DEDI: A Framework for Analyzing Rank Evolution of Random Network Coding in a Wireless Network

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***Abstract*—We develop a framework called DEDI based on differential equations (DE) and differential inclusions (DI) to describe the rank evolution of random network coding (RNC). The DEDI serves as a powerful numerical and analytical tool to study RNC and we demonstrate this via numerical examples as well as an alternate proof of a well known result on RNC – a multicast at rate R exists if and only if a unicast at rate R exists separately for each destination.**

***Index Terms*—Random network coding, rank evolution, differential equation, differential inclusion**

I. Introduction

Since the pioneering work by Ahlswede *et al.* [1] that established the benefits of coding in routers and provided theoretical bounds on the capacity of such networks, the breadth of areas that have been touched by network coding is vast and includes not only the traditional disciplines of information theory, coding theory and networking, but also topics such as routing algorithms [2], distributed storage [3], [4], network monitoring, content delivery [5], [6], and security [7]. Among other variants, random network coding (RNC) [8], [9] has received extensive interest in particular. By allowing routers to perform random linear operations, RNC is shown to be capacity achieving and fault tolerant at the price of low operational complexity. In spite of all the excellent progress previous studies have made in the area of RNC, what is still missing is a simple framework that can be used to describe the evolution of rank/state in a wireless network where RNC is employed. In this paper we present a framework DEDI based on differential equations (DE) and differential inclusions (DI), which are a generalization of DEs to allow for discontinuous right hand sides. The DEDI serves as a powerful numerical and analytical tool to study RNC and we demonstrate this via numerical examples as well as an alternate proof of a well known result on RNC – a multicast at rate R exists if and only if a unicast at rate R exists separately for each destination.

II. Preliminaries

A generic wireless network is modeled as a hypergraph $G = (\mathcal{N}, \mathcal{E})$ consisting of N nodes $\mathcal{N} = \{1, 2, \ldots, N\}$ and hyperarcs $\mathcal{E} = \{(i, \mathcal{K}) | i \in \mathcal{N}, \mathcal{K} \subset \mathcal{N}\}$. Each hyperarc captures the fact that, as any wireless transmission is inherently a broadcast, a packet sent from node i can be received by some or all the nodes in a set $\mathcal{K} \subset \mathcal{N}$. Assume some underlying MAC protocol is already operating in its steady state such that each node i is transmitting at λ_i packets per second. We say that a packet sent from node i is successfully received by a set $\mathcal{K}$ of nodes if the packet is successfully received by at least one node in $\mathcal{K}$. We assume this happens with a probability $P_{i,\mathcal{K}}$ and call it the reception probability of $(i, \mathcal{K})$, which allows the possibility of correlated receptions. Given λ_i and $P_{i,\mathcal{K}}$, we may define the transmission rate $z_{i,\mathcal{K}}$ for $(i, \mathcal{K})$ as

$$z_{i,\mathcal{K}} = \lambda_i P_{i,\mathcal{K}}. \tag{1}$$

By definition we have, for $\mathcal{K} \subset \mathcal{T} \subset \mathcal{N}$, $P_{i,\mathcal{K}} \leq P_{i,\mathcal{T}}$, hence

$$z_{i,\mathcal{K}} \leq z_{i,\mathcal{T}}. \tag{2}$$

Suppose $\mathcal{S}, \mathcal{K} \subset \mathcal{N}$ and $\mathcal{S} \cap \mathcal{K} = \emptyset$. Define a cut for the pair $(\mathcal{S}, \mathcal{K})$ as a set $\mathcal{T}$ satisfying $\mathcal{K} \subset \mathcal{T} \subset \mathcal{S}^c$. Let $C(\mathcal{S}, \mathcal{K})$ denote the collection of all cuts for $(\mathcal{S}, \mathcal{K})$. The size of $\mathcal{T}$ is defined as $c(\mathcal{T}) = \sum_{i \in \mathcal{T}^c} z_{i,\mathcal{T}}$. The min cut $\mathcal{T}_{\min}$ for $(\mathcal{S}, \mathcal{K})$, whose size is denoted as $c_{\min}(\mathcal{S}, \mathcal{K})$ is a cut satisfying

$$c(\mathcal{T}_{\min}) = \min_{\mathcal{T}' \in C(\mathcal{S}, \mathcal{K})} c(\mathcal{T}'). \tag{3}$$

