Performance Analysis of Spread Spectrum CDMA systems

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I. Introduction

In this lecture, we present the mathematical analysis on the average bit error rate for multi-users in a single channel of the spread spectrum, Code Division Multiple Access (SS-CDMA) mobile radio system. We present expressions of BER performance of CDMA systems for a wide range of interference conditions for both synchronous case and asynchronous case, including Gaussian approximations (GA), Improved Gaussian Approximation (IGA), and Simple Improved Gaussian Approximation (SIGA). After the discussion for the single user detection, we discuss the key idea of the multiuser detection in the CDMA systems for high speed data transmission.

The rest of the paper is organized as follows: In section II, we present the multiuser CDMA system model. In section III, we discuss the single user detection for the CDMA systems. In section IV, we present in detail the probability of error of the synchronous case with the single user detection for the CDMA systems. In section V, we discuss the probability of error of the asynchronous case with the single user detection for the CDMA systems, including GA, IGA and SIGA algorithms. Finally, in section VI, we discuss the multiuser detection for the CDMA systems.

II. Multiuser CDMA System Model

In this report, we assume that the spread spectrum, Code Division Multiple Access (SS-CDMA) systems utilize BPSK modulation and the transmitted waveform from the j-th user is given by

$$s_j(t) = \sqrt{2P_j}c_j(t)b_j(t)\cos(\omega_c t + \theta_j), j = 1, ...K \tag{1}$$

where P_j is the transmission power of user j, θ_j is the carrier phase offset of user j relative to reference user 0, $b_j(t)$ is the data sequence for user j and $c_j(t)$ is the spreading or chip sequence for user j given by

$$c_j(t) = \Sigma_{n=-\infty}^{\infty} c_j^{(n)} p_{T_c}(t - nT_c) \tag{2}$$

$$b_j(t) = \Sigma_{n=-\infty}^{\infty} c_j^{(n)} p_T(t - nT) \tag{3}$$

where $c_j^{(n)}$ and $b_j^{(n)}$ are both $\in \{-1, +1\}$, T_c is the chip period of the pseudo noise (PN) sequence $c_j^{(n)}$, T is bit period that satisfy $T = NT_c$.

We assume AWGN channel for the transmission of K users and the system model is shown in Fig. (1). We can get the expression of the received signal at the BS as

$$r(t) = \Sigma_{j=1}^{K} \sqrt{2P_j} c_j(t - \tau_j) b_j(t - \tau_j) \cos(\omega_c t + \phi_j) + n(t) \tag{4}$$

where $\tau_j \in [0, T)$ is the delay of the j-th user to user 0, and $\phi_j \in [0, 2\pi)$ is the received carrier offset of user j given by

$$\phi_j = \theta_j - \omega_c \tau_j \tag{5}$$

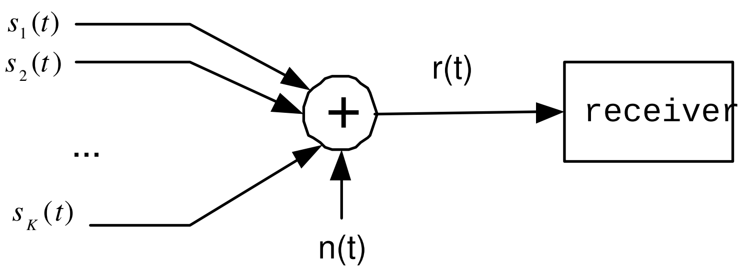

Fig. 1. System Model in our Analysis

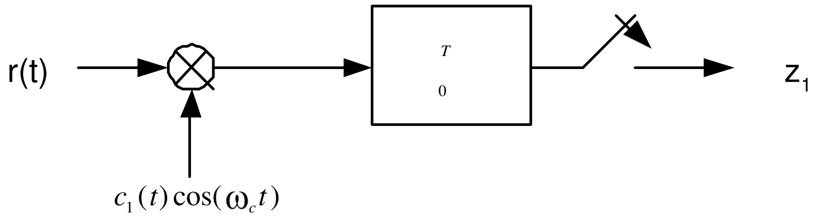

Fig. 2. MF receiver for single user detection

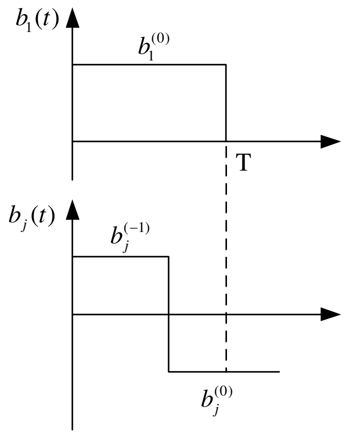

Fig. 3. For the asynchronous case.

