# Analysis of an Approximate Decorrelating Detector

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**Abstract**

In this paper an approximate decorrelating detector is analyzed on the basis of a first order approximation to the inverse crosscorrelation matrix of signature waveforms. The approximation is fairly accurate for systems with low crosscorrelations and is exact in the two-user synchronous case. We present an exact as well as approximate analysis of the bit-error-rate performance of this detector on a channel that is subject to flat fading, and also specifically for the case of random signature waveforms. The detector outperforms the conventional matched filter receiver in terms of BER. The approximate decorrelator (while not being near-far resistant like the decorrelating detector) is fairly robust to imperfections in power control. Power trade-off regions are identified which characterize the significant advantage that the approximate decorrelator provides over the matched filter receiver. The reduced complexity of the approximate decorrelator and performance gains over the conventional matched filter makes it a viable alternative for implementation in practical CDMA systems, in particular in those where the signature waveforms span many symbol intervals.

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# 1 The Approximate Decorrelator

Conventionally, demodulation of DS-CDMA signals is achieved with a matched filter receiver. These receivers are optimum only when there is no multiple access interference in additive Gaussian noise channels, and not only do they suffer from the near-far problem, but they have inferior performance even with perfect power control. Several multiuser detection schemes have been proposed (e.g. [1, 2]), however these are more complex than the matched filter detector and may require explicit knowledge or estimates of various parameters. To overcome this inconvenience, adaptive detectors have been proposed recently (see [3] and references therein) that are still more complex than the matched filter detector to varying degrees. Our approach here is to analyze a simplified multiuser detector that is only slightly more complex than the matched filter detector while still retaining some of the performance advantages of multiuser detectors. We consider a linear multiuser detector, namely the decorrelating detector [1] and analyze an approximation for it which was proposed in [4] for both the synchronous and asynchronous cases.

For simplicity in illustration, we will consider the case of synchronous reception of the users' bits. In this case, the output of the matched filter for the $k^{th}$ user can be written as

$$y_k = A_k b_k + \sum_{j \neq k} A_j b_j \rho_{jk} + n_k, \tag{1}$$

where $A_k$, and $b_k \in \{-1, 1\}$, are the amplitude and the bit of the $k^{th}$ user respectively, and $n_k = \sigma \int_0^T n(t) s_k(t) dt$, with $s_k(t)$ being the signature waveform of the $k^{th}$ user which is assumed to have unit energy. The crosscorrelation between the signature waveforms is defined as

$$\rho_{jk} = \int_0^T s_j(t) s_k(t) dt. \tag{2}$$

The matched filter outputs for the users in the system can be expressed in vector form as

$$\mathbf{y} = \mathbf{RAb} + \mathbf{n}, \tag{3}$$

where $\mathbf{y} = [y_1 \cdots y_K]^\top$, $\mathbf{b} = [b_1 \cdots b_K]^\top$, $\mathbf{A} = \mathrm{diag}\{A_1 \cdots A_K\}$, and $\mathbf{n}$ is a zero-mean Gaussian random vector with covariance matrix equal to $\sigma^2 \mathbf{R}$. The matrix $\mathbf{R}$ of crosscorrelations among the user pairs is such that its $ij^{th}$ entry is $\rho_{ij}$. The exact decorrelating detector gives the following decisions

$$\hat{b}_k = \mathrm{sgn}(\mathbf{R^{-1}y})_k, \tag{4}$$

which can be alternately realized by matched filtering the incoming signal with

$$\sum_{j=1}^{K} \mathbf{R^{+}_{kj}} s_j(t), \tag{5}$$

where $\mathbf{R^{+}_{kj}}$ denotes $(\mathbf{R^{-1}})_{kj}$. One advantage of the decorrelating detector is that it does not require knowledge of amplitudes (or powers) of users in the system. In this paper, we will retain this property of the decorrelating detector while deriving an approximation for it. If, as is usually the case, the crosscorrelations among all the signal pairs are very low with respect to the energy of the signature waveforms, then $\mathbf{R}$ is strongly diagonal and (5) can be approximated by

$$s_k(t) - \sum_{j \neq k} \rho_{jk} s_j(t), \tag{6}$$

on the basis of $(\mathbf{I} + \delta\mathbf{L})^{-1} = \mathbf{I} - \delta\mathbf{L} + o(\delta)$. In fact, for the synchronous two-user case, the above expression is exact. The advantage of this approximation is greater in the asynchronous case, as in that case the crosscorrelations are not known in advance. Another advantage of this approximate decorrelator is that it can be readily implemented at the base station for use in the reverse link of a cellular CDMA system. The crosscorrelations can be computed on-line on a bit by bit basis and are not required to be stored for use in computations as in the case of other multiuser detectors. This also allows the advantage that the spreading sequences need not be confined to the duration of a bit interval as is required in adaptive and other multiuser detectors. This makes it compatible for use in systems with long spreading codes as is being proposed for practical systems [5]. Another advantage is that the approximate decorrelator can be readily implemented in systems that use adaptive single-user matched filters (e.g., rake receivers) by generating the crosscorrelations from the received signature waveforms. In the remainder of the paper, we analyze the approximate decorrelator for both synchronous and asynchronous channels.