We briefly describe RNC for a single multicast session with node 1 being the unique source trying to deliver m packets. Each packet $\underline{w}$ is a row vector from $\mathbb{F}^L$ where $\mathbb{F}$ is a given finite field of size q and L is a positive constant that denotes the length of packet. Every node maintains a reservoir consisting of all the packets the node holds as a source plus all the packets received thus far during a coded session. The reservoir is ever growing and purged only after the associated session is completed. Whenever a node gets to transmit, a coded packet is formed and sent out as follows. Suppose at time t node i needs to form a coded packet $\underline{v}$ from its reservoir $\mathrm{Rsv}(i, t) = \{\underline{w}_{i,1}, \underline{w}_{i,2}, \ldots, \underline{w}_{i,n}\}$, $\underline{v}$ takes the form $\underline{v} = a_1\underline{w}_{i,1} + a_2\underline{w}_{i,2} + \cdots + a_n\underline{w}_{i,n}$, where $[a_1, \ldots, a_n] \in \mathbb{F}^n$ is randomly generated. Since the coding operation is entirely linear, we have $\underline{v} = b_{i,1}\underline{w}_1 + b_{i,2}\underline{w}_2 + \cdots + b_{i,m}\underline{w}_m$ where $\underline{w}_1, \ldots, \underline{w}_m$ are the m source packets and $[b_{i,1}, b_{i,2}, \ldots, b_{i,m}] \in \mathbb{F}^m$ is called the global coefficient vector associated with $\underline{v}$. Each node sends the global coefficient vector along with its associated coded packet in order to enable the receiving nodes to calculate the global coefficient vectors for their own coded packets. Let S_i be the vector space spanned by the global coefficient vectors associated with the packets in $\mathrm{Rsv}(i,t)$ and define $V_i = \dim S_i$, which we call the *rank* of node i. Then S_i and V_i are time dependent as the coded transmissions evolve and once $V_i = m$, decoding can be carried out with a linear inverse operation. Further, for any set $\mathcal{K} \subset \mathcal{N}$, define $S_\mathcal{K} = \sum_{i\in\mathcal{K}} S_i$ and $V_\mathcal{K} = \dim S_\mathcal{K}$. We call $V_\mathcal{K}$ the rank of $\mathcal{K}$. The question we are interested in answering is how the rank V_i of a non-source node i increases from 0 to m over time, i.e., how does the rank evolve.

III. The DEDI Framework

A. Rank Evolution Modeled with DE

While $V_i(t)$ is a complicated incremental stochastic process, under a fluid approximation [10] $V_i(t)$, as represented by $E[V_i(t)]$, can be modeled by a set of differential equations. Thus we drop the notation $E[\cdot]$ and use $V_i(t)$ or $V_\mathcal{K}(t)$ to denote their average values, respectively, as functions of t. Consider a general set $\mathcal{K}$ and a time interval Δt such that

$$\Delta t \sum_{j\notin\mathcal{K}} \lambda_j = 1, \tag{4}$$

i.e., there is one packet that is sent from some node in $\mathcal{K}^c$ in the interval Δt. We wish to calculate the average increase of $V_\mathcal{K}$ caused by this packet. We begin by observing that

1) This packet is actually from node i with probability $\lambda_i / \sum_{j\notin\mathcal{K}} \lambda_j$.
2) This packet from i is successfully received by $\mathcal{K}$ with probability $P_{i,\mathcal{K}}$.
3) The global coefficient vector associated with this packet increases the rank of $\mathcal{K}$ by 1 if and only if it comes from $S_i \setminus (S_i \cap S_\mathcal{K})$. Since
$$|S_i \cap S_\mathcal{K}| = q^{\dim S_i \cap S_\mathcal{K}} = q^{V_i + V_\mathcal{K} - V_{\mathcal{K}\cup\{i\}}}, \tag{5}$$
it follows that the probability is given by
$$(|S_i| - |S_i \cap S_\mathcal{K}|)/|S_i| = 1 - q^{V_\mathcal{K} - V_{\mathcal{K}\cup\{i\}}}. \tag{6}$$

Therefore, in Δt a packet from node i increases $V_\mathcal{K}$ by $\lambda_i P_{i,\mathcal{K}} (1 - q^{V_\mathcal{K} - V_{\mathcal{K}\cup\{i\}}}) / \sum_{j\notin\mathcal{K}} \lambda_j$ on average. Collectively, in Δt the packets from $\mathcal{N}\setminus\mathcal{K}$ increase $V_\mathcal{K}$ by $\sum_{i\notin\mathcal{K}} \lambda_i P_{i,\mathcal{K}} (1 - q^{V_\mathcal{K} - V_{\mathcal{K}\cup\{i\}}}) / \sum_{j\notin\mathcal{K}} \lambda_j$ and

$$V_\mathcal{K}(t+\Delta t) - V_K(t) = \sum_{i\notin\mathcal{K}} \lambda_i P_{i,\mathcal{K}} (1 - q^{V_\mathcal{K} - V_{\mathcal{K}\cup\{i\}}}) / \sum_{j\notin\mathcal{K}} \lambda_j.$$