III. Single User Detection

In this section, we study the single user detection and focus on user 1. Assume the receiver is synchronous to user 1, then without loss of generality, we can assume that $\tau_1 = 0, \rho_1 = 0$. The structure of the matched filter (MF) receiver that is optimal for single user environment is illustrated as the figure below, The output of the MF receiver z_1 shown in Fig. (2) is given by

$$z_1 = \int_0^T c_1(t)\cos(\omega_c t) r(t) dt$$
$$= \sum_{j=1}^{K} \sqrt{P_j/2} \int_0^T c_1(t) c_j(t-\tau_j) b_j(t-\tau_j) \cos(\phi_j) dt + n_1 \quad (6)$$
$$n_1 = \int_0^T n(t) c_1(t) \cos(\omega_c t) dt \quad (7)$$

By introducing the continues time partial cross correlation functions, we can simplify the expression of z_1 as

$$z_1 = \sqrt{\frac{P_1}{2}} T b_1^{(0)} + \sum_{j=2}^{K} \sqrt{\frac{P_j}{2}} [b_j^{(-1)} R_{j,1}(\tau_j) + b_j^{(0)} \hat{R}_{j,1}(\tau_j)] \cos(\phi_j) + n_1 \quad (8)$$

where the continues time partial cross correlation functions are

$$R_{j,1}(\tau) = \int_0^{\tau} c_j(t-\tau) c_1(t) dt, 0 \le \tau \le T \quad (9)$$
$$\hat{R}_{j,1}(\tau) = \int_{\tau}^{T} c_j(t-\tau) c_1(t) dt, 0 \le \tau \le T \quad (10)$$

IV. Synchronous case

For the case of synchronous users, we get $\phi_j = 0$ and $\tau_j = 0$ for any j. In this case, each interference contributes to only one term, i.e.,

$$z_1 = \sqrt{\frac{P_1}{2}} T b_1^{(0)} + \sum_{j=2}^{K} \sqrt{\frac{P_j}{2}} b_j^{(0)} R_{j,1}(\tau_j) + n_1 \quad (11)$$

where

$$R_{j,1} = \underline{c}_j^T c_1 \frac{T}{N} \quad (12)$$

where $n_1 \sim N(0, \frac{N_0 T}{4})$, $\underline{c}_j$ is the spreading code vector of length N for user j, and c_1 is the spreading code vector for user 1.

When users are orthogonal, i.e., $\underline{c}_j^T c_1 = \delta_{ij}$, we obtain

$$z_1 = \sqrt{\frac{P_1}{2}} T b_1^{(0)} + n_1 \quad (13)$$

Thus the bit error rate in this case is given by

$$P_{b,1} = Q(\sqrt{\frac{2E_b}{N_0}}) \quad (14)$$

We can see that it is the same as that of the single user case on AWGN channel. Because of the orthogonality restriction on c_j, it is possible only for $K \leq N$.

On the other hand, in the case where the orthogonality is not satisfied, say, users use random codes $\{\underline{c}_j\}_{j=1}^{K}$ that are random binary vectors, we can use the following two steps to calculate the probability of error,

Step $1.$: Fix codes and bits of other users, and then find conditioned probability of error $P_{b,1}(\{R_{j,1}\}, \{b_j^{(0)}\})$;

Step $2.$: Find the averaged probability of error $\bar{P}_{b,1}$ by averaging over $\{R_{j,1}\}$ and $\{b_j^{(0)}\}$.