The detector performance on flat fading channels is characterized by the gain it offers over the matched filter in terms of BER, number of supportable users, and robustness to imperfections in power control. We derive approximate expressions for the above gains and show via simulations and exact analysis that these are fairly accurate over a wide range of system parameters. The simplicity of the analysis allows easy performance comparison with that of the matched filter.

## 2 Approximations for Bit-Error-Rate

In this section, we analyze the approximate decorrelator for an asynchronous channel which is also subject to slow frequency non-selective Rayleigh fading. Let user 1 be the desired user; the received signal in the interval $[0,T]$ is given by

$$r(t) = \alpha_{0,1}A_1 b_{0,1}s_1(t) + \sum_{k=2}^{K} A_k(\alpha_{-1,k}b_{-1,k}s_k^L(t) + \alpha_{0,k}b_{0,k}s_k^R(t)) + \eta(t), \tag{7}$$

where the $\{\alpha_{i,j}\}$ are the (multiplicative) attenuation factors due to Rayleigh fading, $\{b_{i,j}\}$ are the bits of the respective users, $s_k^L(t)$, $s_k^R(t)$ are the *left* and *right* portions of the $k^{th}$ signature waveform, and $\eta(t)$ is the additive white Gaussian noise process. The simplified decorrelating detector structure for user 1 is given by correlating the received signal with

$$h(t) = s_1(t) - \sum_{k=2}^{K} \rho_{1k}s_k^L(t) - \sum_{k=2}^{K} \rho_{k1}s_k^R(t), \tag{8}$$

where the crosscorrelations $\{\rho_{ij}\}$ between every pair of signature waveforms depend on the offset between the signals and are given by

$$\rho_{ij}(\tau) = \int_0^T s_i(t)s_j(t-\tau)\,dt,\ i<j, \tag{9}$$

and

$$\rho_{ji}(\tau) = \int_0^T s_i(t)s_j(t+T-\tau)\,dt,\ i<j, \tag{10}$$

where $\tau \in [0,T]$. Note that for the conventional matched filter receiver $h(t) = s_1(t)$. The decision statistic, $y = \int_0^T r(t)h(t)\,dt$, is now given by

$$y = \alpha_{0,1}A_1 b_{0,1}S + I_L + I_R + \mathcal{N}, \tag{11}$$

where

$$S = \int_0^T s_1(t)h(t)\,dt, \tag{12}$$

$$I_L = \sum_{k=2}^{K} A_k\alpha_{-1,k}b_{-1,k}\rho_k^L, \tag{13}$$

$$I_R = \sum_{k=2}^{K} A_k \alpha_{0,k} b_{0,k} \rho_k^R, \tag{14}$$

and

$$\mathcal{N} = \int_0^T \eta(t) h(t), \tag{15}$$

with $\mathcal{N}$ being a zero mean Gaussian random variable with variance denoted by $\sigma_{\mathcal{N}}^2$. $\rho_k^L$, $\rho_k^R$ are the correlated outputs due to the left and right signature waveforms and are given by

$$\rho_k^L = \int_0^T s_k^L(t) h(t)\, dt, \tag{16}$$

and

$$\rho_k^R = \int_0^T s_k^R(t) h(t)\, dt. \tag{17}$$

By symmetry, the bit-error-rate is given by

$$P_e = E[Q(\frac{X}{\sigma_{\mathcal{N}}})], \tag{18}$$

where

$$X = \alpha_{0,1} A_1 S + I_L + I_R.$$

The expectation in the above expression is over the interference and fading statistics $\mathcal{I} = (\alpha_{0,1}, \alpha_{-1,2}, \alpha_{0,2}, \cdots, \alpha_{-1,K}, \alpha_{0,K}, b_{-1,2}, b_{0,2}, \cdots b_{-1,K}, b_{0,K})$. Conditioning on the attenuation factors, an exact expression for the above probability of error can be evaluated as follows. Each bit of the desired user is affected by $2K-2$ interfering bits due to the interferers, and the probability of error in (18) is given as

$$P_e = \frac{1}{4^{K-1}} \sum_{b_{-1,2} \in B} \cdots \sum_{b_{-1,K} \in B} \sum_{b_{0,2} \in B} \cdots \sum_{b_{0,K} \in B} Q(\frac{A_1 \alpha_{0,1} S}{\sigma_{\mathcal{N}}} + \sum_{k=2}^{K} \frac{A_k}{\sigma_{\mathcal{N}}} (b_{-1,k} \alpha_{-1,k} \rho_k^L + b_{0,k} \alpha_{0,k} \rho_k^R)) \tag{19}$$

where $B = \{-1, +1\}$. In deriving (19), we have assumed that the bits of all the users in the system are mutually independent. Thus we see that the exact expression for the probability of error can be evaluated although its complexity is increasing exponentially in the number of users in the system. Further, when the fading in the channel is random, the above expression will have to be averaged with respect to the fading statistics as well. This may be tedious, but can be accomplished with some degree of difficulty.

