Network Coding as a Dynamical System

Narayan B. Mandayam

IEEE Distinguished Lecture

(joint work with Dan Zhang and Kai Su)

WINLAB

Department of Electrical and Computer Engineering

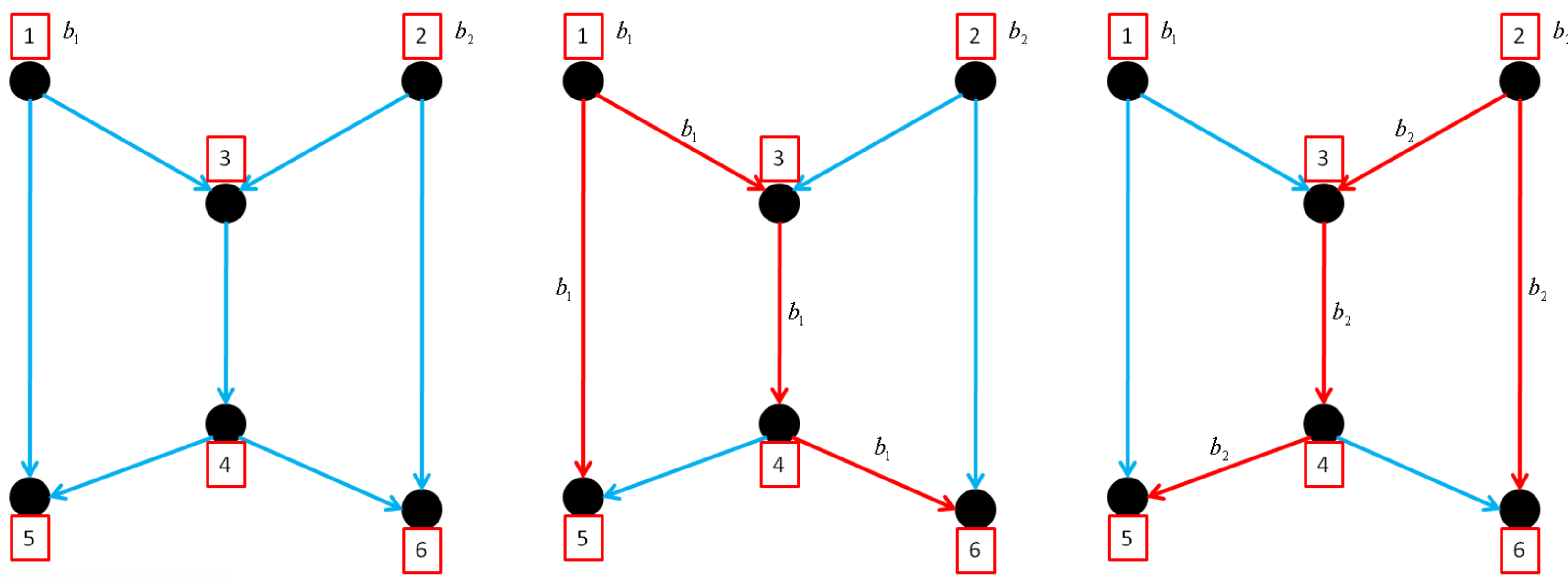
Rutgers University

Outline

1. Introduction
2. Differential Equation (DE) framework for RNC
3. Use Cases of DE framework
4. Concentration Property
5. Applications in Resource Allocation
6. Conclusion

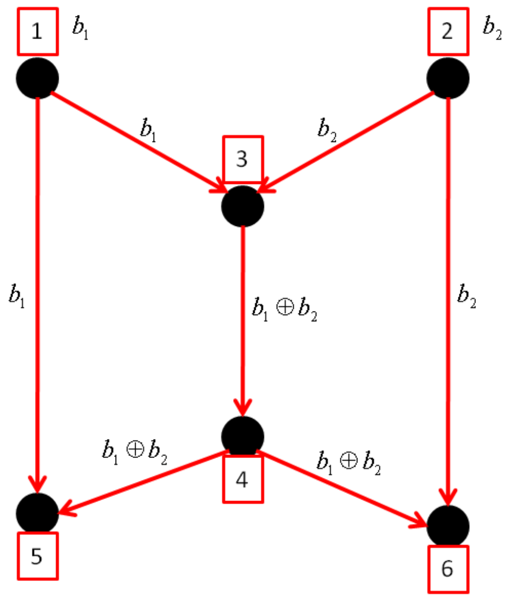
What is Network Coding?
Butterfly Example

- How to achieve multicast capacity?
  - Each link has unit capacity
  - Node 1, 2 want to deliver b_1, b_2 to 5 and 6
- Take 5 seconds and center link used twice



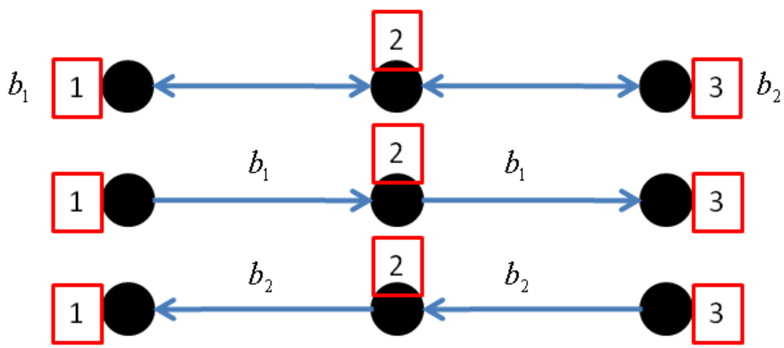
What is Network Coding?
Butterfly Example Contd.

- Can we do better?
  - XOR at node 3
  - Center link used once
  - Finish transmission in 3 seconds



Wireless Network Coding & Multicast Advantage

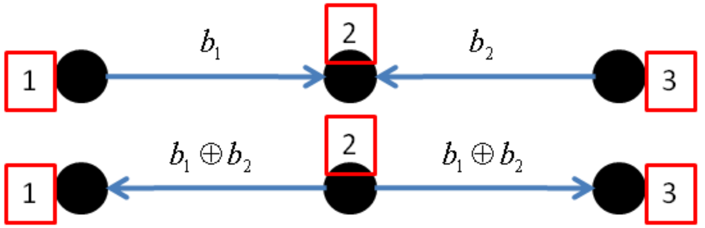
- How to explore multicast advantage?
  - Each wireless link is broadcasting with unit capacity
  - Node 1 and 3 want to exchange a bit
  - Node 1 and 3 can reach each other ONLY through node 2



Takes 4 Transmissions

Multicast Advantage Contd.

- How to exploit multicast advantage?
  - XOR at node 2
  - 3 transmissions – previously 4

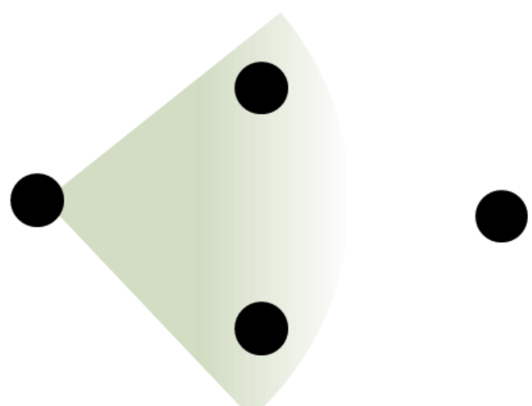


Random Linear Network Coding

- Every outgoing packet $\mathbf{c}$ is a coded packet
- $\mathbf{c}$ is a linear combination of source packets $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_m$ over $\mathrm{GF}(q)$

$$\underbrace{\mathbf{c}}_{\text{coded packet}} = \underbrace{\begin{bmatrix}\mathbf{s}_1 & \mathbf{s}_2 & \cdots & \mathbf{s}_m\end{bmatrix}}_{\text{source packets}} \underbrace{\mathbf{b}}_{\text{coefficient vector}}$$

- Every wireless node performs the same coding operation



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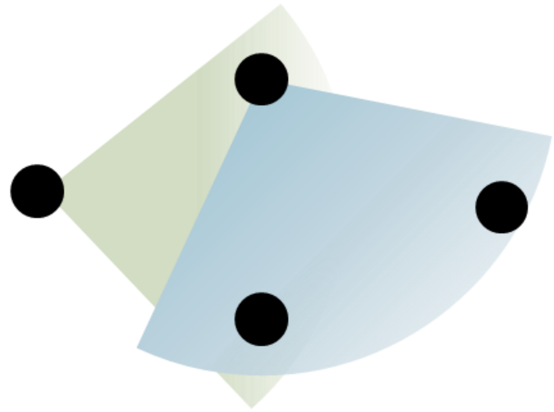

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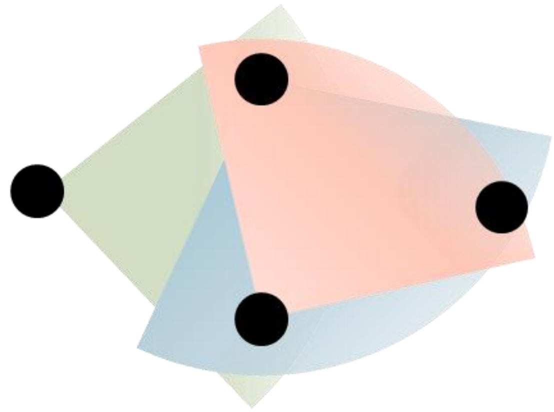


