Spectral Characteristics of Digitally Modulated Signals

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Abstract: This lecture first introduces the standard representation of complex baseband signal of digital modulated signal and its power spectral density (PSD). Then it shows how the PSD of bandpass signals can always be obtained by its complex baseband envolope. Finally, two important modulation schemes OFDM and PSK are introduced and the spectral characteristics of these two modulated signals are discussed.

I. PSD of Modulated Bandpass Signal

As we know that bandwidth efficiency is a very important factor for choosing a modulation scheme. To learn the bandwidth efficiency of one modulation scheme, we need to get the power spectral density of the corresponding modulated signal.

Generally, a digitally modulated signal can be written as

$$s(t) = \mathrm{Re}\{v(t)e^{j(2\pi f_c t+\phi_T)}\}$$

,where *v(t)* is the baseband equivalent complex envolope and ϕ_T is the random phase.

To calculate the PSD of *s(t)*, we can rewrite it as:

$$s(t) = \frac{1}{2}\{v(t)e^{j(2\pi f_c t+\phi_T)} + v^*(t)e^{-j(2\pi f_c t+\phi_T)}\}$$

Then the PSD of s(t) can be derived by:

$$\phi_{ss}(\tau) = E[s(t+\tau)s(t)]$$

$$= \frac{1}{4}E[v(t+\tau)v(t)e^{j(4\pi f_c t+2\pi f_c\tau+2\phi_T)}$$

$$+ v^*(t+\tau)v(t)e^{-j2\pi f_c \pi}$$

$$+ v(t+\tau)v^*(t)e^{j2\pi f_c \pi}$$

$$+ v^*(t+\tau)v^*(t)e^{j(4\pi f_c t+2\pi f_c\tau+2\phi_T)}]$$

Observe that E[$e^{\pm j(4\pi f_c t+2\pi f_c\tau+2\phi_T)}$] = 0

We got

$$\phi_{ss}(\tau) = \frac{1}{4}\phi^*_{vv}(\tau)e^{-j2\pi f_c\tau} + \frac{1}{4}\phi_{vv}(\tau)e^{j2\pi f_c\tau}$$

As, $\phi_{vv}(\tau) = \phi^*_{vv}(-\tau)$, we got,

$$s_{ss}(f) = \frac{1}{4}[s_{vv}(f-f_c) + s_{vv}^*(-f-f_c)]$$

As $s_{vv}()$ is real and even, we got

$$s_{ss}(f) = \frac{1}{4}[s_{vv}(f-f_c) + s_{vv}(f+f_c)]$$

So, the PSD of bandpass signal *s(t)* is completely determined by the PSD of its complex baseband envelope *v(t).* As a consequence, the results of performance of various modulation and demodulation techniques are independent of carrier frequencies and channel frequency bands.

II. Standard Representation of Complex Baseband Signal

1. Standard Format:

We now move on to study the **s**tandard representation of complex baseband signal of a digitally modulated signal. Generally, the baseband signal *v(t)* can be written as:

$$v(t) = A\sum_k b(t-kT,\underline{x_k})$$

,where A is the carrier amplitude, $\underline{x_k} = (x_k, x_{k-1}, x_{k-2}, \ldots, x_{k-K})$ is the source symbol sequence, K is the memory length, which depends on modulation scheme, T is the symbol duration, and $b(t-kT,\underline{x_k})$ is an equivalent shaping function of duration T.

This is the standard representation of the complex baseband signal.

2. An Example of Amplitude-Shift Keying (ASK) Modulation

For ASK modulation, we have

$$v(t) = A\sum_n x_n h_a(t-nT)$$

,where $\{x_n\} = \{x_n{}^I + jx_n{}^Q\}$ is the source symbol sequence, $h_a(t)$ is the amplitude shaping pulse (square wave or something else)

Here, k=0 which means no memory. And the equivalent shaping function:

$$b(t,\underline{x_k}) = x_k h_a(t)$$

The bandpass signal s(t) can be written as:

$$s(t) = A\sum_n \{|x_n| h_a(t-nT)\cos[2\pi f_c t + \arg(x_n)]\}$$

,where $|x_n| = \sqrt{(x_n^I)^2 + (x_n^Q)^2}$

and $\arg(x_n) = \tan^{-1}(\frac{x_n^Q}{x_n^I})$

III. PSD of Baseband Signal

As we have given the standard representation of the baseband complex envelope of the modulated bandpass signal. We can use it to get the PSD of the baseband signal .