Using $z_{i,\mathcal{K}} = \lambda_i P_{i,\mathcal{K}}$ and equation (4), we can now make an approximation that $\forall \mathcal{K} \subset \mathcal{N}$

$$\dot{V}_\mathcal{K} \approx \frac{V_\mathcal{K}(t+\Delta t) - V_\mathcal{K}(t)}{\Delta t} = \sum_{i\notin\mathcal{K}} z_{i,\mathcal{K}} (1 - q^{V_\mathcal{K} - V_{\mathcal{K}\cup\{i\}}}), \tag{7}$$

which is accurate under the fluid model assumption where Δt is usually much smaller than the total transmission time and forms the basis for the DEDI framework. Note that (7) is the rank evolution equation for an arbitrary set $\mathcal{K} \subset \mathcal{N}$ and there are $2^N - 1$ such DEs that completely describe the rank evolution of the system. Assuming node 1 is the unique source with m packets to deliver, the boundary conditions (B.C.) for this systems of DEs are

$$V_\mathcal{K}(0) = \begin{cases} m, & 1 \in \mathcal{K}, \\ 0, & \text{o.w.} \end{cases} \tag{8}$$

Equation (7) is defined for every nonempty $\mathcal{K} \subset \mathcal{N}$ and it turns out the $V_\mathcal{K}$'s are interdependent in a layered structure. Specifically, to determine $V_\mathcal{K}(t)$, one only needs to know $V_{\{i\}\cup\mathcal{K}}(t)$ for every $i \notin \mathcal{K}$ and this hierarchical structure can be used to prove general theorems on RNC by induction. For example, the layered structure of a 3-node network is shown in Fig. 1 where a quantity depends solely on quantities in the immediate upper layer indicated by arrows.

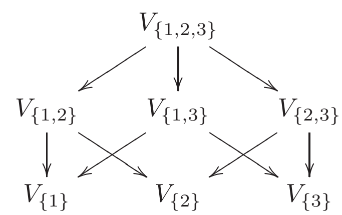

Fig. 1. Layered structure for the rank evolution of a 3-node network

Practical choices of q (usually an integral power of 2) allows a further simplication of the system of DEs. Specifically, we may approximate $1 - q^{V_\mathcal{K} - V_{\mathcal{K}\cup\{i\}}}$ by

$$1 - q^{V_\mathcal{K} - V_{\mathcal{K}\cup\{i\}}} \approx \begin{cases} 1, & V_\mathcal{K} < V_{\mathcal{K}\cup\{i\}}, \\ 0, & V_\mathcal{K} = V_{\mathcal{K}\cup\{i\}}, \end{cases} \tag{9}$$

where the approximation gets better with increasing q.

Consequently we may rewrite (7) as

$$\dot{V}_\mathcal{K} \approx \sum_{i\notin\mathcal{K}} z_{i,\mathcal{K}} (V_{\mathcal{K}\cup\{i\}} \ominus V_\mathcal{K}), \quad \forall \mathcal{K} \subset \mathcal{N} \tag{10}$$

with the same boundary conditions as in (8), where we define the binary operation $\ominus$ as (see Fig. 3(a))

$$x \ominus y = \begin{cases} 1, & x > y, \\ 0, & \text{o.w.} \end{cases} \tag{11}$$

We now illustrate the power of the DEDI framework by considering an example of RNC in the wireless P2P network shown in Fig. 2 where the source node 1 has 100 packets to transmit to nodes 2, 3, and 4. Suppose each node transmits

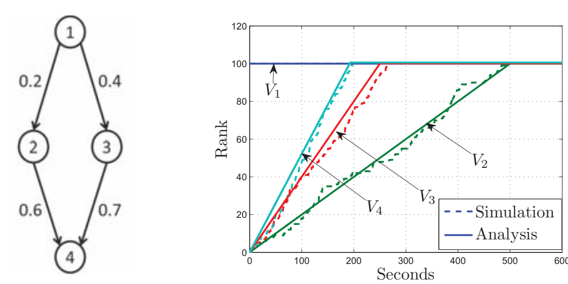

Fig. 2. Simulation and DE based solution for a 4-node wireless P2P network

at 1 packet/second uniformly, but node 1's transmission is successfully received by node 2 and node 3 with probability 0.2 and 0.4, respectively. The transmission of node 2 and node 3 are successfully received by node 4 with probability 0.6 and 0.7, respectively. Fig. 2 shows a good match between the simulated rank increase at each node as well as the rank evolution by solving the corresponding system of DEs.