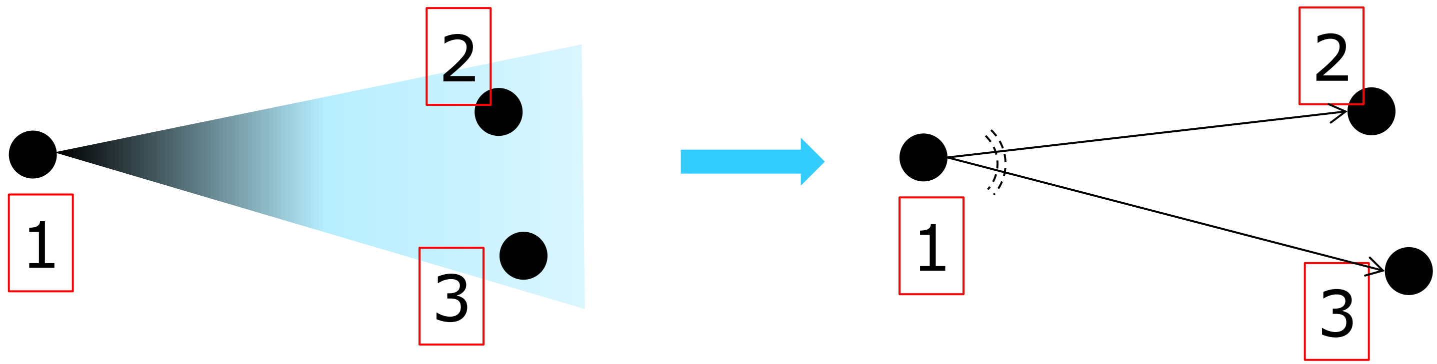
Decoding: Random Linear Network Coding

- Coding coefficient vector $\mathbf{b}$ sent along with coded packet
- Key quantity to track
  - Number of linearly independent coefficient vectors
  - Call it rank
- Decodability
  - m linearly independent coefficient vectors
  - Full rank

$$[\mathbf{s}_1 \quad \mathbf{s}_2 \quad \cdots \quad \mathbf{s}_m] = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \cdots \quad \mathbf{c}_m][\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_m]^{-1}$$

Modeling: Hypergraph for Wireless Networks

$$G = (N, E), \quad E = \{(i, K) \mid i \in N, K \subset N\}$$



Hyperarc (1,{2,3})

- Hyperarc (i, K): sender → multiple receivers
  - Broadcast nature of wireless
  - Transmission range
  - Directional antenna (beamforming)

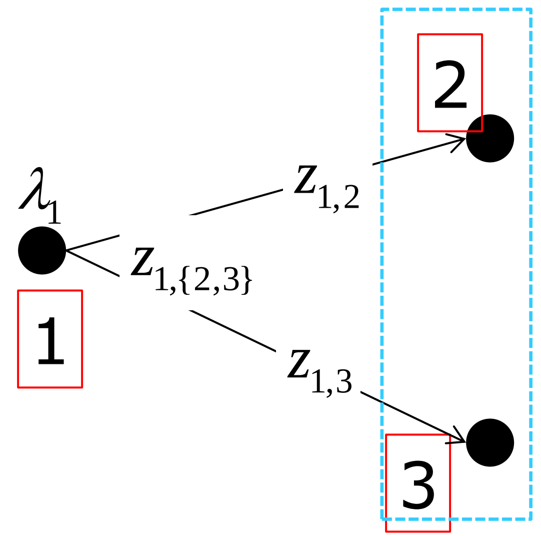
Definition of Capacitated Hypergraph

- A packet goes through the hyperarc (i, K) with probability $P_{i,K}$
  - $P_{i,K}$: received by at least one node in set K
  - Reception can be correlated

    e.g., channel correlation, joint detection

For independent reception

$$P_{i,K} = 1 - \prod_{j \in K} \left(1 - P_{i,j}\right)$$



- Capacity of (i, K)

$$z_{i,K} = \lambda_i P_{i,K}$$

Tx rate of node i (MAC)

Rank Evolution of RNC

- Decoding relies on the rank of node
  - from which also define *rank* of an arbitrary set K

$$S_i = \text{span}\{\mathbf{b}_1, \mathbf{b}_1, \cdots, \mathbf{b}_n\}, \quad N_i = \text{rank at } \boldsymbol{i} = \dim(S_i)$$

$$S_K = \sum_{i \in K} S_i, \quad N_K = \text{rank at } K = \dim(S_K)$$

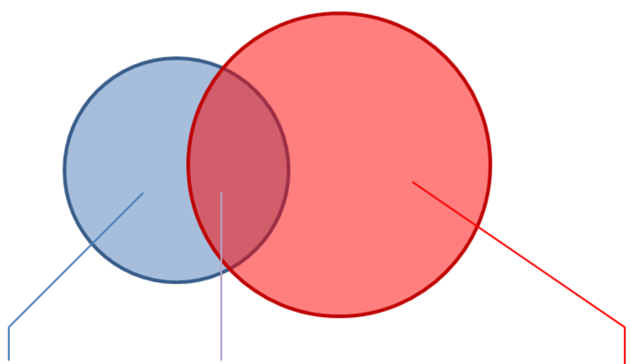
- Define $V_i = E[N_i]$ and $V_K = E[N_K]$
  - Fluid approximation $V_i \approx N_i$ and $V_K \approx N_K$
- How will V_i and V_K evolve over time?
  - When will V_i reach m?

Differential Equations for RNC

- In Δt a packet is sent from node i in K^c with probability $\lambda_i \Delta t$
- It is received by at least one node in K with probability $P_{i,K}$
- This incoming packet increments V_K with probability

$$\frac{|S_i| - |S_i \cap S_K|}{|S_i|} = \frac{q^{\dim S_i} - q^{\dim S_i \cap S_K}}{q^{\dim S_i}} = \frac{q^{V_i} - q^{V_i + V_K - V_{\{i\} \cup K}}}{q^{V_i}} = 1 - q^{V_K - V_{\{i\} \cup K}}$$

- To see this …



Differential Equations for RNC

- On average, the packet increments V_K by

$$V_K(t+\Delta t)-V_K(t)=\Delta t\sum_{i\in K^c}\lambda_i P_{i,K}\left(1-q^{V_K-V_{\{i\}\cup K}}\right)$$

$\Rightarrow$ A Differential Equation (DE) for rank evolution:

$$\dot{V}_K \approx \left(V_K(t+\Delta t)-V_K(t)\right)/\Delta t=\sum_{i\in K^c} z_{i,K}\left(1-q^{V_K-V_{\{i\}\cup K}}\right)$$

Differential Equations for RNC

$$\dot{V}_K = \sum_{i \in K^c} z_{i,K} \left(1 - q^{V_K - V_{\{i\} \cup K}}\right)$$

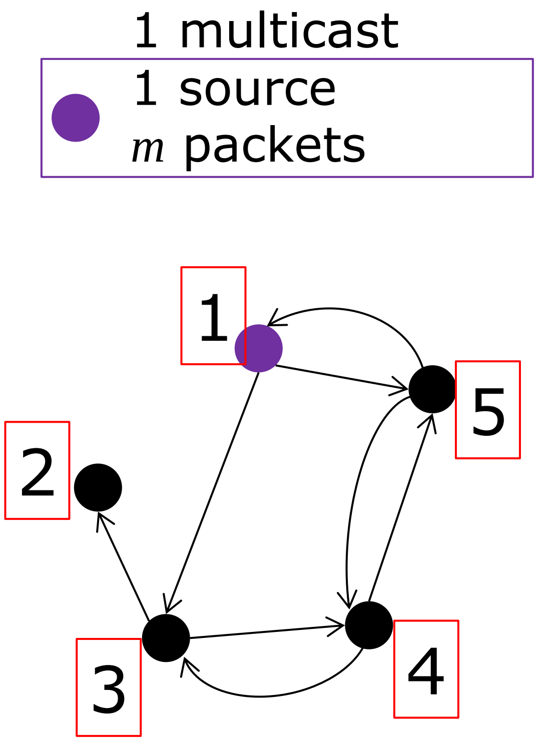
Packet reception rate

Innovative packet probability

- A system of $2^{|N|}-1$ DE's, one for every nonempty K
- Can be solved numerically with initial conditions
  - Good for performance evaluation of various RNC schemes
- Can be analyzed
  - Good for theoretical insights too
- Enables Crosslayer Design