Recall: $v(t) = A\sum_k b(t-kT,\underline{x_k})$

Then its autocorrelation is given by:

$$\phi_{vv}(t+\tau,t) = \frac{1}{2}E[v(t+\tau)v^*(t)]$$

$$= \frac{A^2}{2}\sum_i\sum_k E[b(t+\tau-iT,\underline{x_i})b^*(t-kT,\underline{x_k})]$$

Claim: *v(t)* is a cyclostantionary process, i.e. $\phi_{vv}(t+\tau,t)$ is periodic in t with period T

Proof:

$$\phi_{vv}(t+T+\tau,t+T) = \frac{1}{2}E[v(t+T+\tau)v^*(t+T)]$$

$$= \frac{A^2}{2}\sum_i\sum_k E[b(t+T+\tau-iT,\underline{x_i})b^*(t+T-kT,\underline{x_k})]$$

Let *i'=i-1* and *k'=k-1*

$$\phi_{vv}(t+T+\tau,t+T)]$$

$$= \frac{A^2}{2}\sum_{i'}\sum_{k'} E[b(t+\tau-i'T,\underline{x_{i'+1}})b^*(t-k'T,\underline{x_{k'+1}})$$

Assume the source symbols be stationary, i.e. $\underline{x_i} = \underline{x_{i+1}}$, then, we have

$$\phi_{vv}(t+T+\tau,t+T)$$

$$= \frac{A^2}{2}\sum_{i'}\sum_{k'} E[b(t+\tau-i'T,\underline{x_{i'}})b^*(t-k'T,\underline{x_{k'}})]$$

$$= \phi_{vv}(t+\tau,t)$$

So, *v(t)* is cyclostationary. Thus, $\phi_{vv}(\tau)$ can be obtained by taking the time average of $\phi_{vv}(t+\tau,t)$ i.e. $\phi_{vv}(\tau) =$

$$\frac{A^2}{2}\sum_i\sum_k \frac{1}{T}\int_0^T E[b(t+\tau-iT,\underline{x_i})b^*(t-kT,\underline{x_k})dt]$$

Change variable: *t-kT=z*, we got $\phi_{vv}(\tau) =$

$$\frac{A^2}{2T}\sum_i\sum_k \int_{-kT}^{-kT+T} E[b(z+\tau-(i-k)T,\underline{x_i})b^*(z,\underline{x_k})dz$$

Let *i-k=m,* $\phi_{vv}(\tau) =$

$$\frac{A^2}{2T}\sum_{m+k}\sum_k \int_{-kT}^{-kT+T} E[b(z+\tau-mT,\underline{x_{m+k}})b^*(z,\underline{x_k})dz$$

$$= \frac{A^2}{2T}\sum_m \int_{-\infty}^{-\infty} E[b(z+\tau-mT,\underline{x_m})b^*(z,\underline{x_0})dz$$

So, its PSD is given by

$$S_{vv}(f) =$$

$$E[\frac{A^2}{2T}\sum_m \int_{-\infty}^{-\infty}\int_{-\infty}^{-\infty} b(z+\tau-mT,\underline{x_m})b^*(z,\underline{x_0})dz\cdot e^{-j2\pi f_c\tau}d\tau$$

$$= E[\frac{A^2}{2T}\sum_m \int_{-\infty}^{-\infty} b(z+\tau-mT,\underline{x_m})\cdot e^{-j2\pi f_c(z+\tau-mT)}d\tau]$$

$$\cdot\int_{-\infty}^{-\infty} [b^*(z,\underline{x_0})\cdot e^{j2\pi f_c z}dz\cdot e^{-j2\pi f_c mT}]$$

$$= \frac{A^2}{2T}\sum_m E[\int_{-\infty}^{-\infty} b(\tau', \underline{x_m}) \cdot e^{-j2\pi f_c \tau'} d\tau'$$

$$\cdot \int_{-\infty}^{-\infty} [b^*(z, \underline{x_0}) \cdot e^{j2\pi f_c z} dz \cdot e^{-j2\pi f_c mT}]$$

$$= \frac{A^2}{2T}\sum_m E[B(f, \underline{x_m}) B^*(f, \underline{x_0})] e^{-j2\pi f_c mT}$$