In step 1, we have

$$\begin{aligned} P_{b,1}(\{R_{j,1}\}, \{b_j^{(0)}\}) &= Pr(z_1 < 0 | b_1^{(0)} = 1) \\ &= Q(\frac{\sqrt{\frac{P_1}{2}}T + \sum_{j=2}^{K}\sqrt{\frac{P_j}{2}}b_j^{(0)}R_{j,1}}{\sqrt{\frac{N_0 T}{4}}}) \\ &= Q(\sqrt{\frac{2E_b}{N_0}} + \sum_{j=2}^{K}\sqrt{\frac{2E_{b,j}}{N_0}}b_j^{(0)}\frac{R_{j,1}}{T}) \end{aligned} \tag{15}$$

And in step 2, we get

$$\bar{P}_{b,1} = E_{\{b_j^{(0)}\}\{R_{j,1}\}}[Q(\sqrt{\frac{2E_b}{N_0}} + \sum_{j=2}^{K}\sqrt{\frac{2E_{b,j}}{N_0}}b_j^{(0)}\frac{R_{j,1}}{T})] \tag{16}$$

We can see that the expression above is not convenient for analysis, unless some approximation is introduced.

A. Gaussian Approximation (GA)

Now we use Gaussian approximation to analyze the BER performance of the synchronous case. We assume equal power for each user, i.e., $P_j = P$ for all j, and a very large K. Let I_1 present the interferences from all the other users, i.e., $I_1 = \sum_{j=2}^{K}\sqrt{\frac{P}{2}}b_j^{(0)}R_{j,1}$. The main idea of this approach is to approximate I_1 as Gaussian variable, with mean and variance as

$$E[I_1] = \sum_{j=2}^{K}\sqrt{\frac{P}{2}}E[b_j^{(0)}]E[R_{j,1}] = 0 \tag{17}$$

$$Var[I_1] = \frac{P}{2}\sum_{j=2}^{K}E[b_j^{(0)}]^2 Var[R_{j,1}] \tag{18}$$

Consider $R_{j,1} = \underline{c}_j^T \underline{c}_1 \frac{T}{N}$, we can easily obtain

$$Var[R_{j,1}] = (\frac{T}{N})^2 N = \frac{T^2}{N} \tag{19}$$

$$Var[I_1] = \frac{P}{2} \frac{T^2}{N} (K-1) \tag{20}$$

We get a very useful approximation of $\bar{P}_{b,1}$ by

$$\bar{P}_{b,1} = Q(\frac{\sqrt{\frac{P}{2}}T}{\sqrt{\frac{N_0 T}{4} + \frac{PT^2}{2N}(K-1)}}) \tag{21}$$

$$\approx Q(\frac{\sqrt{\frac{P}{2}}T}{\sqrt{\frac{PT^2}{2N}(K-1)}}) = Q(\sqrt{\frac{N}{K-1}}) \tag{22}$$

when K is large.

V. Asynchronous case

For the case of asynchronous users, the output of the MF receiver of user 1 is given by

$$z_1 = \sqrt{\frac{P_1}{2}} T b_1^{(0)} + \sum_{j=2}^{K} \sqrt{\frac{P_j}{2}} T I_{j,1}(\underline{b}_j, \tau_j, \phi_j) + n_1 \tag{23}$$

where

$$I_{j,1}(\underline{b}_j, \tau_j, \phi_j) = \frac{1}{T}[b_j^{(-1)} R_{j,1}(\tau_j) + b_j^{(0)} R_j(\tau_j)] \cos(\phi_j) \tag{24}$$

$$\underline{b}_j = [b_j^{(-1)} b_j^{(0)}] \tag{25}$$

Assume for any j, $P_j = P$, then Eq.(23) can be simplified by

$$z_1 = \sqrt{\frac{P}{2}} T [b_1^{(0)} + I_1(\underline{b}, \underline{\tau}, \underline{\phi})] + n_1 \tag{26}$$

where

$$I_1(\underline{b}, \underline{\tau}, \underline{\phi}) = \sum_{j=2}^{K} I_{1,j}(\underline{b}_j, \tau_j, \phi_j) \tag{27}$$

$$\underline{b} = (\underline{b}_2, \underline{b}_3, ..., \underline{b}_K) \tag{28}$$

$$\underline{\tau} = (\underline{\tau}_2, ..., \underline{\tau}_K), \underline{\phi} = (\underline{\phi}_2, ..., \underline{\phi}_K) \tag{29}$$