Illustration1: Boundary Conditions

- Specific networking scenarios yield boundary condition



$$V_K(0)=\begin{cases} m, & 1\in K, \\ 0, & \text{o.w.} \end{cases}$$

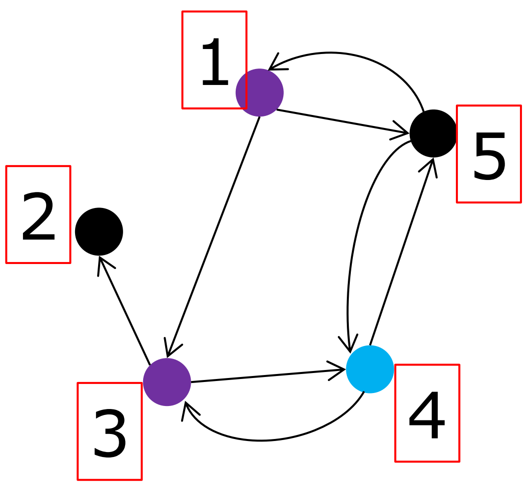
Illustration 2: Boundary Conditions

- Specific networking scenarios yield boundary condition

2 multicasts

2 image sources
m_1 packets

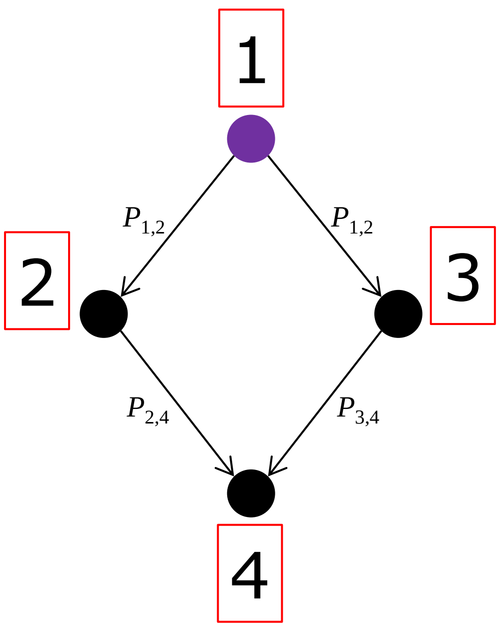
1 single source
m_2 packets



$$V_K(0)=\begin{cases} m_1, & \{1,3\}\cap K \neq \phi, 4 \notin K, \\ m_2, & 4 \in K, \{1,3\}\cap K = \phi, \\ m_1+m_2, & \{1,3\}\cap K \neq \phi, 4 \in K, \\ 0, & \text{o.w.} \end{cases}$$

A Toy Example

- A wireless network with independent receptions
- Transmission of *m* packets start at $t = 0$, compute $V_4(t)$



$$\dot{V}_4 = z_{2,4}\left(1-q^{V_4-V_{\{2,4\}}}\right)+z_{3,4}\left(1-q^{V_4-V_{\{3,4\}}}\right)$$

$$\dot{V}_{\{2,4\}} = z_{1,\{2,4\}}\left(1-q^{V_{\{2,4\}}-m}\right)+z_{3,\{2,4\}}\left(1-q^{V_{\{2,4\}}-V_{\{2,3,4\}}}\right)$$

$$\dot{V}_{\{3,4\}} = z_{1,\{3,4\}}\left(1-q^{V_{\{3,4\}}-m}\right)+z_{2,\{3,4\}}\left(1-q^{V_{\{3,4\}}-V_{\{2,3,4\}}}\right)$$

$$\dot{V}_{\{2,3,4\}} = z_{1,\{2,3,4\}}\left(1-q^{V_{\{2,3,4\}}-m}\right)$$

$$z_{2,4} = \lambda_2 P_{2,4}, \quad z_{3,4} = \lambda_3 P_{3,4}, \quad z_{1,\{2,4\}} = \lambda_1 P_{1,\{2,4\}} = \lambda_1 P_{1,2}$$

$$z_{1,\{3,4\}} = \lambda_1 P_{1,\{3,4\}} = \lambda_1 P_{1,3}, \quad z_{1,\{2,3,4\}} = \lambda_1 P_{1,\{2,3,4\}} = \lambda_1 P_{1,\{2,3\}}$$

$$P_{1,\{2,3\}} = 1-\left(1-P_{1,2}\right)\left(1-P_{1,3}\right)$$

$$\text{B.C.} \quad V_4(0) = V_{\{2,4\}}(0) = V_{\{3,4\}}(0) = V_{\{2,3,4\}}(0) = 0$$

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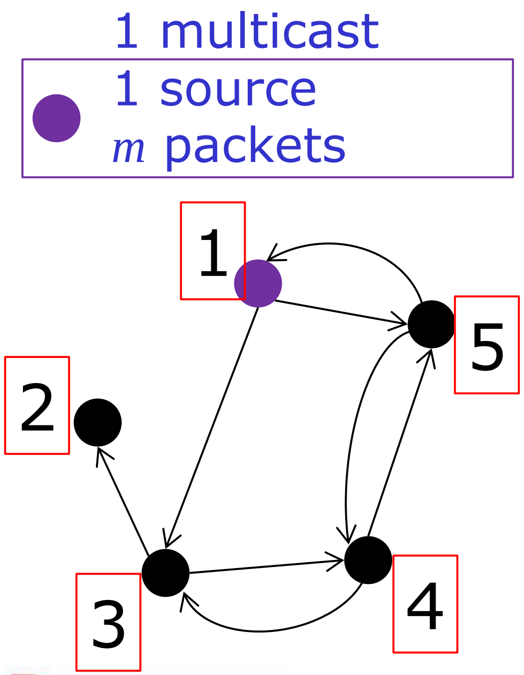
Analytical Result: Multicast Throughput Using RNC

- Solve the DEs (with B.C.) for $V_K(t)$
- Throughput is the derivative of $V_K(t)$
- Throughput given by min cut!



$$\dot{V}_K = \sum_{i \notin K} z_{i,K}\left(1 - q^{V_K - V_{\{i\} \cup K}}\right)$$

$$\text{B.C.} \quad V_K(0) = \begin{cases} m, & 1 \in K, \\ 0, & \text{o.w.} \end{cases}$$

$1 \notin K$:

$$V_K(t) = \begin{cases} c_{\min}(\{1\}, K)t, & t \in [0, m/c_{\min}(\{1\}, K)), \\ m, & t \in [m/c_{\min}(\{1\}, K), \infty). \end{cases}$$

$1 \in K$:

$$V_K(t) = m, \quad t \in [0, \infty).$$

What is Min Cut of a Hypergraph?

- A cut for (S, K) is a set T s.t.

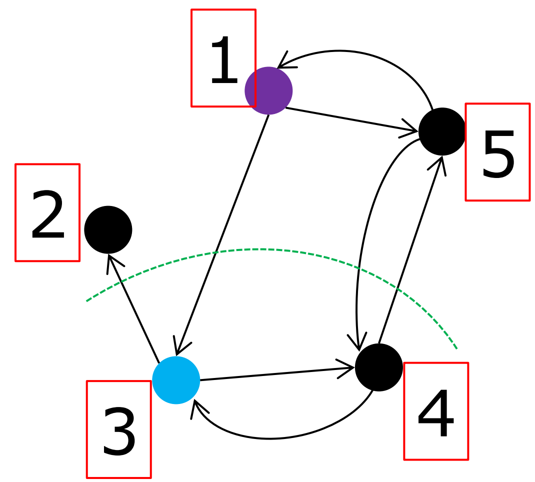
$$K \subset T \subset S^c$$

- Cut size

$$c(T) = \sum_{i \in T^c} z_{i,T}$$

- Min cut: smallest cut size

$$c_{\min}(S,K) = \min_{T \text{ is a } (S,K) \text{ cut}} c(T)$$



Example:

Cut T for $(\{1\},\{3\})$, $T=\{3,4\}$

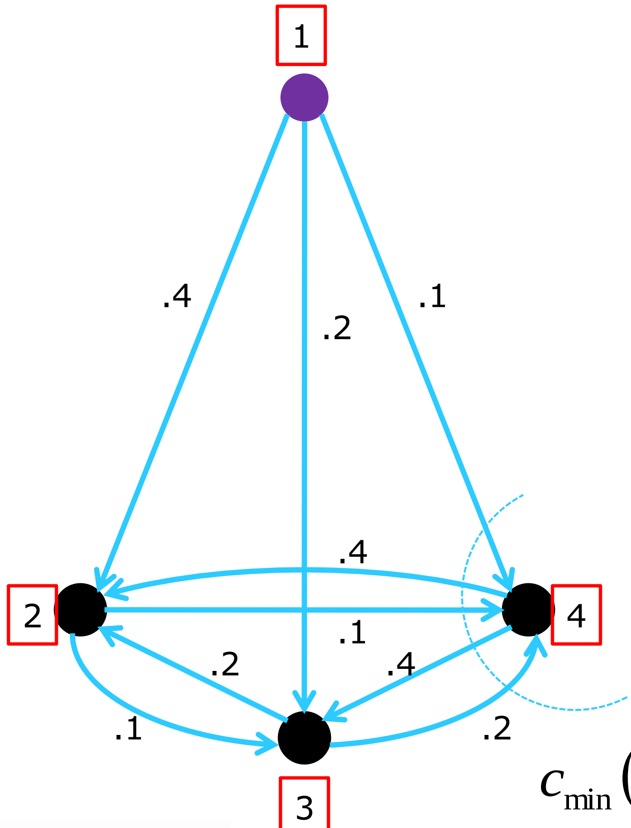
$$c(T) = z_{1,\{3,4\}} + z_{2,\{3,4\}} + z_{5,\{3,4\}}$$