,where $B(f, \underline{x_m})$ is the Fourier transform of $b(t, \underline{x_m})$

Observations: The PSD of *v(t)* depends on:

a) The form of the equivalent pulse shaping fucntion $b(t, \underline{x_m})$.

b) The correlation properties of source sequence $\underline{x_m}$.

While the above gives a frequency domain representation or characteristics for a specific candidate *v(t)*. What about the representation of whole class of feasible waveforms?

Generally, *v(t)* during a symbol interval belong to a $\{v_l(t)\}_{l=1}^{M}$, represented in terms a set of orthonormal basis functions $\{\phi_n(t)\}_{n=1}^{N}$, *N<M*.

i.e. $v_i(t) = \sum_{m=1}^{N} v_{im}\phi_m(t)$, i=1…M.

,where $v_{im} = \int_0^T v_i(t)\phi_m(t)dt$

and $\int_0^T \phi_i(t)\phi_j(t)dt = \delta_{ij}$

For the ASK case, $s(t) = \mathrm{Re}\{v(t)e^{j2\pi f_c t}\}$.

A set of basis for the signal space is:

$$\phi_1(t) = \sqrt{\frac{2}{T} h_a(t)\cos(2\pi f_c t)} ,$$

$$\phi_2(t) = \sqrt{\frac{2}{T} h_a(t)\sin(2\pi f_c t)}$$

In terms of $\phi_1(t)$ and $\phi_2(t)$, we have

$$s_i(t) = \sqrt{E_A} x_i^I \phi_1(t) + \sqrt{E_A} x_i^Q \phi_2(t),$$

where $E_A = A^2T/2$ is the symbol energy in the waveform.

One popular ASK is M-QAM with source symbols as: $x_i = x_i^I + jx_i^Q$, where x_i^I , $x_i^Q \in \{\pm 1, \pm 3, \pm 5, ..., \pm(N-1)\}$ and $N = \sqrt{M}$.

For the case of 16-QAM, the signal constellation is shown in Fig 1

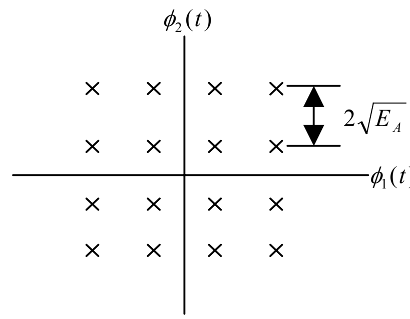

Fig 1 Signal Constellation of 16-QAM

The PSD of 16-QAM signal is:

$$S_{vv}(f) = \frac{A^2}{T} E[x^2] |H_a(f)|^2 ,$$

Where $E[x^2]$ is the variance of symbols, $H_a(f)$ is the Fourier transform of $h_a(t)$

If we choose $h_a(t)$ be raised cosine with rolloff factor $\beta = 0.5$, then Fig 2 shows the PSD of this 16QAM signal.