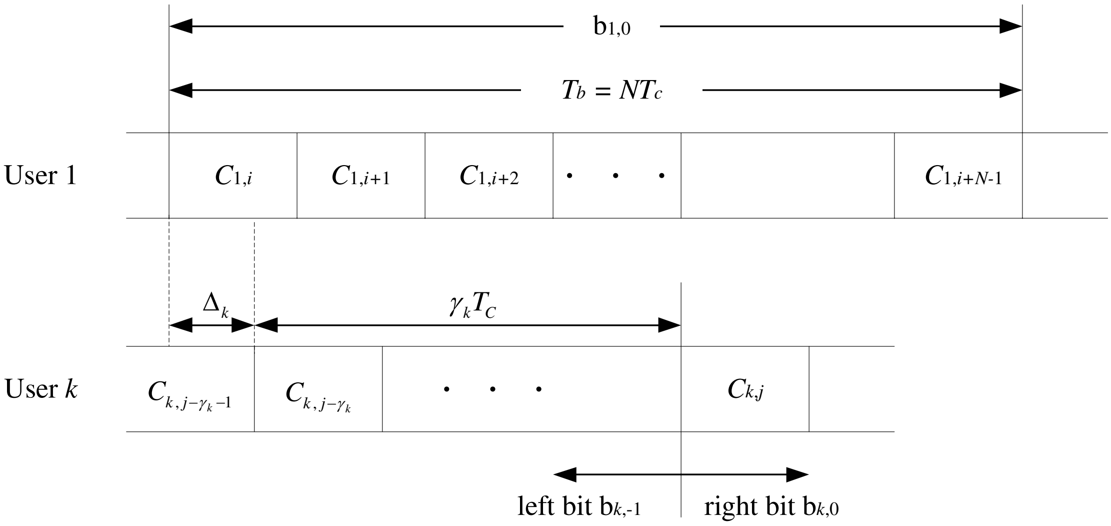

Fig. 4. Timing of PN sequence in the case of asynchronous

For fixed $\underline{b}, \underline{\tau}, \underline{\phi}$, we can get the conditioned probability of error when user 1 transmits $b_1^{(0)} = -1$,

$$\begin{aligned} P_{b,-1} &= Pr(z_1 > 0 | b_1^{(0)} = -1) \\ &= Pr\{\sqrt{\frac{P}{2}} T[-1 + I_1(\underline{b}, \underline{\tau}, \underline{\phi})] + n_1 > 0\} \\ &= Q(\sqrt{\frac{2E_b}{N_0}}(1 - I_1(\underline{b}, \underline{\tau}, \underline{\phi})) \end{aligned} \tag{30}$$

$$\bar{P}_{b,-1} = E_{\underline{b},\underline{\tau},\underline{\phi}}[Q(\sqrt{\frac{2E_b}{N_0}}(1 - I_1(\underline{b}, \underline{\tau}, \underline{\phi}))] \tag{31}$$

Otherwise, if user 1 transmits $b_1^{(0)} = +1$,

$$\begin{aligned} P_{b,1} &= Pr(z_1 < 0 | b_1^{(0)} = +1) \\ &= Pr\{\sqrt{\frac{P}{2}} T[1 + I_1(\underline{b}, \underline{\tau}, \underline{\phi})] + n_1 > 0\} \\ &= Q(\sqrt{\frac{2E_b}{N_0}}(1 + I_1(\underline{b}, \underline{\tau}, \underline{\phi})) \end{aligned} \tag{32}$$

$$\bar{P}_{b,1} = E_{\underline{b},\underline{\tau},\underline{\phi}}[Q(\sqrt{\frac{2E_b}{N_0}}(1 + I_1(\underline{b}, \underline{\tau}, \underline{\phi}))] \tag{33}$$

We can see that the error probability $P_{b,-1}$ and $P_{b,+1}$ are difficult to compute. However, there exists good bounds to approximate it.

A. Gaussian Approximations (GA)

In the Gaussian approximations, we model $I_1(\underline{b},\underline{\tau},\underline{\phi})$ as Gaussian variable. $b_j^{(m)}$ and $b_i^{(n)}$ are assumed to be independent for any $i \neq j, m \neq n$. For any $i \neq l$, we assume τ_i and τ_l are independent and uniformly distributed in $[0, T)$. ϕ_i and ϕ_l are independent for any $i \neq l$, and are distributed uniformly in $[0, 2\pi)$. Then we can get

$$E[I_1(\underline{b},\underline{\tau},\underline{\phi})] = 0 \tag{34}$$

$$Var[I_1(\underline{b},\underline{\tau},\underline{\phi})] = \sum_{j=2}^{K} \sigma_{j,1}^2 \tag{35}$$

$$Var[z_1|b_1^{(0)} = -1] = N_0 T/4 + \frac{PT^2}{2} \sum_{j=2}^{K} \sigma_{j,1}^2 \tag{36}$$