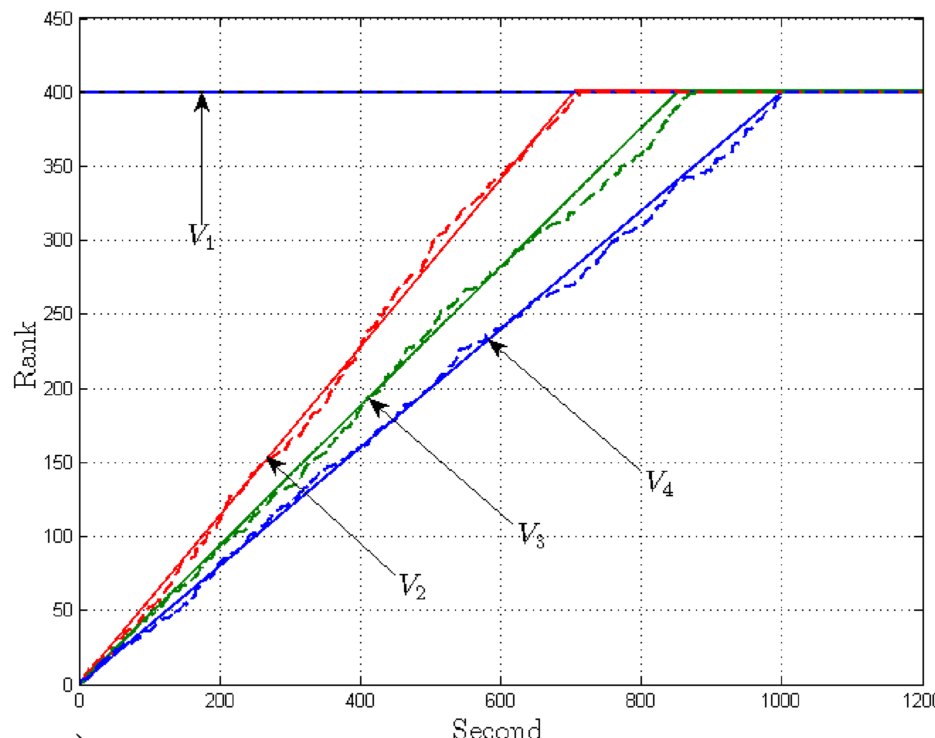
Example - Wireless P2P

$\lambda_i = 1, \quad i = 1,2,3,4$

$m = 400$

- Peers download from server
- Peers transmit to each other to enhance throughput





$$c_{\min}(\{1\},\{4\}) = .1 + .1 + .2 = .4$$

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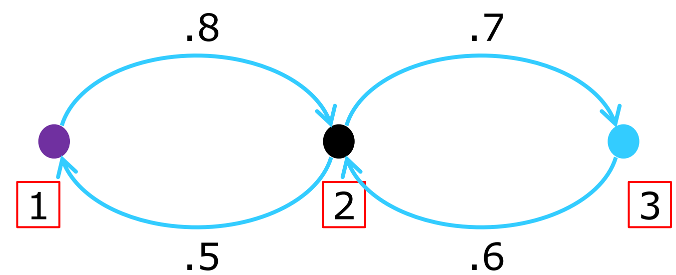
Example – Multiple Sources

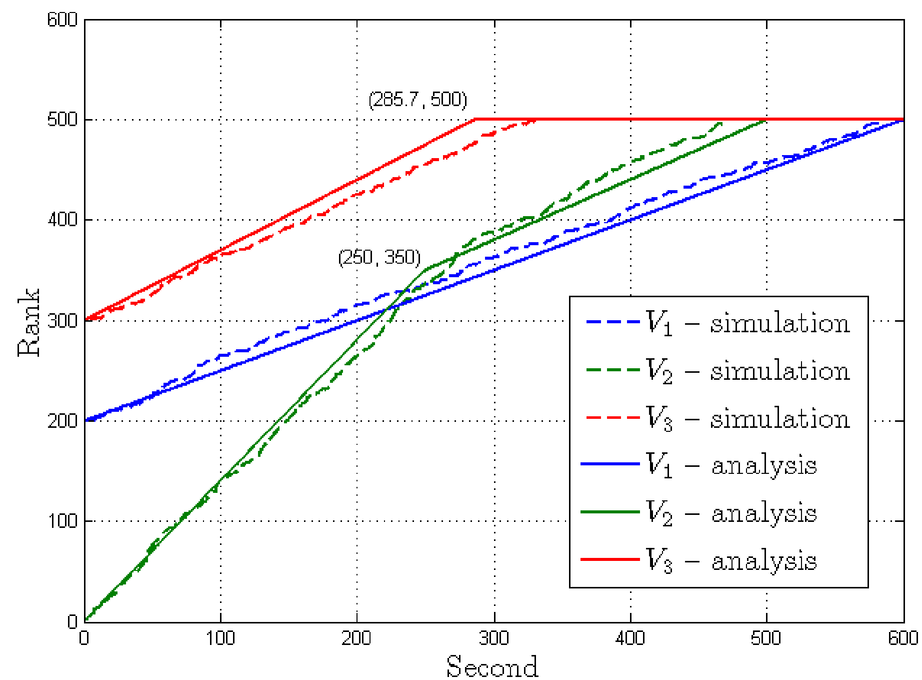
$\lambda_i = 1, \quad i = 1,2,3$

● $m_1 = 200,$ ● $m_2 = 300$



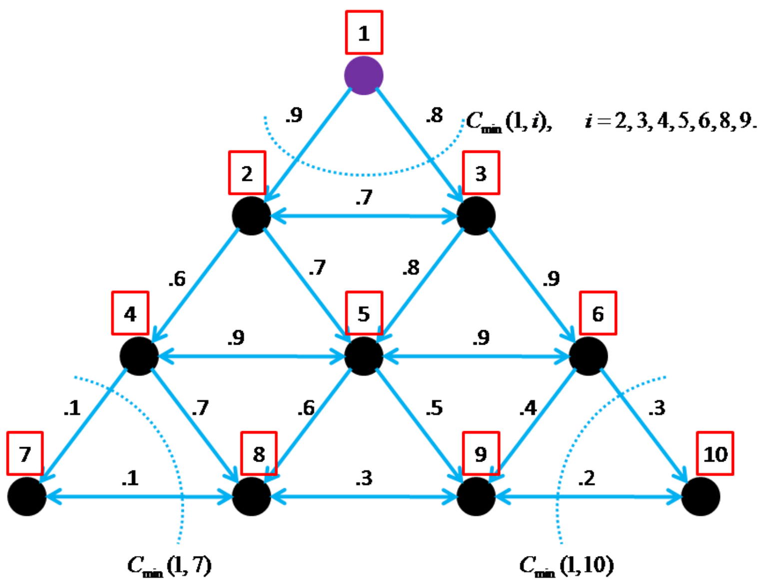
$$\dot{V}_K = \sum_{i \notin K} z_{i,K} \left(1 - q^{V_K - V_{\{i\} \cup K}}\right)$$

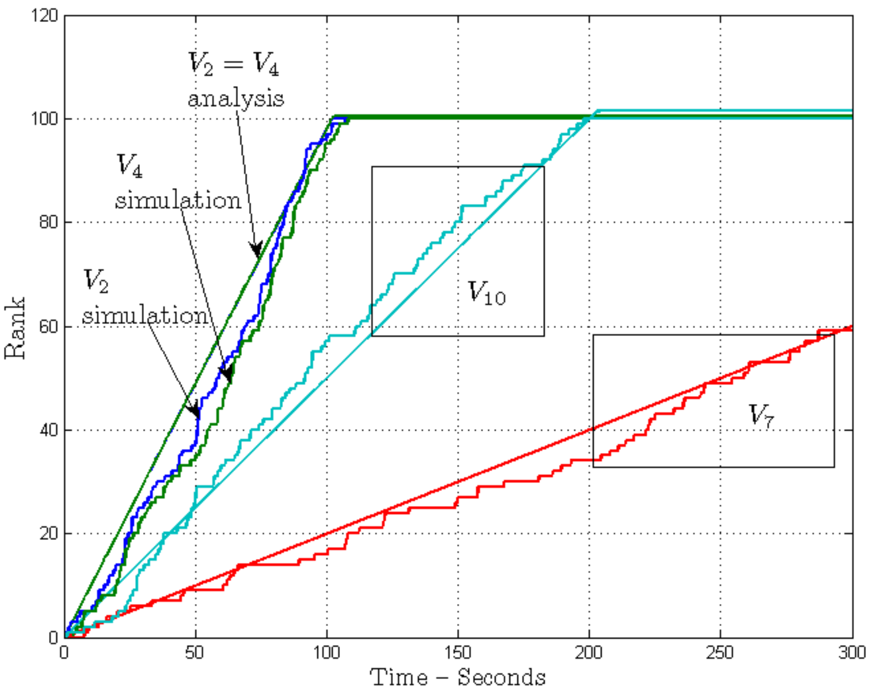
$$\text{B.C.}\quad V_K(0) = \begin{cases} 200, & K = \{1\}, \{1,2\}, \\ 300, & K = \{3\}, \{2,3\}, \\ 500, & K = \{1,2,3\}, \\ 0, & K = \{2\}. \end{cases}$$