Owing to the analytical intractability in exactly evaluating the performance, we now derive an approximation for the bit-error-rate of the approximate decorrelator in an asynchronous channel with slow frequency non-selective Rayleigh fading. Without loss of generality, let us assume that the fading parameters $\{\alpha\}$ are i.i.d. random variables with $E[\alpha] = m$, and $Var(\alpha) = \xi^2$. Since the modulation is antipodal, $E[I_L] = E[I_R] = 0$. The variance of $I_L$ is now given as

$$Var(I_L) = \sum_{k=2}^{K} A_k^2 (\rho_k^L)^2 (\xi^2 + m^2), \tag{20}$$

where once again we have used the independence assumption amongst users and fading parameters. A similar expression results for

$$Var(I_R) = \sum_{k=2}^{K} A_k^2 (\rho_k^R)^2 (\xi^2 + m^2), \tag{21}$$

Furthermore, it follows that $E[X] = mA_1S$, and

$$Var(X) = A_1^2 S^2 \xi^2 + Var(I_R) + Var(I_R). \tag{22}$$

Using just the first and second moments of the fading and interference statistics, an approximation for $P_e$ is given by

$$P_e \approx \{\frac{2}{3}Q(\frac{E[X]}{\sigma_{\mathcal{N}}}) + \frac{1}{6}Q(\frac{E[X] + \sqrt{3Var(X)}}{\sigma_{\mathcal{N}}}) + \frac{1}{6}Q(\frac{E[X] - \sqrt{3Var(X)}}{\sigma_{\mathcal{N}}})\}. \tag{23}$$

The details of the derivation of the above equation are given in the Appendix. In Figure 1, we analyze the bit-error-rate performance of the approximate decorrelator for an asynchronous system with 5 users. In this example, the signature waveforms chosen are $m$-sequences of length 31. Simulations, exact analysis, and approximations are shown for the approximate decorrelator. It is seen that the approximations derived in (23) are fairly accurate over a wide range of interferer power levels. A comparative simulation of the matched filter detector performance shows that the approximate decorrelator clearly outperforms the matched filter detector. In Figure 2, we consider 5 asynchronous users each using $m$-sequences of length 31, and Rayleigh distributed fading coefficients. Once again the approximations for the bit-error-rate are fairly accurate compared to the simulations. The above results illustrate that the approximation derived in (23) is fairly accurate over a wide range of interference power levels, background noise power levels, and the number of users in the system. Further, we can easily incorporate asynchronous users as well as fading in the analysis. Thus we have an approximation that is relatively simple in that it uses only means and variances of the relevant random variables in the system. In the remainder of the paper, we will work with the above approximation and do a comparative analysis of the advantages that the approximate decorrelator offers over a conventional matched filter receiver.

# 3 Analysis for Random Signature Waveforms

We will consider a synchronous DS-CDMA system employing random signature (PN) sequences, and analyze the performance of the approximate decorrelating detector in terms of bit-error-rate, power control, SNR gain, and sensitivity to imperfections in power control. Specifically, we will compare the relative performance to that of a conventional matched filter receiver. We will parameterize the performance of the two detectors in terms of the number of users $K$, the length of the random signatures $N$, the powers of the users $P_k (= A_k^2)$, and the noise variance $\sigma_{\mathcal{N}}^2$. We will set the fading coefficients $\{\alpha\}$ to unity although the random fading case can be easily incorporated in the analysis.

## 3.1 BER Analysis

We will use the approximation in (23) to derive the bit-error-rates for both the conventional and the approximate decorrelating detector. In order to accomplish this, we need to derive the mean and variance of $X$, and the corresponding noise variance. For the conventional matched filter receiver, we use the approximation (cf. Appendix)

$$P_e^{mf} \approx \{\frac{2}{3}Q(\frac{E[X_{mf}]}{\sigma_{\mathcal{N}}^{mf}}) + \frac{1}{6}Q(\frac{E[X_{mf}] + \sqrt{3Var(X_{mf})}}{\sigma_{\mathcal{N}}^{mf}}) + \frac{1}{6}Q(\frac{E[X_{mf}] - \sqrt{3Var(X_{mf})}}{\sigma_{\mathcal{N}}^{mf}})\}, \quad (24)$$

where

$$E[X_{mf}] = \sqrt{P_1}, \quad (25)$$

and

$$Var(X_{mf}) = \frac{1}{N}\sum_{k=2}^{K} P_k, \quad (26)$$

and

$$\sigma_{\mathcal{N}}^{mf} = \sigma_{\mathcal{N}}. \tag{27}$$

The above relations are derived by setting $h(t) = s_1(t)$, and averaging over all the random signature sequences of length $N$. It is seen that $X_{mf}$ is given by