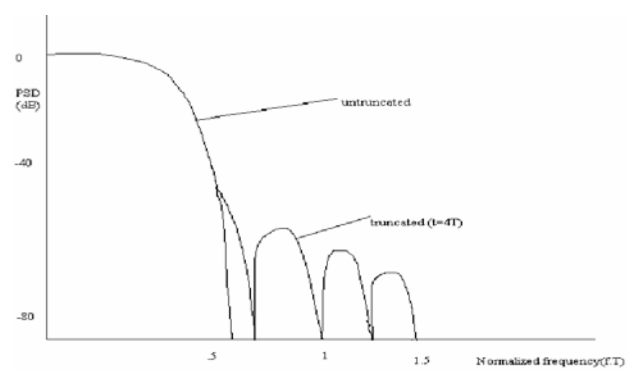

Fig 2 PSD of 16-QAM with $h_a(t)$ as a raised cosine function

IV. OFDM

1. What's OFDM

Orthogonal Frequency Division Modulation (OFDM) is a block modulation scheme designed to combat the effect of multipath frequency selective fading. In OFDM, a block of N signal source symbols, each of duration Ts, is combined into a block of N parallel modulated symbols each of with duration *T=NTs.* N is chosen such that: $NT_s >> \delta_\tau$ (the RMS delay spread). So, the channel will look like a flat fading channel and the need for equalization is avoid.

Also, as each source symbol in the block of length N is transmitted in parallel by employing N orthogonal subcarriers, the symbol rate on each subcarrier is much less than the serial source rate. As a result the effect of delay spread is reduced. These are the advantages of OFDM.

2. Representation of OFDM signal

The complex envelope of OFDM modulated signal is:

$$v(t) = A\sum_k \sum_{n=0}^{N-1} x_{k,n}\phi_n(t-kT),$$

,where $\{\phi_n(t)\}$ are orthogonal waveforms chosen as $\phi_n(t) = h_a(t)\{j\dfrac{2\pi(n-\frac{N-1}{2})t}{T}\}$, *n=0,1,...,N-1.* And $h_a(t) = u_T(t)$, which is a rectangle pulse. As the frequency separation is $\dfrac{1}{T}$, $\{\phi_n(t)\}$ are orthogonal.

We can see that at time instant k, N source symbols are transmitted using the N distinct subcarriers.

Usually, $x_{k,n}$ are chosen from QAM constellation. The standard format is then given by

$$v(t) = A\sum_k b(t-kT, \underline{x_k}),$$

where $b(t-kT, \underline{x_k})$

$$= h_a(t)\sum_{n=0}^{N-1} x_{k,n} \exp\{\frac{j2\pi(n-\frac{N-1}{2})t}{T}\}$$

, $\underline{x_k} = \{x_{k,0}, x_{k,1}, \ldots, x_{k,N-1}\}$

3. Why OFDM attractive?

OFDM modulation is attractive because it can be achieved by using either inverse discrete Fourier transform (IDFT) or inverse fast Fourier transform (IFFT).

Consider *k=0* and ignore the frequency effect term $\exp\{\dfrac{j2\pi(n-\frac{N-1}{2})t}{T}\}$. Also choose $h_a(t) = u_T(t)$, we have

$$v(t) = A\sum_{n=0}^{N-1} x_{0,n} \exp\{j\frac{2\pi nt}{T}\}, \quad 0 \le t \le T = NT_s$$

If we sample *v(t)* at *t=kTs,* we got

$$v(kT_s) = A\sum_{n=0}^{N-1} x_{0,n} \exp\{j\frac{2\pi nkT_s}{T}\}$$

$$= A\sum_{n=0}^{N-1} x_{0,n} \exp\{j\frac{2\pi nk}{N}\}$$

Denote $\{x_{0,k}\} = \{v(kT_s)\}$, *k=0, 1, ... , N-1*

Then $\{x_{0,k}\}$ are just the IFFT of the block $A\underline{x_0}$, where $\underline{x_0} = (x_{0,1}, x_{0,2}, \ldots, x_{0,N-1})$

So, the transmitter is easy to implement. Fig 3 shows the scheme of an OFDM transmitter.