Example – Complex Networks

$$\lambda_i = 1, \quad i = 1,2,\cdots,10, \quad m = 100$$





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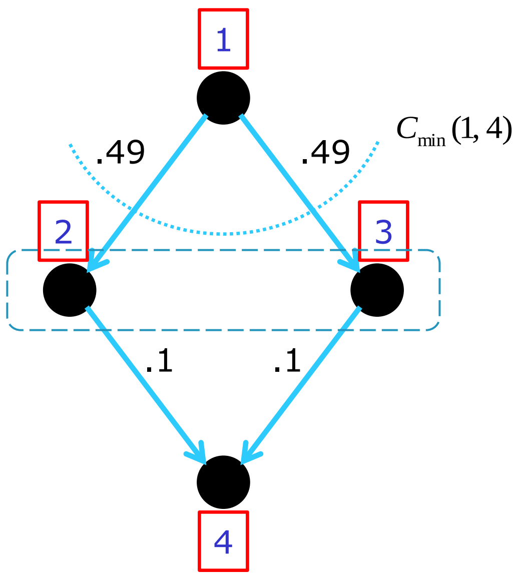
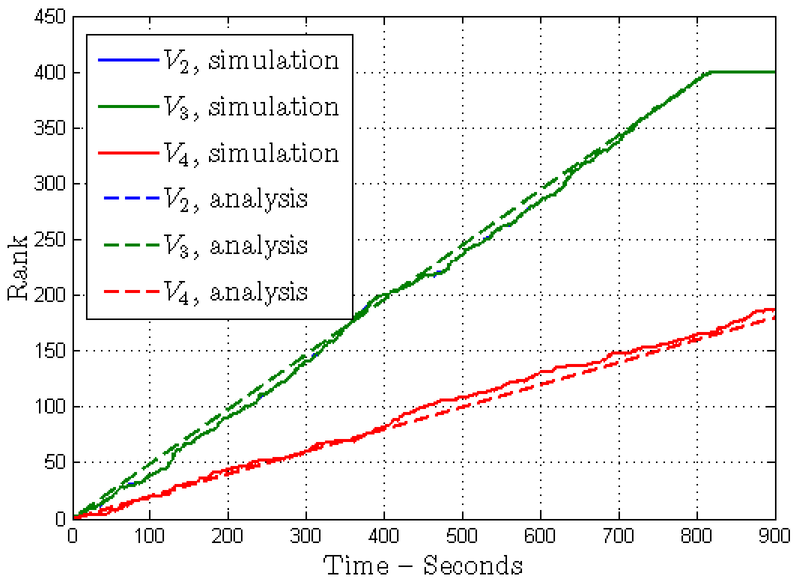
Example – Correlated Reception

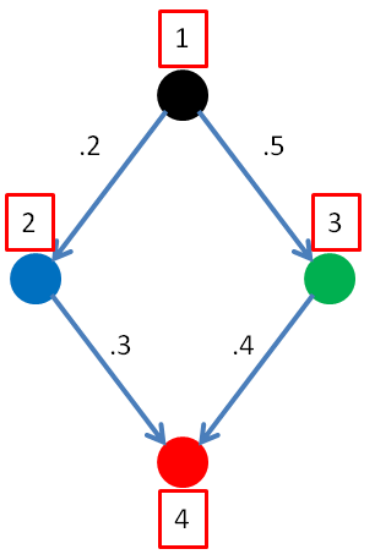


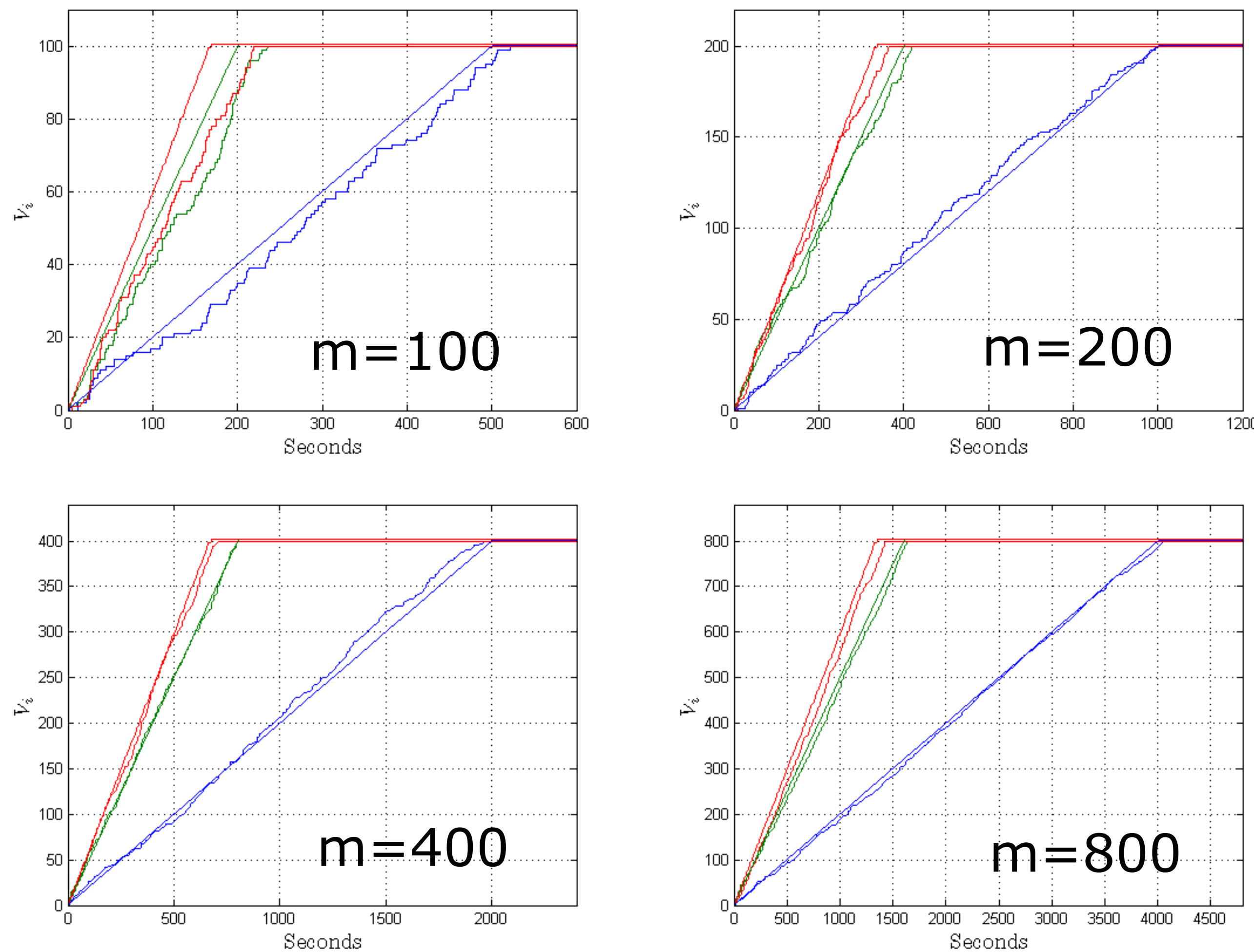
$P_{1,2} = 0.49, \quad P_{1,3} = 0.49, \quad P_{1,\{2,3\}} = 0.5$



Channel 1→2 and channel 1→3 highly correlated

Effect of Number of Source Packets m?





Is there a Concentration Result?

- Numerical examples suggest concentration property
  - As no. of source packets increase, DE solution becomes increasingly accurate
  - Rank processes concentrate to DE solution
- Previous studies showed throughput given by

$$c_{\min}(\mathrm{src}, \mathrm{dst})$$

  is achieved asymptotically with no. of source packets

- Can we prove this result with DE framework?

Yes We Can!

Let node 1 be source, D the destination set, m packets to deliver and T the total Tx time

- DE for the variance

$$\frac{\mathrm{d}\,\mathrm{var}[N_K]}{\mathrm{d}t} = \frac{\mathrm{d}E[N_K]}{\mathrm{d}t} + 2\sum_{i \notin K} z_{i,K} \,\mathrm{cov}[N_K, 1 - q^{N_K - N_{\{i\}\cup K}}]$$

- Use it to bound variance

$$\limsup_{t\to\infty} \frac{\mathrm{var}[N_K]}{t^{2-\delta_K}} \le d_K \qquad \text{for some } \delta_K, d_K > 0$$

- Use Chebyshev inequality to show throughput converges to min cut in probability

$$\lim_{m\to\infty} P(m/T > \varepsilon c_{\min}(1, D)) = 1, \quad \forall 0 < \varepsilon < 1$$

Can we use DE Framework for Cross-layer Resource Allocation in RNC?