$$X_{mf} = A_1 + \sum_{k=2}^{K} A_k b_k \rho_{1k}, \tag{28}$$

where $\rho_{1k}$ is the crosscorrelation between user 1 and user $k$. Since $E[b_k] = 0$, it follows that $E[X_{mf}] = A_1 = \sqrt{P_1}$. We will now derive the variance expression in (26) as follows:

$$\begin{aligned} Var(X_{mf}) &= E[(\sum_{k=2}^{K} A_k b_k \rho_{1k})^2], \\ &= E[\sum_{k=2}^{K} P_k \rho_{1k}^2 + \sum_{\substack{j,k \\ j \neq k}} A_k A_j b_k b_j \rho_{1k} \rho_{1j}], \\ &= \sum_{k=2}^{K} P_k E[\rho_{1k}^2], \end{aligned} \tag{29}$$

where we have used the independence assumption among the transmitted bits. We will now derive an expression for $E[\rho_{1k}^2]$ from first principles as follows. Let $d_{ki} \in \{+1, -1\}$ denote the polarity of the $i^{th}$ chip of the $k^{th}$ user's signature sequence. Then by definition,

$$\rho_{jk} = \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}\{d_{ji} = d_{ki}\} - \mathbf{1}\{d_{ji} \neq d_{ki}\} = -1 + \frac{2}{N} \sum_{i=1}^{N} \mathbf{1}\{d_{ji} = d_{ki}\},$$

where the indicator function is defined as

$$\mathbf{1}\{a = b\} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

Using the above relation, and conditioning on the signature waveforms of the interferers and taking expectations with respect to the desired user, it can be shown that [6]

$$E_1[\rho_{1k}\rho_{1l}] = \frac{\rho_{kl}}{N},$$

from which it follows that $E[\rho_{1k}^2] = 1/N$. Note that the standard deviation of the crosscorrelation terms is $\frac{1}{\sqrt{N}}$, justifying the use of the approximate decorrelator if random signature sequences of sufficiently high processing gain are used.

For the case of the approximate decorrelator, a similar approximation for the bit-error-rate results in

$$P_e^{ad} \approx \{\frac{2}{3}Q(\frac{E[X_{ad}]}{\sigma_{\mathcal{N}}^{ad}}) + \frac{1}{6}Q(\frac{E[X_{ad}] + \sqrt{3Var(X_{ad})}}{\sigma_{\mathcal{N}}^{ad}}) + \frac{1}{6}Q(\frac{E[X_{ad}] - \sqrt{3Var(X_{ad})}}{\sigma_{\mathcal{N}}^{ad}})\}. \quad (30)$$

As for the matched filter, the following expressions can be derived for the approximate decorrelating detector by setting $h(t) = s_1(t) - \sum_{k=2}^{K} \rho_{1k} s_k(t)$, resulting in

$$E[X_{ad}] = \sqrt{P_1}(1 - \frac{K-1}{N}), \quad (31)$$

and

$$Var(X_{ad}) = (\frac{(K-2)}{N^2} + \frac{(K-2)(K-3)}{N^3})(\sum_{k=2}^{K} P_k), \quad (32)$$

and

$$\sigma_{\mathcal{N}}^{ad} = \sigma_{\mathcal{N}} \sqrt{(1 + \frac{(K-1)(K-2)}{N^2} - \frac{(K-1)}{N})} \quad (33)$$

Thus, we have analytical approximations for the bit-error-rates of the matched filter ($P_e^{mf}$), and the approximate decorrelating detector ($P_e^{ad}$) as functions of the powers $P_k$, background noise variance $\sigma_{\mathcal{N}}^2$, number of users $K$, and the length of the signature sequences $N$. The gain in BER offered by the approximate decorrelating detector over the matched filter detector is illustrated in Figure 3. In the range of Figure 3, we see that the approximate decorrelator accommodates $10 - 15$ users more than the conventional detector in a system with processing gain equal to 127.

## 3.2 Power Trade-Off Regions for fixed BER

In this section, we characterize the robustness of the approximate decorrelator with respect to imperfections in power control. Since we have an analytical expression for the bit-error-rate of the detector, it can be inverted to yield the user powers (or alternately the signal-to-background noise ratio) for a fixed level of bit-error-rate. Specifically, we characterize the power trade-off regions for both the matched filter and the approximate decorrelator for a fixed desired bit-error-rate. In Figure 4, we consider a system with $K = 30$ users, and concentrate on the effect of the variation of the power of user 2 on the other users. We assume that all users except user 2 have the same power. The power trade-off curves are plotted in terms of the SNRs required for user 1 (and users $k = 3, \cdots, 30$) and user 2 so that the users in the system achieve a bit-error-rate not exceeding $10^{-3}$. The exact decorrelator as well as the line of perfect power control are also shown for reference. The allowable operating region for the exact decorrelating detector is, as expected, insensitive to the imperfections in power control. It is seen that the approximate decorrelating detector is tolerant to a wider range of imperfections in power control than the matched filter detector which is sensitive to even slight imbalances in the respective powers of the users in the system.