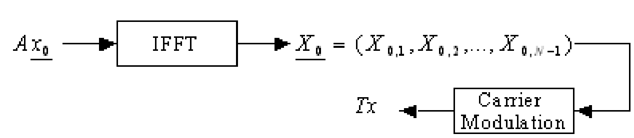

Fig 3 A OFDM transmitter

4. PSD of OFDM Signal

Assume the source symbols are zero mean and the amplitude shaping pulse is $h_a(t)$, then the PSD of *s(t)* is given by

$$S_{vv}(f)$$
$$= \frac{A^2}{T}\delta_x^2 \sum_{n=0}^{N-1} | H_a(f - \frac{1}{T}(n - \frac{N-1}{2})) |^2$$

, where $\delta_x^2 = \frac{1}{2}E[| x_{k,n} |^2]$

If $h_a(t) = u_T(t)$, then $H_a(f) = \sin c(f)$. Fig 4 shows the PSD of *s(t)* in this case:

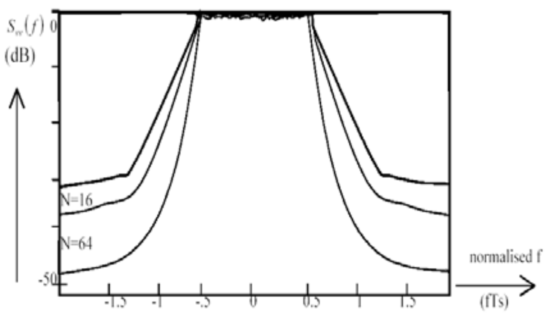

Fig 4 PSD of OFDM signal

We can see from the figure that the main lope carries more part of the energy as N increases. This implies that we can get better spectral efficiency as N increases.

V. Phase Shifting Keying (PSK)

1 What's PSK

PSK is perhaps the generic form of modulation most widely utilized in contemporary practice, ranging from voice-band modems to high-speed satellite transmission. As the name suggest, the signal set is generated by phase modulation of a sinusoidal carrier to one of M equispaced phase positions.

For *M-ary* PSK signal, the standard format of its complex baseband signal is given by

$$v(t) = A\sum_k b(t - kT, \underline{x_k}),$$

where

$$b(t - kT, \underline{x_k}) = h_a(t)\exp\{j\frac{\pi}{M}x_k h_p(t)\},$$

where $\underline{x_k} = x_k$ (no memory), $h_a(t)$ is the amplitude shaping pulse, $h_p(t)$ is the phase shaping pulse, M is the size of alphabet, $x_k = \{\pm 1, \pm 3, \ldots, \pm(M-1)\}$

Fig 5 shows the signal constellation of the 8PSK,

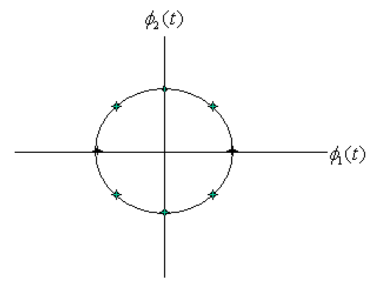

Fig 5 Constellation of 8PSK

Often we choose $h_p(t) = u_T(t)$ and $h_a(t) = u_T(t)$ or raised cosine pulse.

2. PSD of PSK signal

Let's assume uncorrelated source symbols and $h_p(t) = u_T(t)$, Also we assume source symbols are equal probable and defined by set:

$$x_k \in \{2i - 1 - M : i = 1,2,\ldots M\}$$

Then $b(t, \underline{x_k}) = h_a(t)\exp\{j\frac{\pi}{M}x_k h_p(t)\}$,

And $E[b(t, \underline{x_k})] = h_a(t)\sin f(\frac{\pi}{M}\ h_p(t))$

,where $\sin f \overset{\Delta}{=} \frac{\sin(Mx)}{M\sin x}$

The result followed from the following:

$$E[\exp\{j\frac{\pi}{M}x_k\alpha(t)\}] = \sin f(\frac{\pi}{M}\alpha(t))$$

Recall: $S_{vv}(f) =$

$$= \frac{A^2}{2T}\sum_m E[\int_{-\infty}^{-\infty} b(\tau', \underline{x_m}) \cdot e^{-j2\pi f_c \tau'} d\tau'$$
$$\cdot \int_{-\infty}^{-\infty} [b^*(z, \underline{x_0}) \cdot e^{j2\pi f_c z} dz \cdot e^{-j2\pi f_c mT}]$$