B. Rank Evolution Modeled with DI

While we have shown the power of the DE approach, one of the challenges that needs to be addressed is the possibly discontinuous right-hand sides (due to the $\ominus$ operation) that make the system of DEs in (10) difficult to handle. Consider Fig. 3(a) where $x \ominus y$ is shown as a function of $x-y$, Fig 3(b) shows an approximation to it using the upper semicontinuous function $\mathrm{Sgn}^+ : \mathbb{R} \to 2^{\mathbb{R}}$ defined as

$$\mathrm{Sgn}^+(x) = \begin{cases} \{0\}, & x < 0 \\ [0,1], & x = 0 \\ \{1\}, & x > 0 \end{cases}. \tag{12}$$

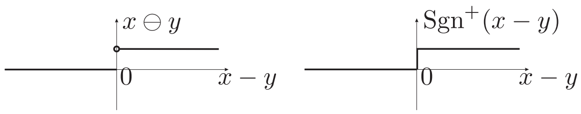

Fig. 3. (a) Plot of $x \ominus y$ as a function of $x-y$. (b) Plot of Sgn^+ as a set-valued function of $x-y$.

We may rewrite (10) in a new form

$$\dot{V}_{\mathcal{K}} \in \sum_{i \notin \mathcal{K}} z_{i,\mathcal{K}} \mathrm{Sgn}^+(V_{\mathcal{K}\cup\{i\}} - V_{\mathcal{K}}), \quad \forall \mathcal{K} \subset \mathcal{N} \tag{13}$$

with the same boundary conditions shown in (8). To be compatible with (10), when $\mathcal{K} = \mathcal{N}$, we define the right-hand side of (13) to be $\{0\}$ instead of $\emptyset$. In mathematical literature, (13) plus (8) is called a system of differential inclusions (DI) [11]. Any solutions to (10) are necessarily solutions to (13).

The generalization from (10) to (13) allows an interpretation of the solution to (10) without any discrepancy, as illustrated by the next example. Suppose we use RNC in the network

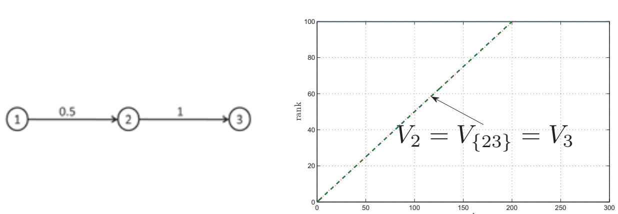

Fig. 4. A 3-node network where node 1 tries to deliver 100 packets to node 2 and node 3

shown in Fig. 4 to deliver 100 packets from node 1 to node 2 and 3. Each node transmits at 1 packet/second uniformly. Suppose $P_{1,2} = 0.5$ and $P_{2,3} = 1$, then $z_{12} = z_{1,\{23\}} = 0.5$, $z_{23} = 1$. We wish to know at what rates $V_2(t)$ and $V_3(t)$ increase by solving the corresponding DEs as given in (10):

$$\dot{V}_2 = z_{12}(m \ominus V_2), \tag{14}$$

$$\dot{V}_3 = z_{23}(V_{\{23\}} \ominus V_3), \tag{15}$$

$$\dot{V}_{\{23\}} = z_{1,\{23\}}(m \ominus V_{\{23\}}), \tag{16}$$

with boundary conditions $V_2 = V_3 = V_{\{23\}} = 0$, for which the solutions obtained by a numerical DE solver are shown in Fig. 4. In fact, Fig. 4 shows $V_2(t) = V_{\{23\}}(t) = V_3(t)$, $\forall t \geq 0$ and $\dot{V}_2(t) = \dot{V}_{\{23\}}(t) = \dot{V}_3(t) = 0.5$, $\forall t \in [0, 200)$. However, if we plug the solution back into (15), we get $\dot{V}_3(t) = 0$, $\forall t \in [0, 200)$. This discrepancy arises due to the discontinuous right-hand sides of the system of DEs in (14)–(16). If we recast (14)–(16) into DIs as follows.

$$\dot{V}_2 \in z_{12}\mathrm{Sgn}^+(m - V_2) = 0.5\mathrm{Sgn}^+(100 - V_2),$$

$$\dot{V}_3 \in z_{23}\mathrm{Sgn}^+(V_{\{23\}} - V_3) = \mathrm{Sgn}^+(V_{\{23\}} - V_3),$$

$$\dot{V}_{\{23\}} \in z_{1,\{23\}}\mathrm{Sgn}^+(m - V_{\{23\}}) = 0.5\mathrm{Sgn}^+(100 - V_{\{23\}}),$$

with the same boundary conditions, it is trivial to see that $V_2(t) = V_{\{23\}}(t) = V_3(t) = 0.5t$ is a solution for the system of DIs for $t \in [0, 200)$. Thus, as illustrated in this example, the DIs offer a way to handle the anomalies that arise in DEs due to discontinuities.