- Model RNC as a dynamical system:

$$\dot{V}_{\mathcal{K}} = \sum_{i \notin \mathcal{K}} z_{i,\mathcal{K}} (1 - q^{V_{\mathcal{K}} - V_{\{i\} \cup \mathcal{K}}}), \ \forall \mathcal{K} \subset \mathcal{N} \text{ and } \mathcal{K} \neq \emptyset$$

Packet reception rate

Innovative packet probability

- Hyperarc capacity of $(i, \mathcal{K})$
  - $z_{i,\mathcal{K}} = \lambda_i P_{i,\mathcal{K}}$

Tx rate at node i

Reception probability (A pkt from *i* can be recvd by at least 1 node in *K*)

MAC

PHY

DE framework closely models rank evolution of RNC in terms of PHY and MAC parameters

How does Power Control Impact RNC?

- In general, power control to achieve certain objective, e.g.:
  - Maintain certain SINR value
  - Minimize total power
- With RNC, Tx powers influence network coding performance
  - Larger transmit power:

Good for its own transmission ✔

Increases interference for other transmissions in the same network ✗

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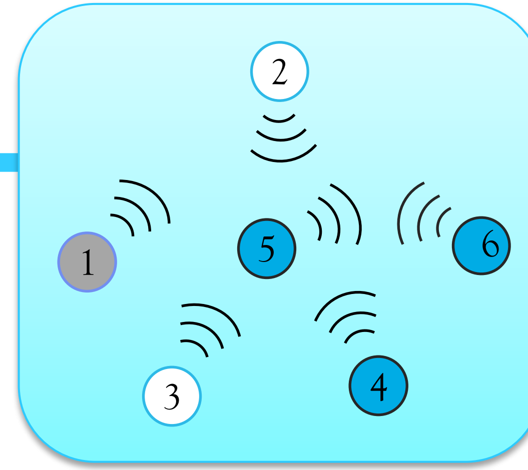
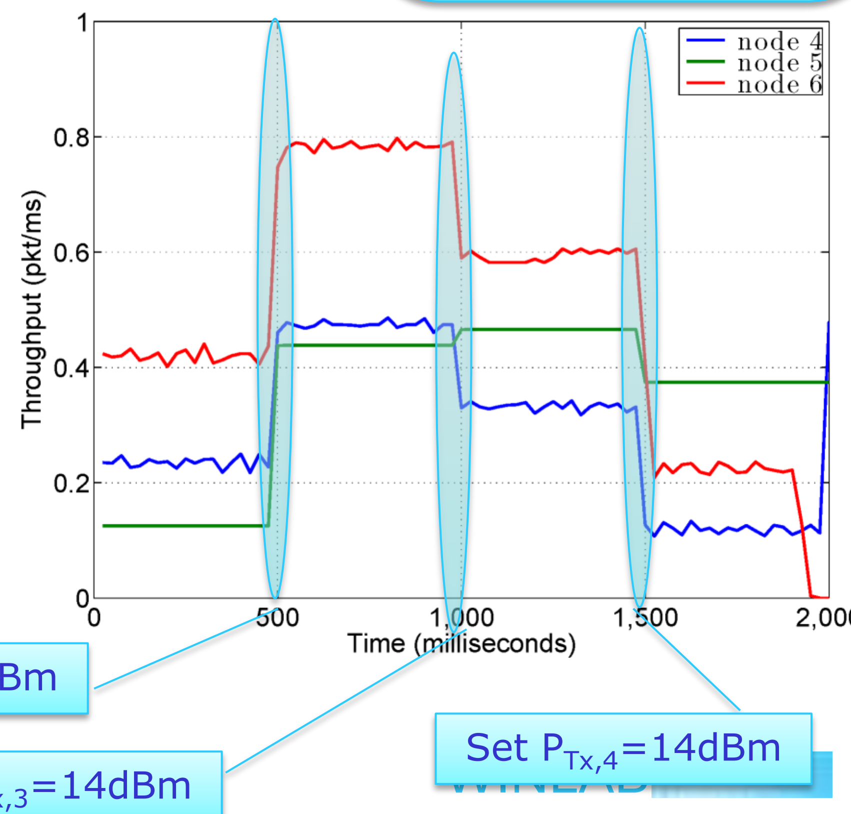
Motivation - Necessity of Power Control in RNC



- Consider
  - 6-node topology, RNC
  - Node 1 delivers pkts to {4,5,6}
  - Each node Tx at 1pkt/ms
  - t=0, every node's Tx power set to 13dBm
  - Tx powers $P_{\mathrm{Tx},1}, P_{\mathrm{Tx},3}, P_{\mathrm{Tx},4}$ are increased sequentially
- Observations
  - Incrementing power isn't always good for throughput
  - Is there an optimal strategy?



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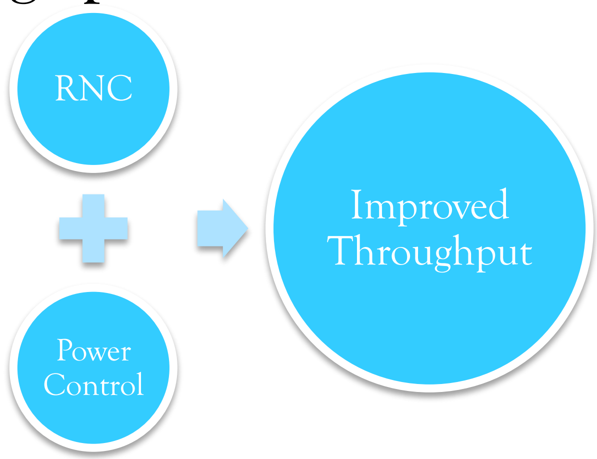
Network Coding Aware Power Control

- Find an optimal allocation of Tx powers such that multicast throughput can be maximized



- Challenge: Need a good model of RNC performance in terms of Tx powers
  - DE framework!

Problem Setup - Max-min-throughput Power Control

- Formulate max-min throughput problem using DE:

$$
\begin{aligned}
\text{maximize} \quad & \min_{j \in \mathcal{D}} \dot{V}_j \\
\text{subject to} \quad & \dot{V}_{\mathcal{K}} = \sum_{i \notin \mathcal{K}} z_{i,\mathcal{K}} \cdot (1 - q^{V_{\mathcal{K}} - V_{\{i\} \cup \mathcal{K}}}) \\
& z_{i,\mathcal{K}} = \lambda_i P_{i,\mathcal{K}}(\mathbf{P}_{\mathrm{Tx}}) \\
& 0 \preceq \mathbf{P}_{\mathrm{Tx}} \preceq \mathbf{P}_{\mathrm{Tx}}^{\max} \\
\text{variables} \quad & \mathbf{P}_{\mathrm{Tx}}
\end{aligned}
$$

Min. throughput among the dests. set $\mathcal{D}$

DE framework for RNC

Size-N Tx power vector

Objective: adjusting powers so that minimum throughput is maximized

Gradient-based Algorithm

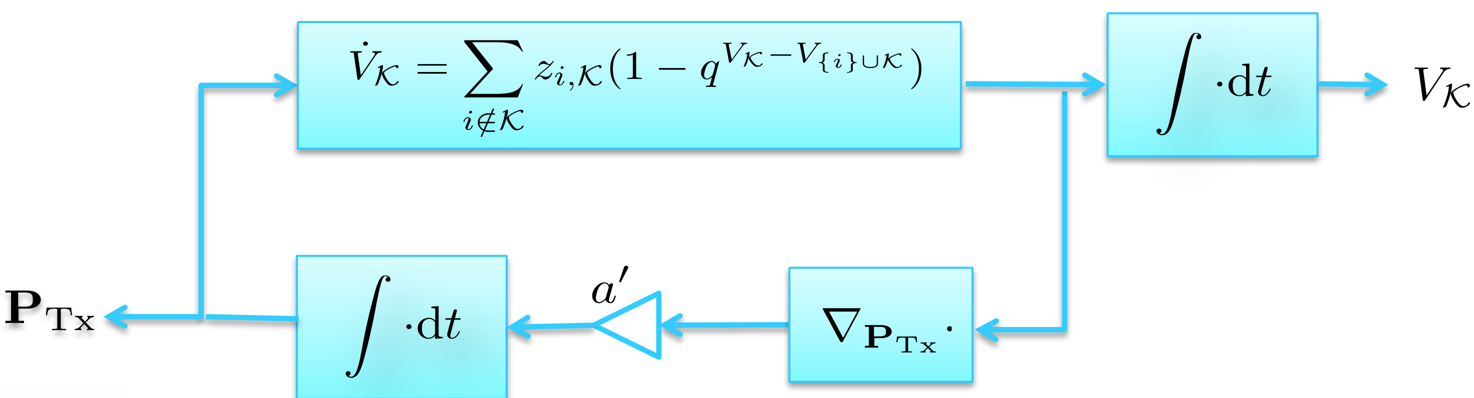
- Idea :
  - Suppose dest k has the minimum instantaneous throughput
  - Find the gradient of the throughput of k
  - Adapt powers towards the direction of the gradient

$$\dot{\mathbf{P}}_{\mathrm{Tx}} = a' \cdot \nabla \dot{V}_k \left(\mathbf{P}_{\mathrm{Tx}}\right).$$

Positive constant serving as gain parameter

Gradient of minimum throughput

- Power control feedback loop:



Gradient-based Algorithm: Approximation of the Gradient

$$\dot{\mathbf{P}}_{\mathrm{Tx}} = a' \cdot \nabla \dot{V}_k \left(\mathbf{P}_{\mathrm{Tx}}\right).$$

- Challenge: hard to directly evaluate $\nabla \dot{V}_k$
- Recall $\dot{V}_{\mathcal{K}} = \sum_{i \notin \mathcal{K}} \lambda_i P_{i,\mathcal{K}} (1 - q^{V_{\mathcal{K}} - V_{\{i\} \cup \mathcal{K}}})$
- Why hard?
  - Underlying PHY schemes may change over time
    - Result in changed $P_{i,\mathcal{K}}$
  - The explicit expression for $P_{i,\mathcal{K}}$ may even be unknown in practice
    - Depends on PHY specifics like modulation, FEC, diversity ...