## 3.3 Perfect Power Control Analysis

In this section, we will consider the case of perfect power control, and analyze the advantages the approximate decorrelator yields in terms of relative performance to the conventional matched filter receiver. Specifically, we consider the case of fixed bit-error-rate for both detectors and analyze the trade-off between SNR and the number of users that can be supported. First, we compare the gain offered by the approximate decorrelator in terms of number of users for a fixed SNR level over that of a matched filter receiver. In Figure 5, the number of users that can be supported in the system with a bit-error-rate of $10^{-3}$, is shown for both the matched filter receiver and the approximate decorrelator as a function of SNR. It is seen that in the range considered in the figure, the approximate decorrelator supports more than twice the number of users that a matched filter receiver can support for the same SNR level.

The output signal-to-interference ratio (SIR) for random signature sequences is also of interest for both the matched filter ($\mathrm{SIR}_{mf}$), and the approximate decorrelator ($\mathrm{SIR}_{ad}$) for the synchronous case. The SIR is defined as the average power in the decision statistic due to the desired user divided by the average power in the decision statistic due to the interference and background noise. Specifically, the expressions for both quantities can be derived as

$$\mathrm{SIR}_{mf} = \frac{P_1}{\frac{1}{N}\sum_{k=2}^{K} P_k + \sigma_{\mathcal{N}}^2}, \tag{34}$$

and

$$\mathrm{SIR}_{ad} = \frac{P_1(1 - \frac{2(K-1)}{N} + \frac{(K-1)^2}{N^2})}{(\frac{(K-2)}{N^2} + \frac{(K-2)(K-3)}{N^3})(\sum_{k=2}^{K} P_k) + \sigma_{\mathcal{N}}^2(1 + \frac{(K-1)(K-2)}{N^2} - \frac{(K-1)}{N})}, \tag{35}$$

where the SIR's are parameterized as functions of the powers $P_k$, background noise variance $\sigma_{\mathcal{N}}^2$, number of users $K$, and the length of the signature sequences $N$. The gain of the approximate decorrelator over the matched filter in terms of SIR is shown in Figure 6. The reader should be cautioned that the SIR gain is pessimistic relative to the bit-error-rate gain because of the fact that the interference is not Gaussian.

In the asymptotic case of large number of users and vanishing background noise: $K \to \infty$, $\frac{P_1}{\sigma_{\mathcal{N}}^2} \to \infty$, and $\frac{1}{K}\sum_{k=2}^{K} \frac{P_k}{\sigma_{\mathcal{N}}^2} \to \infty$, we have

$$\lim \frac{\text{SIR}_{ad}}{\text{SIR}_{mf}} = \frac{(1-\lambda)^2}{\lambda(1+\lambda)}, \tag{36}$$

where $\lambda = \frac{K}{N}$. We see from (36) and Figure 6 that the gain of the approximate decorrelator with respect to the matched filter decreases with $\lambda$, until it vanishes for $\lambda = \frac{1}{3}$, a load factor usually considered too high for the capabilities of the conventional detector.

## 3.4 Sensitivity Analysis

In this section, we will derive an analytical expression for the sensitivity of the approximate decorrelator to variations in power levels of interferers. In other words, this is a measure of the sensitivity to imperfections in power control in the system. This analysis could be used to study the robustness of this detector to time delays in implementing the power control mechanism. Let us vary the power of only one interferer labeled $P_2$ (keeping all other users at fixed power $P$). Using the analyticity of the $Q$-function, and (24) and (30), it can be shown that

$$\frac{dP_e^{mf}}{dP_2} = \frac{\sqrt{3}}{\sqrt{2\pi\sigma_{\mathcal{N}}^{mf}\,Var(X_{mf})}} \exp\left(\frac{-(E^2[X_{mf}] + 3Var(X_{mf}))}{(2\sigma_{\mathcal{N}}^{mf})^2}\right) \sinh\left(\frac{E[X_{mf}]\sqrt{3Var(X_{mf})}}{(\sigma_{\mathcal{N}}^{mf})^2}\right)\left\{\frac{1}{N}\right\}, \tag{37}$$

and

$$\frac{dP_e^{ad}}{dP_2} = \frac{\sqrt{3}}{\sqrt{2\pi\sigma_{\mathcal{N}}^{ad}\,Var(X_{ad})}} \exp\left(\frac{-(E^2[X_{ad}] + 3Var(X_{ad}))}{(2\sigma_{\mathcal{N}}^{ad})^2}\right) \sinh\left(\frac{E[X_{ad}]\sqrt{3Var(X_{ad})}}{(\sigma_{\mathcal{N}}^{ad})^2}\right)\left\{\frac{(K-2)}{N^2} + \frac{(K-2)(K-3)}{N^3}\right\}, \tag{38}$$