IV. DEDI Framework as an Analytical Tool

We will use the system of DIs in (13) to give an alternate and simpler proof to a well known result on RNC which states that a multicast session at rate R exists in an arbitrary and possibly lossy wireless network if and only if a unicast session at rate R exists to each destination separately. This statement was proved in [1] for deterministic network coding in a wired network and in [8] for RNC in lossy wireless networks.

Note that the highest unicast rate is known to be determined by the size of the min cut [12], [13] that separates the source and the destination. If a unicast at rate R exists for each destination d, we must have $c_{\min}(1, d) \geq R$. Using the DEDI framework we show a slightly stronger statement that also provides insights into the operation of RNC:

Theorem 1: In a wireless network $G = (\mathcal{N}, \mathcal{E})$ where node 1 is the source node of a multicast session trying to deliver m packets to some destinations and each node carries out RNC, the solution to the associated system of DIs (13) is given as

1) $\forall \mathcal{K} \subset \mathcal{N}$ and $1 \in \mathcal{K}$, $V_{\mathcal{K}}(t) = m, \forall t \in [0, \infty)$;
2) $\forall \mathcal{K} \subset \mathcal{N}$ and $1 \notin \mathcal{K}$,

$$V_{\mathcal{K}}(t) = \begin{cases} c_{\min}(1,\mathcal{K})t, & \forall t \in [0, m/c_{\min}(1,\mathcal{K})), \\ m, & \forall t \in [m/c_{\min}(1,\mathcal{K}), \infty). \end{cases} \tag{17}$$

Theorem 1 (see Appendix for proof) states that the rank of K increases until it reaches m at the rate allowed by the min cut that separates K from the source.

Corollary 1: For $1 \notin \mathcal{K}$, $\dot{V}_{\mathcal{K}} = c_{\min}(1, \mathcal{K})$, when $V_{\mathcal{K}} < m$.

Specializing Corollary 1 to an arbitrary destination node d, we obtain the same result shown in [8]:

Corollary 2: For $d \neq 1$, $\dot{V}_d = c_{\min}(1, \{d\})$, when $V_d < m$.

Corollary 2 shows that if a unicast at rate R exists for each destination d separately, i.e., $c_{\min}(1,d) \geq R$, then the proposed coding scheme is sufficient to implement a multicast at rate R. Theorem 1 is a little more general than the statements made in [8] and [14] since it reveals that, not only the rank at a single node, but also the rank at any subset $\mathcal{K} \subset \mathcal{N}$ increases at its min cut size $c_{\min}(1,\mathcal{K})$.

It should be pointed out that in the proof to Theorem 1, typical difficulties with cycles in the network topology do not arise due to the layered structure of the DIs that includes all topological information.

V. Concluding Remarks

We presented a framework called DEDI, based on differential equations and/or differential inclusions, which allows prediction of the rank/state evolution in an arbitrary wireless network where RNC is employed. We gave numerical examples and an alternate proof to a well known result on RNC with a single multicast to demonstrate the capability of DEDI – results on multiple multicast sessions will be presented in an upcoming submission [15]. We believe that the DEDI framework has wide ranging applications from studying network dynamics to cross-layer design to nonlinear and hybrid network coding schemes. Further, numerical DE solvers allow network practitioners to follow the dynamics of network coding, thereby impacting real network design.

Appendix
Proof of Theorem 1

We begin with the following lemma:

Lemma 1: In a hypergraph $G=(\mathcal{N},\mathcal{E})$, if $1 \notin \mathcal{K} \subset \mathcal{N}$ and $\mathcal{T} \supset \mathcal{K}$ is a min cut for $(1,\mathcal{K})$, then

a) $\forall i \in \mathcal{T} \setminus \mathcal{K}$, $\mathcal{T}$ is a min cut for $(1,\{i\}\cup\mathcal{K})$;
b) $\forall j \in \mathcal{T}^c \setminus \{1\}$, $c_{\min}(1,\{j\}\cup\mathcal{K}) \geq c_{\min}(1,\mathcal{K})$.