Gradient-based Algorithm: Approximation of the Gradient

- Approach: Estimate in a centralized manner
  - For node i to adjust power
    - Increment i's power, others remain the same.
    - Construct a power vector

$$\mathbf{q}_i = \mathbf{P}_{\mathrm{Tx}} + \Delta q \mathbf{e}_i .$$

Power increment for node i

Vector with i-th component being 1 and 0 elsewhere

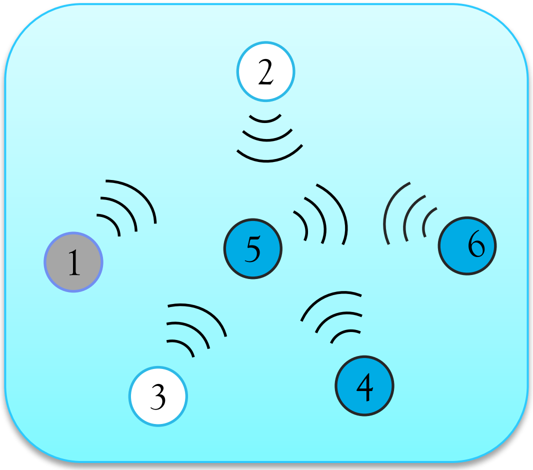
- Estimate the gradient:

$$\frac{\partial \dot{V}_k(\mathbf{P}_{\mathrm{Tx}})}{\partial P_{\mathrm{Tx},i}} = \frac{1}{\Delta q}\left(\sum_{j \neq k} (z_{j,k}(\mathbf{q}_i) - z_{j,k}(\mathbf{P}_{\mathrm{Tx}}))(1 - q^{V_{\mathcal{K}} - V_{\{i\}\cup\mathcal{K}}})\right).$$

- Adjust the power in the direction of gradient

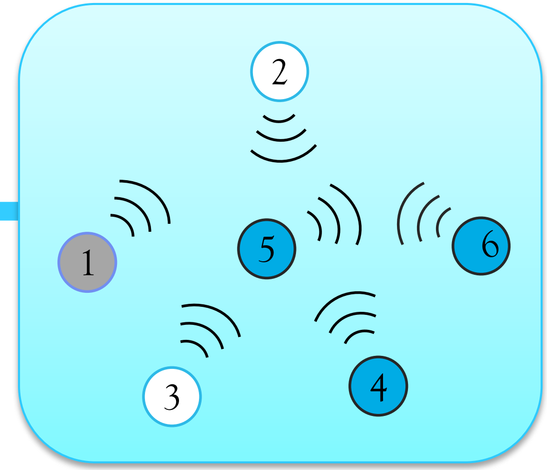
Numerical Results

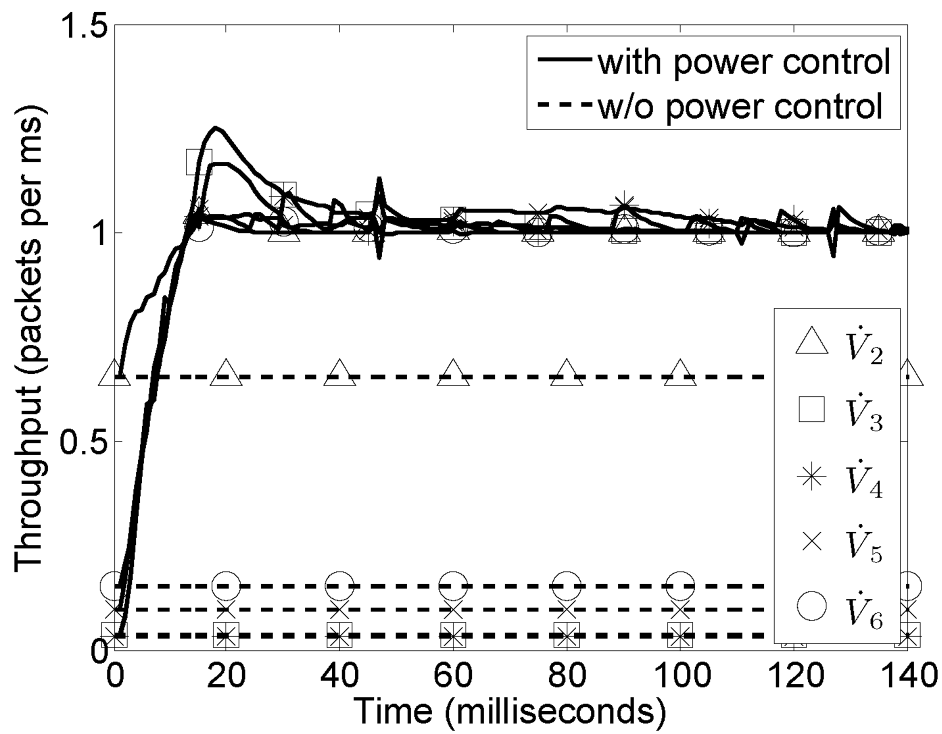
- Simulation setup:
  - Use numerical DE solver to evaluate the algorithm
  - 6-node topology, nodes perform RNC
  - Source: node 1, and dest. nodes {4,5,6}
  - src has 2000 packets to multicast to dests.
  - Assumption:
    - Certain MAC: each node transmits at 1pkt/ms
    - Certain PHY: BPSK signaling and Gaussian interference model
    - Tx power varies between 0dBm and 15dBm



Throughput Performance

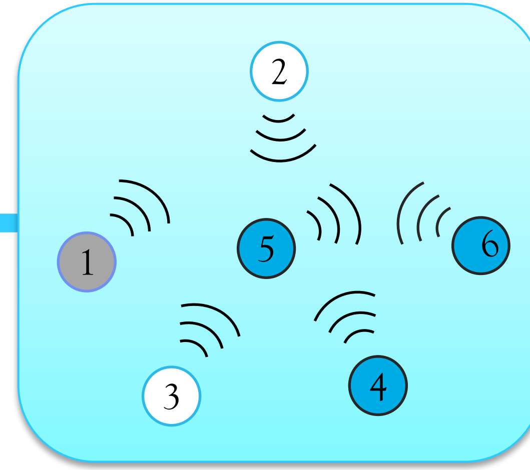
- Without Power Control (PC), throughput is a constant
- With PC: instantaneous throughput is improved compared to no PC
- With PC: throughput Converges around t=60ms
- Each node Tx at 1pkt/ms, min cut between. src. and dst. is 1 pkt/ms



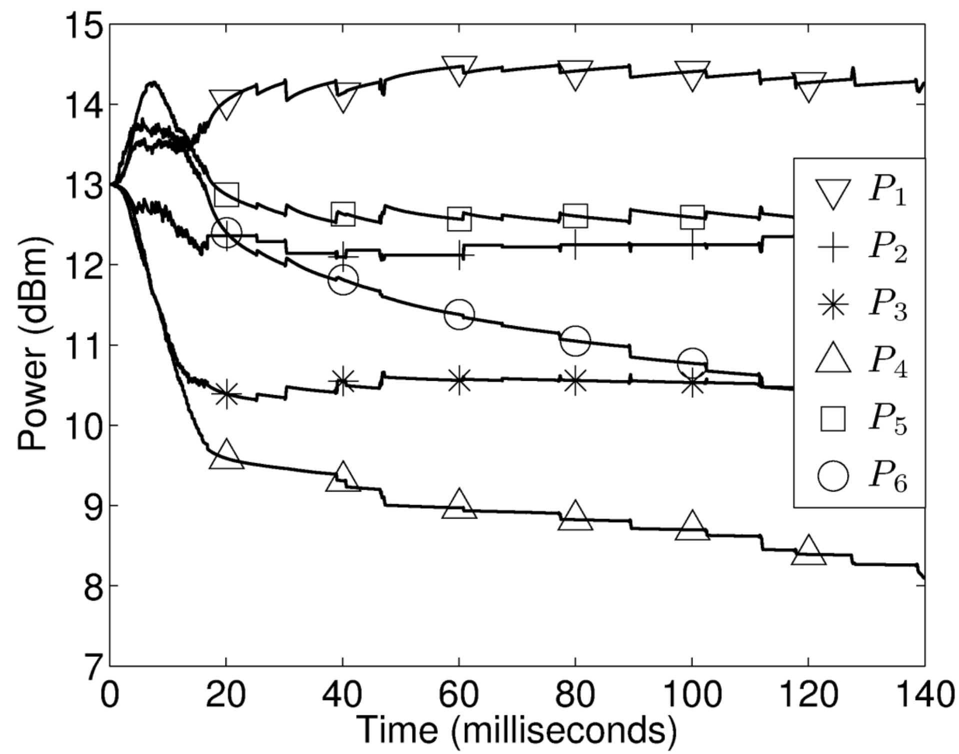


RNC with PC achieves optimal throughput!