where the above derivatives are parameterized as functions of the number of users, power levels, the length of the random signature sequences, and the background noise level. In Figure 7, the derivative of the bit-error rate is shown (for both the receivers) with respect to the relative strength of user 2. It is seen that at the point corresponding to perfect power control ($0\ dB$), the derivative of the approximate decorrelator is almost three orders of magnitude smaller than that of the conventional one. This is an operating point of interest since it characterizes the tolerance to deviations from perfect power control. Further, the derivative remains at the low constant level up to even a 10 dB increase in the interferer power. This implies that in contrast to the conventional matched filter receiver, the approximate decorrelator works well even in a system that has a coarse power control strategy. This is well suited to tolerate tracking delays in power control loops. The sensitivity of the bit-error rate ceases to be meaningful as the bit-error rate approaches $\frac{1}{2}$. This is the reason for the steep decline in the sensitivity of the conventional detector above $P_2/P_1 = 15dB$ in Figure 7.

# 4 Conclusions

In this paper we have analyzed an approximate decorrelating detector derived on the basis of a first-order approximation to the inverse crosscorrelation matrix of signature waveforms. The approximation is fairly accurate for systems with low crosscorrelations and is exact in the two-user synchronous case. An approximate analysis that results in simple expressions for the bit-error-rate of this detector was presented. These results were shown to be fairly accurate compared to an exact analysis as well as simulations. Specifically, for the case of random signature sequences, the detector was shown to outperform the conventional matched filter receiver in terms of gain in BER (for a fixed number of users) as well as gain in number of users (for a fixed level of performance). The approximate decorrelator (while not being near-far resistant like the decorrelating detector) was also seen to be fairly robust to imperfections in power control. Power trade-off regions were identified to characterize the significant advantage that the approximate decorrelator provides over the matched filter receiver. The advantages are multi-fold in terms of increased capacity, lower power requirements, and less stringent power control. The reduced complexity of the approximate decorrelator and performance gain over the conventional matched filter makes it a viable alternative for implementation in cellular CDMA systems, in particular in those systems where the period of the signature waveform is larger than the bit-period.

# Appendix

We will now derive the approximation in (23). Consider

$$P_e = \int Q(\frac{x}{\sigma_{\mathcal{N}}}) f_X(x) dx = E[Q(\frac{X}{\sigma_{\mathcal{N}}})], \tag{39}$$

where $f_X$ is the probability density function of the interference and fading statistics. In other words, the random variable $X$ is distributed with the underlying density function being $f_X$, and let us denote its mean and variance, $E[X]$ and $Var(X)$, respectively. We will obtain an approximation for $P_e$ in terms of just the mean and variance of $X$ as follows. Let $\theta$ be a random variable with mean $\mu$, and variance $\sigma^2$, then assuming existence of derivatives, we can rewrite a function $P(\theta)$ using a Taylor series as follows

$$P(\theta) = P(\mu) + (\theta - \mu)P'(\mu) + \frac{1}{2}(\theta - \mu)^2 P''(\mu) + \cdots \tag{40}$$

By truncating the series to just terms of second order, and taking expectations one gets

$$E[P(\theta)] \approx P(\mu) + \frac{1}{2}P''(\mu)\sigma^2.$$

If instead of using a Taylor series, one uses an expansion in central differences (Stirling formula) [7, 8], then we arrive at the approximation

$$E[P(\theta)] \approx P(\mu) + \frac{1}{2}\frac{P(\mu + h) - 2P(\mu) + P(\mu - h)}{h^2}\sigma^2 \tag{41}$$

for small $h$. The value of $h$ for which (41) holds with equality depends strongly on $\sigma$, the standard deviation of $\theta$. It was shown in [7], using the Gauss-Hermite quadrature [9], that $h = \sqrt{3}\sigma$ makes the approximation *exact* for fifth degree polynomials and normally distributed $\theta$. It was also shown that the above approximation is fairly robust to non-Gaussian distributions and deviations from the above assumptions. Using the above approximation on (39) results in

$$P_e \approx \{\frac{2}{3}Q(\frac{E[X]}{\sigma_{\mathcal{N}}}) + \frac{1}{6}Q(\frac{E[X] + \sqrt{3Var(X)}}{\sigma_{\mathcal{N}}}) + \frac{1}{6}Q(\frac{E[X] - \sqrt{3Var(X)}}{\sigma_{\mathcal{N}}})\}. \tag{42}$$