Proof:

a) If it is not true, since $\mathcal{T} \in C(1,\{i\}\cup\mathcal{K})$, we must have $c_{\min}(1,\{i\}\cup\mathcal{K}) < c_{\min}(1,\mathcal{K})$. But the min cut for $(1,\{i\}\cup\mathcal{K})$ is also a cut for $(1,\mathcal{K})$, so $c_{\min}(1,\mathcal{K}) \leq c_{\min}(1,\{i\}\cup\mathcal{K})$, a contradiction.
b) Since a min cut for $(1,\{j\}\cup\mathcal{K})$ is also a cut for $(1,\mathcal{K})$, the conclusion follows.

■

Proof of Theorem 1: The proof to Theorem 1 is by induction on $|\mathcal{N}\setminus\mathcal{K}|$, i.e., the cardinality of $\mathcal{N}\setminus\mathcal{K}$. Recall that a solution $V_{\mathcal{K}}(t)$ to (13) is a continuous function for each nonempty subset $\mathcal{K} \subset \mathcal{N}$. From the right-hand side of (13) and the definition of Sgn^+ in (12), we know $V_{\mathcal{K}}(t)$ is increasing, which implies that $V_{\mathcal{K}}(t)$ has bounded variation and differentiable *a.e.* Therefore, for arbitrary t_1, t_2, we have

$$V_{\mathcal{K}}(t_2) = V_{\mathcal{K}}(t_1) + \int_{t_1}^{t_2} \dot{V}_{\mathcal{K}} \mathrm{d}t. \tag{18}$$

1) Case $1 \in \mathcal{K}$, *base step* $|\mathcal{N}\setminus\mathcal{K}|=0$: Since $\mathcal{K}=\mathcal{N}$, it follows from (8) and our remark after (13) that

$$\dot{V}_{\mathcal{K}} \in \{0\}, \quad \text{B.C.} \quad V_{\mathcal{K}}(0)=m. \tag{19}$$

Clearly the solution is $V_{\mathcal{K}}(t)=m, t\geq 0$.

2) Case $1 \in \mathcal{K}$, *induction step* $|\mathcal{N}\setminus\mathcal{K}|=k-1$ *to* $|\mathcal{N}\setminus\mathcal{K}|=k$: From (13) and (8), we have

$$\dot{V}_{\mathcal{K}} \in \sum_{i\notin\mathcal{K}} z_{i,\mathcal{K}} \mathrm{Sgn}^+(V_{\{i\}\cup\mathcal{K}} - V_{\mathcal{K}}), \text{ B.C. } V_{\mathcal{K}}(0)=m. \tag{20}$$

Since $1 \in \{i\}\cup\mathcal{K}$ and $|\mathcal{N}\setminus(\{i\}\cup\mathcal{K})| = k-1$, by induction hypothesis, (20) can be rewritten as

$$\dot{V}_{\mathcal{K}} \in \sum_{i\notin\mathcal{K}} z_{i,\mathcal{K}} \mathrm{Sgn}^+(m - V_{\mathcal{K}}). \tag{21}$$

Suppose $\exists t_2 > 0$ such that $V_{\mathcal{K}}(t_2) \neq m$. Since $V_{\mathcal{K}}(0)=m$, by monotonicity, we have $V_{\mathcal{K}}(t_2) > m$. Let

$$t_1 = \sup\{t\geq 0 | V(\mathcal{K}) = m\}, \tag{22}$$

then t_1 exists because the set on the right hand side of (22) is nonempty and $t_1 < t_2$. By continuity of $V_{\mathcal{K}}$, $V_{\mathcal{K}}(t_1)=m$. By definition of t_1, $V_{\mathcal{K}}(t) > m, \forall t \in (t_1,t_2]$. So (21) reduces to,

$$\dot{V}_{\mathcal{K}}(t) \in \{0\}, \quad \forall t \in (t_1,t_2], \tag{23}$$

which is equivalent to $\dot{V}_{\mathcal{K}}(t) = 0, \forall t \in (t_1,t_2]$. With $V_{\mathcal{K}}(t_1)=m$, we have $V_{\mathcal{K}}(t_2)=m$, a contradiction.

3) Case $1 \notin \mathcal{K}$, *base step* $|\mathcal{N}\setminus\mathcal{K}|=1$: From (13) and the case we have proved for $1 \in \mathcal{K}$, we have

$$\begin{aligned}\dot{V}_{\mathcal{K}} &\in z_{1,\mathcal{K}} \mathrm{Sgn}^+(V_{\mathcal{N}} - V_{\mathcal{K}}) \\ &= z_{1,\mathcal{K}} \mathrm{Sgn}^+(m - V_{\mathcal{K}}) = c_{\min}(1,\mathcal{K}) \cdot \{1\}\end{aligned} \tag{24}$$

when $V_{\mathcal{K}}(t) < m$ (Recall $V_{\mathcal{K}}(t)$ is a continuous increasing function starting from $V_{\mathcal{K}}(0)=0$ according to (8)), which is equivalent to $\dot{V}_{\mathcal{K}} = c_{\min}(1,\mathcal{K})$. Therefore

$$V_{\mathcal{K}}(t) = c_{\min}(1,\mathcal{K})t, \quad \forall t \in [0, m/c_{\min}(1,\mathcal{K})). \tag{25}$$

