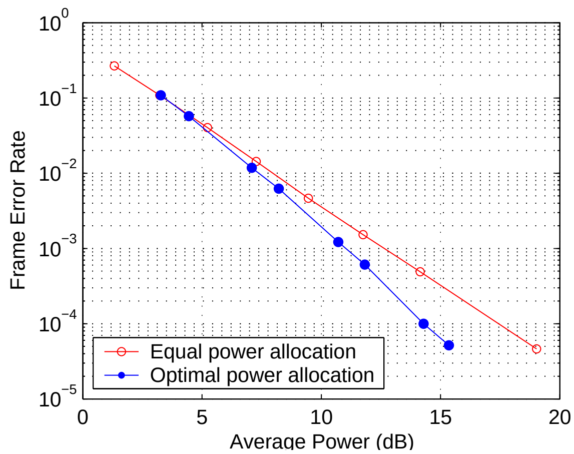


Figure 4: Performance comparison of the optimal power allocation strategy and equal power allocation strategy

After obtaining the parameters necessary to solve the optimization problem and substituting them into equation (22), we apply the optimal power allocation strategy to the simulated Hybrid–ARQ system. In Figure 4, the average power for different FER targets for the optimal power allocation strategy and the equal[2] power allocation strategy are compared. The simulation results show that the optimal scheme can provide a gain up to 4dB at a target FER of 1E-4. However, at very high target FERs, there are not any significant gains to be obtained over an equal power allocation scheme.

V. Conclusion and Future Work

In this paper, we provided a method to optimize the average power consumption for a type-II Hybrid–ARQ system over an i.i.d Rayleigh block fading channel. The optimization problem was formulated and solved as a geometric programming problem. For the special case of two (re)transmissions, a closed form optimal power allocation solution was also derived.

Compared with a traditional Hybrid–ARQ scheme with equal power allocation for all transmissions in one ARQ round, the optimal scheme achieved significant power savings for an RCPC coded Hybrid–ARQ system. From the simulation results, a gain up to 4dB, at a target FER of 1E-4, was achieved.

In our future work, we will apply the techniques proposed here to the case of Turbo and LDPC codes as well. The performance of the optimized power allocation strategies will be analyzed and compared to equal power allocation strategies. We will also explore the dual problem approach for solving this optimal resource allocation problem.

[2]Note that the equal power allocation strategy is a solution to the optimization problem in equation (10) with the symbol energy being the same for each (re)transmission.

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