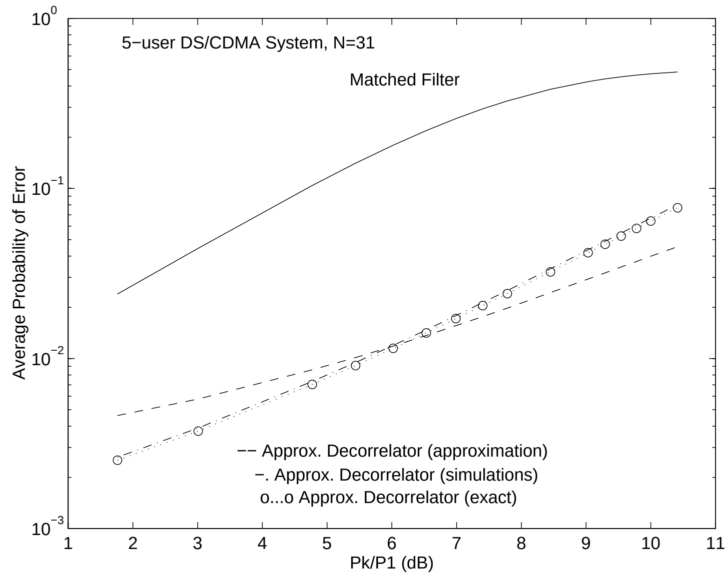


Figure 1: Performance of Approximate Decorrelating Detector

SNR (of the desired user) = 10 dB, $P_k = P_j$ for $j, k = 2, 3, 4, 5$

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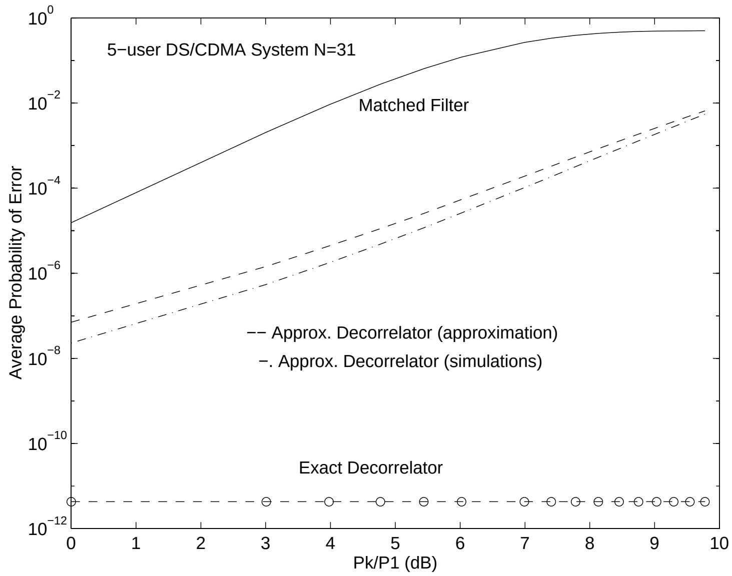


Figure 2: Performance of Approximate Decorrelating Detector (Fading)