For uncorrelated source symbols, we have

$$S_{vv}(f) =$$

$$= \frac{A^2}{2T}\int_{-\infty}^{-\infty}\int_{-\infty}^{-\infty} E[b(\tau', \underline{x_0}) \cdot b^*(z, \underline{x_0}) \cdot e^{-j2\pi f_c(\tau'-z)} dz d\tau'$$

$$= \frac{A^2}{2T}\int_{-\infty}^{-\infty}\int_{-\infty}^{-\infty} e^{j\frac{\pi}{M}[h_p(\tau')-h_p(z)]x_0} h_a(\tau')h_a(z) dz d\tau'$$

$$= \frac{A^2}{2T}\int_{-\infty}^{-\infty}\int_{-\infty}^{-\infty} \frac{\sin\{M\frac{\pi}{M}[h_p(\tau')-h_p(z)]\}}{M\sin\{\frac{\pi}{M}[h_p(\tau')-h_p(z)]\}} h_a(\tau')h_a(z$$

As $\lim_{t->\infty} \frac{\sin \pi t}{M\sin(\frac{\pi}{M}t)} dt = 1$

We have $S_{vv}(f) =$

$$= \frac{A^2}{2T}\int_{-\infty}^{-\infty}\int_{-\infty}^{-\infty} h_a(\tau')h_a(z) \cdot e^{-j2\pi f_c(\tau'-z)} dz d\tau'$$

$$= \frac{A^2}{2T}|H_a(f)|^2$$

Choose $h_p(t) = u_T(t)$, then

$$S_{vv}(f) = \frac{A^2}{2T}[\frac{\sin(\pi f T)}{\pi f}]^2 = \frac{A^2 T}{2}[\frac{\sin(\pi f T)}{\pi f T}]^2$$

$$= E_0[\frac{\sin(\pi f T)}{\pi f}]^2$$

,where E_0 is the symbol energy and $T = (\log_2^M)T_b$.

For a fair comparison of bandwidth efficiency with different M, we substitute T with $(\log_2^M)T_b$ and get:

$$S_{vv}(f) = \frac{A^2(\log_2^M)T_b}{2}[\frac{\sin(\pi f(\log_2^M)T_b)}{\pi f(\log_2^M)T_b}]^2$$

Fig 6 shows the PSD of the PSK signals with different M. Here, the bandwidth efficiency is defined as $\eta_B = \frac{R_b}{B}$ and the bandwidth B here is defined as the "null-to-null" bandwidth.

From the figure, we can get table 1 which shows the bandwidth efficiency of PSK with different M.

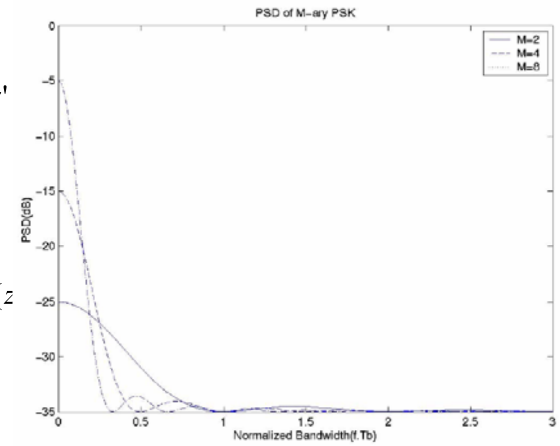

Fig 6 PSD of PSK with different M

Table 1 Bandwidth efficiency of PSK

M	2	4	8	16	32	64
η_B	0.5	1	1.5	2.0	2.5	3.0

From the table, we can see that the bandwidth efficiency η_B increases as M increases for *M-ary* PSK. However, its the power efficiency decreases as M increases due to the closer distant between different signals in the constellation.

VI. **Conclusion**

In this lecture, the PSD of modulated signals are discussed. First, we showed that the PSD of bandpass signal *s(t)* is completely determined by the PSD of its complex baseband envelope *v(t)*. Then two important modulation schemes, OFDM and PSK, are introduced. OFDM modulation is not only good to fight multi-path fading but also easy to implement, making it very attractive for high bit-rate wireless applications in a multipath radio environment. M-ary PSK is also widely used in contemporary practice, ranging from voice-band modems to high-speed satellite transmission.

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