By continuity, $V_{\mathcal{K}}(m/c_{\min}(1,\mathcal{K})) = m$. By Monotonicity, $V_{\mathcal{K}}(t) \geq m$ for $t \geq m/c_{\min}(1,\mathcal{K})$. To show $V_{\mathcal{K}}(t)=m, \forall t \in [m/c_{\min}(1,\mathcal{K}),\infty)$, argue by contradiction. Suppose $\exists t_2 > m/c_{\min}(1,\mathcal{K})$ such that $V_{\mathcal{K}}(t_2) \neq m$, then we must have $V_{\mathcal{K}}(t_2) > m$. Let

$$t_1 = \sup_{t\geq m/c_{\min}(1,\mathcal{K})} \{V_{\mathcal{K}}(t) \leq m\}, \tag{26}$$

then $t_1 < t_2$ and $V_{\mathcal{K}}(t_1) = m/c_{\min}(1,\mathcal{K})$ by continuity. By definition of t_1, $V_{\mathcal{K}}(t) > m, \forall t \in (t_1,t_2]$. By induction hypothesis, $V_{\{i\}\cup\mathcal{K}} \leq m$. So (13) reduces to $\dot{V}_{\mathcal{K}} \in \{0\}, \forall t \in (t_1,t_2]$. It then follows from (18) that $V_{\mathcal{K}}(t_2)=m$, a contradiction.

4) Case $1 \notin \mathcal{K}$, *induction step* $|\mathcal{N}\setminus\mathcal{K}|=k-1$ *to* $|\mathcal{N}\setminus\mathcal{K}|=k$ *for* $t \in [0, m/c_{\min}(1,\mathcal{K}))$: Without loss of generality, assume

$$\mathcal{T}^c = \{1,2,\ldots,\ell\}, \; \mathcal{T} = \{\ell+1,\ell+2,\ldots,k\}\cup\mathcal{K}. \tag{27}$$

Then $c_{\min}(1,\mathcal{K}) = \sum_{i\in\mathcal{T}^c} z_{i,\mathcal{T}}$. To show $V_{\mathcal{K}}(t) = c_{\min}(1,\mathcal{K})t$, $\forall t \in [0, m/c_{\min}(1,\mathcal{K}))$, argue by contradiction. Suppose this is not true, then $\exists t_2 \in (0, m/c_{\min}(1,\mathcal{K}))$ such that $V_{\mathcal{K}}(t_2) \neq c_{\min}(1,\mathcal{K})t_2$. Let

$$t_1 = \sup\{0 \leq t < t_2 | V_{\mathcal{K}}(t) = c_{\min}(1,\mathcal{K})t\}. \tag{28}$$

Because the set on the right-hand side of (28) is not empty and upper bounded by t_2, t_1 exists and

$$V_{\mathcal{K}}(t_1) = c_{\min}(1,\mathcal{K})t_1 \tag{29}$$

by continuity. There are two possibilities:
$V_{\mathcal{K}}(t_2) > c_{\min}(1,\mathcal{K})t_2$: Then

$$V_{\mathcal{K}}(t) > c_{\min}(1,\mathcal{K})t, \forall t \in (t_1, t_2]. \tag{30}$$

Otherwise, since $V_{\mathcal{K}}(t_2) > c_{\min}(1,\mathcal{K})t_2$, there is $t_3 \in (t_1, t_2)$ such that $V_{\mathcal{K}}(t_3) = c_{\min}(1,\mathcal{K})t_3$, contradicting (28). By Lemma 1(a), $\forall j = \ell+1,\ldots,k$, $c_{\min}(1,\{j\}\cup\mathcal{K}) = c_{\min}(1,\mathcal{K})$ and by by induction hypothesis

$$V_{\{j\}\cup\mathcal{K}}(t) = c_{\min}(1,\mathcal{K})t,\ \forall t \in [0, m/c_{\min}(1,\mathcal{K})). \tag{31}$$