SNR (of the desired user) = 20 dB, $P_k = P_j$ for $j, k = 2, 3, 4, 5$

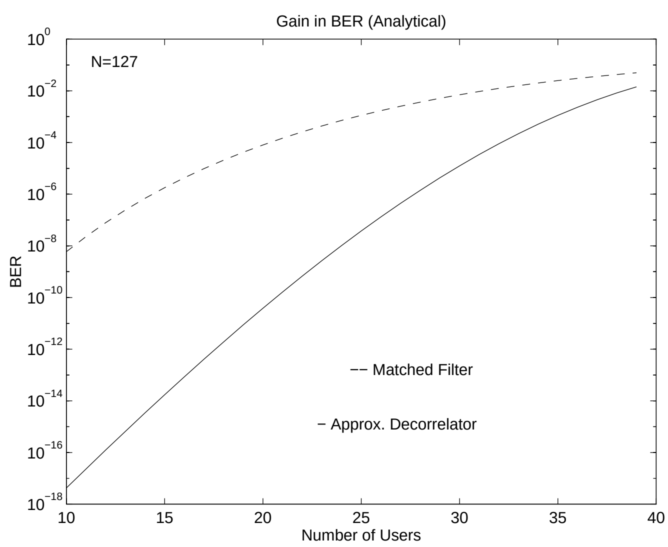


Figure 3: BER Gain

$P_k = P_1 = P$, and $P/\sigma_{\mathcal{N}}^2 = 20dB$

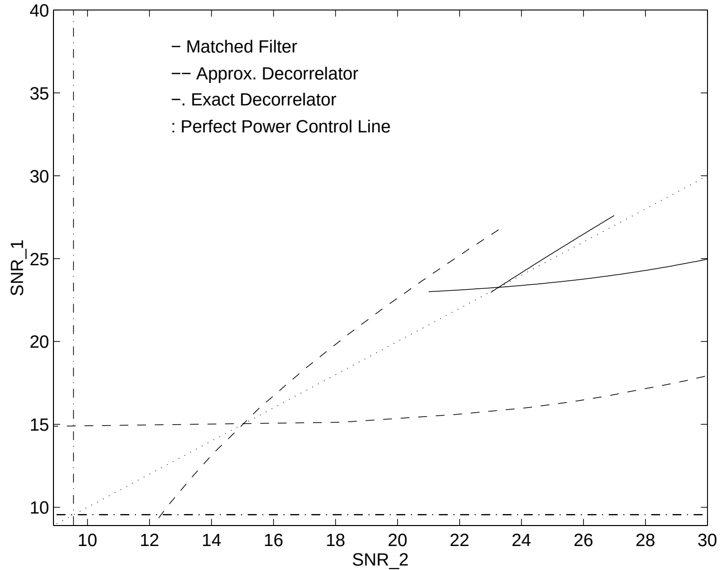


Figure 4: Power Trade-off Regions

$N = 127$, $K = 30$, and BER $\leq 0.001$

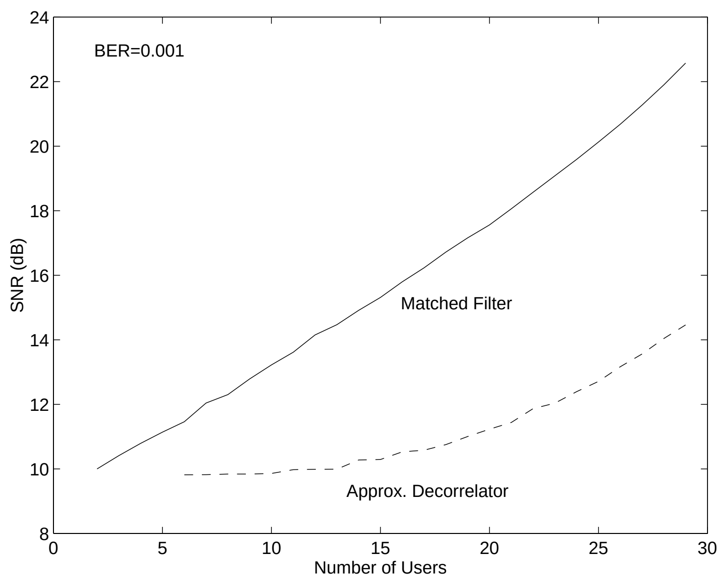


Figure 5: Required SNR (Perfect Power Control)

$N = 127$, BER $= 0.001$

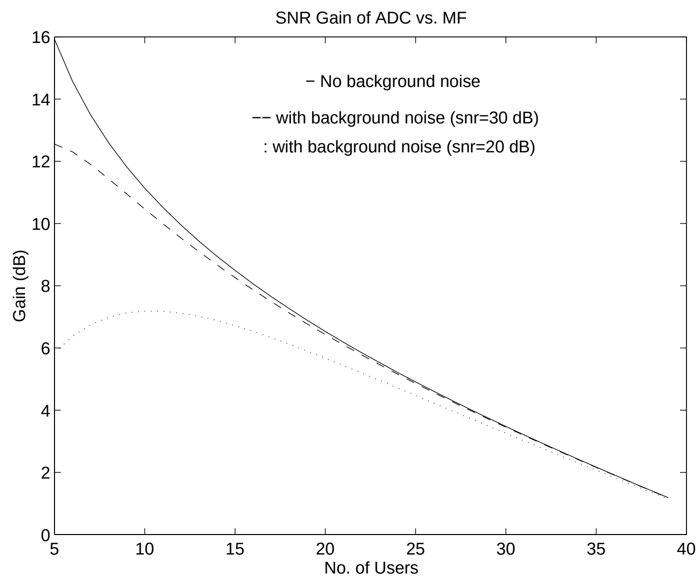


Figure 6: SIR Gain (Perfect Power Control)

$N = 127$

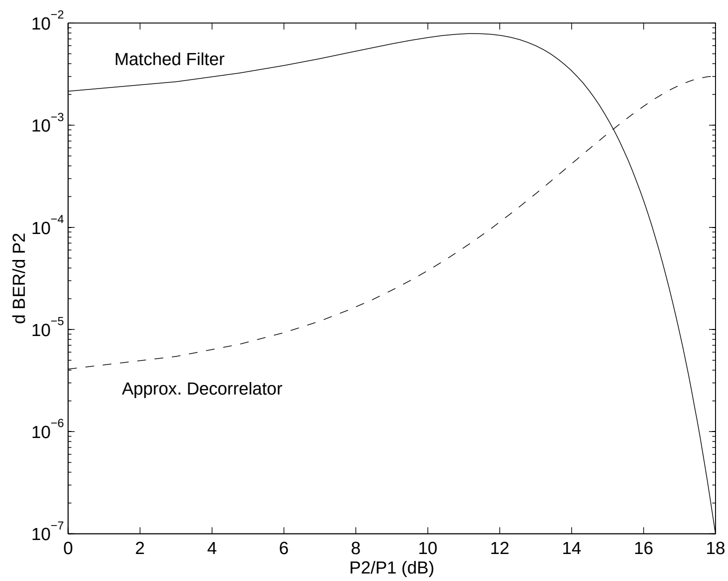


Figure 7: Sensitivity to Interferer Power Level
The rate of change of the BER with respect to the power of user # 2 is shown for the matched filter receiver and the approximate decorrelator. The number of chips is $N = 127$. All the users $k = 1, 3, \cdots, 30$ are held at equal power levels, while $P_2$ is varied. $P/\sigma_{\mathcal{N}}^2 = 20dB$.