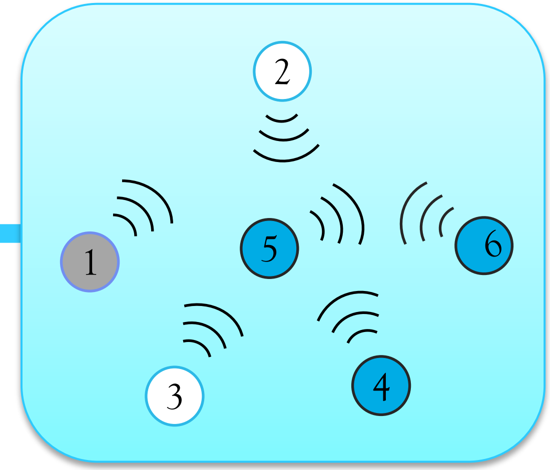
Transmit Powers



- Tx power initially set to 13dBm
- Powers are adjusted with an upper bound of 15dBm
- Tends to converge around t=140ms

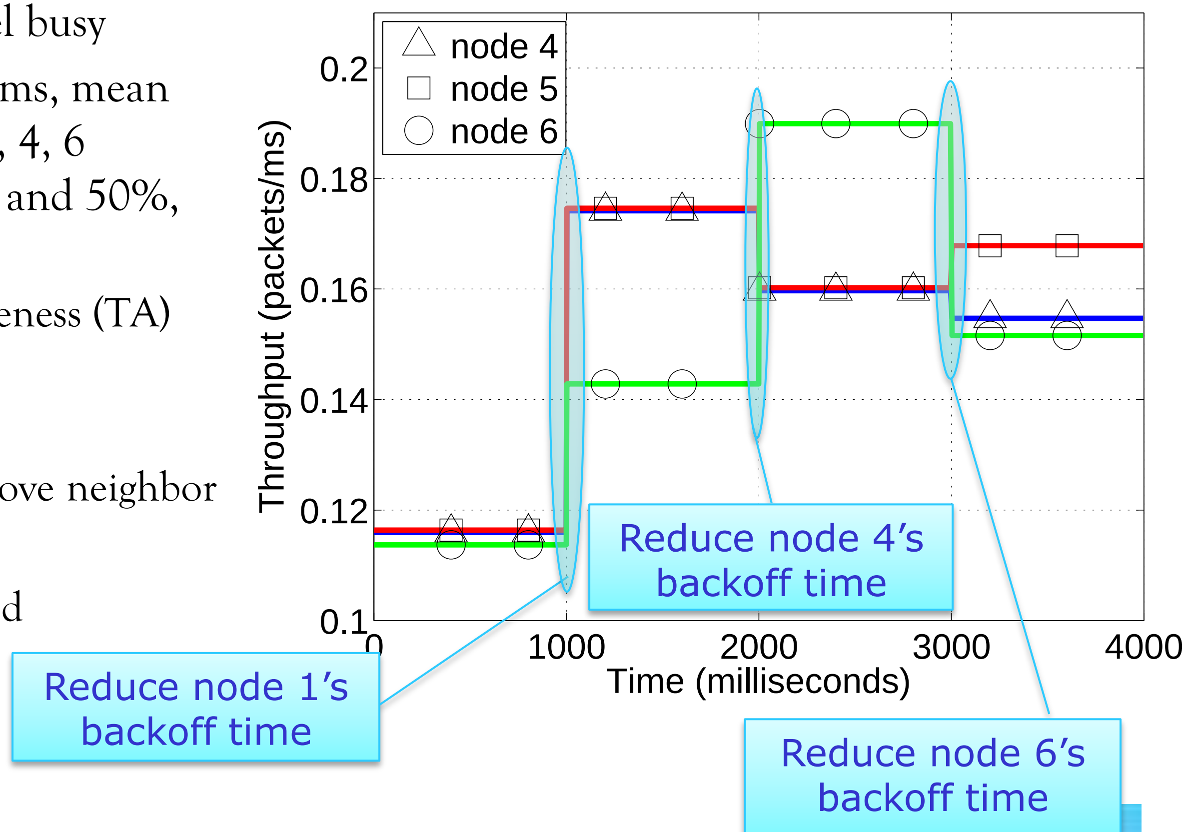


Motivation - Necessity of CSMA backoff control in RNC



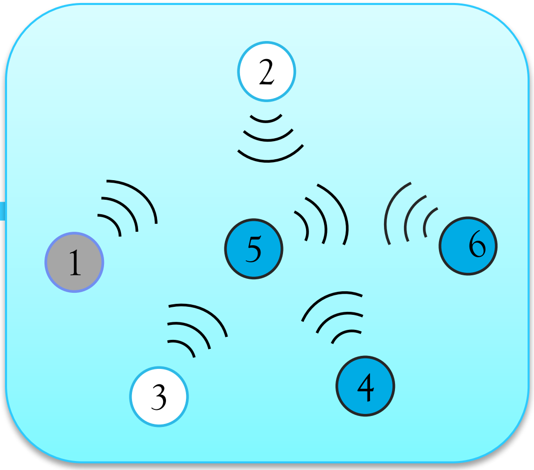
- Consider
  - Node 1 delivers pkts to {4,5,6}
  - CSMA as the MAC protocol
  - Exp. backoff if channel busy
  - At t=1000, 2000, 3000ms, mean backoff time of node 1, 4, 6 reduced, by 30%, 40% and 50%, respectively

  → Transmission aggressiveness (TA) increased
- Observations
  - Increased TA: May improve neighbor throughput
  - May also be bad: reduced channel availability

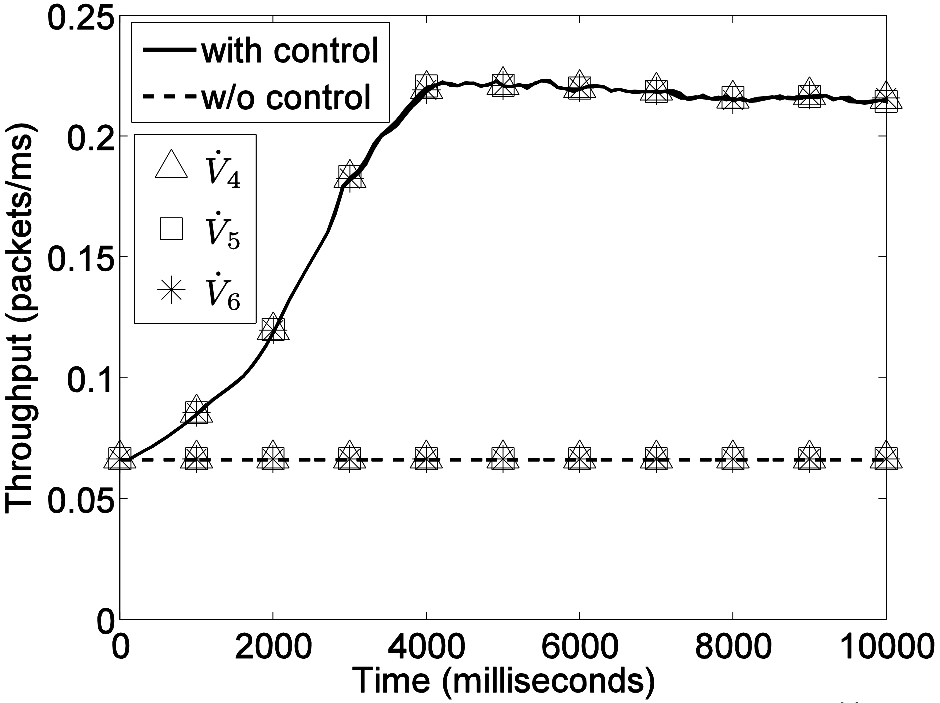


RUTGERS

CSMA backoff control - Throughput Performance



- Without control:
  Mean backoff time of nodes is fixed
- Without control:
  throughputs remain at 0.06pkt/ms.
- With control:
  instantaneous throughput is improved compared with no control case
- With control:
  throughput converges to 0.22 around t=4000ms
- Throughput gain: >200%.



Concluding Remarks

- Impact of PHY algorithms on network coding performance
  - *Rate of Rank Evolution*
- Impact of MAC on network coding performance
  - *Interference effects*
- Resource Allocation
  - *Power Control, Scheduling*

$$\dot{V}_K \approx \sum_{i \notin K} z_{i,K} \left(1 - q^{V_K - V_{\{i\} \cup K}}\right)$$

- Crosslayer Design problems
  - *Solving systems differential of equations*
  - *Appropriate boundary conditions*

- Emulation/Software Utility under Development
  - *Network Topologies*
  - *Radio Channels*
  - *PHY, MAC and Resource Allocation Algorithms*

References on DE Framework for Network Coding

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