Inserting (30), (31) into (13), we get $\forall t \in (t_1, t_2]$,

$$\dot{V}_{\mathcal{K}} = \sum_{i=1}^{\ell} z_{i,\mathcal{K}} \mathrm{Sgn}^+(V_{\{i\}\cup\mathcal{K}} - V_{\mathcal{K}}) \tag{32}$$

$$\leq \sum_{i=1}^{\ell} z_{i,\mathcal{K}} \leq \sum_{i=1}^{\ell} z_{i,\mathcal{T}} = c_{\min}(1,\mathcal{K}), \tag{33}$$

where the last inequality follows from (2). Inserting (29) and (33) into (18), we get $V_{\mathcal{K}}(t_2) \leq c_{\min}(1,\mathcal{K})t_2$, a contradiction.
$V_{\mathcal{K}}(t_2) < c_{\min}(1,\mathcal{K})t$: Arguing similarly with above, we get

$$V_{\mathcal{K}}(t) < c_{\min}(1,\mathcal{K})t, \forall t \in (t_1, t_2]. \tag{34}$$

Inserting (34), (31) into (13), we get $\forall t \in (t_1, t_2]$,

$$\dot{V}_{\mathcal{K}}(t) = \sum_{i=1}^{\ell} z_{i,\mathcal{K}} \mathrm{Sgn}^+(V_{\{i\}\cup\mathcal{K}} - V_{\mathcal{K}}) + \sum_{j=\ell+1}^{k} z_{j,\mathcal{K}}. \tag{35}$$

We claim

$$V_{\{i\}\cup\mathcal{K}}(t) > V_{\mathcal{K}}(t), \quad \forall t \in (t_1, t_2]. \tag{36}$$

In fact, by induction hypothesis, for any t, either $V_{\{i\}\cup\mathcal{K}}(t) = c_{\min}(1,\{i\}\cup\mathcal{K})t$ or $V_{\{i\}\cup\mathcal{K}}(t) = m$. If $V_{\{i\}\cup\mathcal{K}}(t) = m$, we certainly have that $\forall i \in \mathcal{T}^c, \forall t \in (t_1, t_2]$,

$$V_{\{i\}\cup\mathcal{K}}(t) > c_{\min}(1,\mathcal{K})t > V_{\mathcal{K}}(t);$$

if $V_{\{i\}\cup\mathcal{K}}(t) = c_{\min}(1,\{i\}\cup\mathcal{K})t$, we have that for $\forall i \in \mathcal{T}\setminus\{1\}$, $\forall t \in (t_1, t_2]$,

$$V_{\{i\}\cup\mathcal{K}}(t) = c_{\min}(1,\{i\}\cup\mathcal{K})t \geq c_{\min}(1,\mathcal{K})t > V_{\mathcal{K}}(t),$$

where the first inequality follows from Lemma 1(b).
Inserting (36) into (35), we get

$$\dot{V}_{\mathcal{K}}(t) = \sum_{i=1}^{\ell} z_{i,\mathcal{K}} + \sum_{j=\ell+1}^{k} z_{j,\mathcal{K}} = c_{\min}(1,\mathcal{K}). \tag{37}$$

Inserting (29), (37) into (18), we get $V_{\mathcal{K}}(t_2) = c_{\min}(1,\mathcal{K})t_2$, a contradiction.

5) *Case $1 \notin \mathcal{K}$, induction step $|\mathcal{N}\setminus\mathcal{K}| = k-1$ to $|\mathcal{N}\setminus\mathcal{K}| = k$ for $t \in [m/c_{\min}(1,\mathcal{K}),\infty)$*: We already know, by continuity, $V_{\mathcal{K}}(m/c_{\min}(1,\mathcal{K})) = m$ and, by monotonicity, $V_{\mathcal{K}}(t) \geq m$ for $t \geq m/c_{\min}(1,\mathcal{K})$. We only need to show that $\nexists t_2 > m/c_{\min}(1,\mathcal{K})$ such that $V_{\mathcal{K}}(t_2) > m$. Suppose this is not true, let

$$t_1 = \sup\{t \geq m/c_{\min}(1,\mathcal{K}) | V_{\mathcal{K}}(t) = m\} \tag{38}$$

since the set ont the right-hand side of (38) is nonempty and upper bounded by t_2, t_1 exists. By continuity,

$$V_{\mathcal{K}}(t_1) = m. \tag{39}$$

By monotonicity, $V_{\mathcal{K}}(t) > m, \forall t \in (t_1, t_2]$. By induction hypothesis,

$$V_{\{i\}\cup\mathcal{K}}(t) \leq m, \quad \forall i \notin \mathcal{K}. \tag{40}$$

Inserting (40) into (13), we get

$$\dot{V}_{\mathcal{K}} \in \{0\}, \quad \forall t \in (t_1, t_2]. \tag{41}$$

Inserting (39), (41) into (18), we get $V_{\mathcal{K}}(t_2) = m$, a contradiction.

■

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