Thus, the probability of error is

$$\bar{P}_{b,1} = Q(\frac{\sqrt{P/2}T}{\sqrt{(N_0 T/4) + \frac{PT^2}{2} \sum_{j=2}^{K} \sigma_{j,1}^2}}) \tag{37}$$

$$= Q(\sqrt{\frac{2E_b}{N_0}(\frac{1}{1 + \frac{2E_b}{N_0}\sum_{j=2}^{K} \sigma_{j,1}^2})}) \tag{38}$$

Interference from user k to user 1,

$$I_{k,1} = \int_0^T \sqrt{2P_k} b_k(t-\tau_k) c_k(t-\tau_k) c_1(t) \cos(\omega_c t + \phi_k)\cos(\omega_c t) dt \tag{39}$$

The delay of user k to relative to user 1 as

$$\tau_k = \nu_k T_c + \triangle_k, 0 \leq \triangle_k \leq T_c \tag{40}$$

$$I_{k,1} = T_c \sqrt{\frac{P_k}{2}} \cos(\phi_k)\{x_k + (1 - \frac{2\triangle_k}{T_c}) y_k + (1 - \frac{\triangle_k}{T_c}) u_k + \frac{\triangle_k}{T_c} \upsilon_k\} \tag{41}$$

x_k,y_k,u_k and v_k have distribution conditioned on A and B as follows,

$$p_{x_k}(l) = \begin{pmatrix} A \\ \frac{l+A}{2} \end{pmatrix} 2^{-A}, l = -A, -A+2, ..., A-2, A \tag{42}$$

$$p_{y_k}(l) = \begin{pmatrix} B \\ \frac{l+B}{2} \end{pmatrix} 2^{-B}, l = -B, -B+2, ..., B-2, B \tag{43}$$

$$p_{u_k}(l) = 1/2, l = -1, +1 \tag{44}$$

$$p_{v_k}(l) = 1/2, l = -1, +1 \tag{45}$$

where A and B are defined as follows, A is the number of integers in $[0, N-2]$ for which

$$c_{1,l+i}c_{1,l+i+1} = 1 \tag{46}$$

In another word, A measures for a given code the number of successive no transitions from $+1$ to -1 in the code c_1.

Similarly, B is the number of integers in $[0, N-2]$ for which

$$c_{1,l+i}c_{1,l+i+1} = -1 \tag{47}$$

In another word, A measures for a given code the number of successive transitions from $+1$ to -1 in the code c_1.

A and B are disjoint and span the set of total possible signature sequences of length N where there are a total of $N-1$ possible chip level transitions, that is to say,

$$A + B = N - 1 \tag{48}$$

For random signature sequence, statistic of

$$\sigma_{j,1}^2 = \frac{1}{T^2} E[\cos^2(\phi_j)] E[(b_j^{(-1)} R_{j,1}(\tau_j) + b_j^{(0)} \hat{R}_{j,1}(\tau_j))^2] \tag{49}$$

For random signature sequence, it is possible to model

$$b_j^{(-1)} R_{j,1}(\tau_j) + b_j^{(0)} \hat{R}_{j,1}(\tau_j) = \sum_{l=1}^{N} x_l \tag{50}$$

$$x_l = \begin{cases} \text{Uniform} & (-\frac{T}{N}, \frac{T}{N}) & \text{w.p.} \frac{1}{2} \\ \text{Bernoulli} & (-\frac{T}{N}, \frac{T}{N}, \frac{1}{2}) & \text{w.p.} \frac{1}{2} \end{cases} \tag{51}$$

The mean and variance of x_l are given by

$$E[x_l] = 0 \tag{52}$$

$$Var[x_l] = \frac{1}{2}\frac{2}{3}(\frac{T}{N})^2 + \frac{1}{2}(\frac{T}{N})^2 = \frac{2}{3}(\frac{T}{N})^2 \tag{53}$$

Then we can get

$$\sigma_{j,1}^2 = \frac{1}{3N} \tag{54}$$

Thus for the high SNR area, we get a good approximation of the probability as

$$\bar{P}_{b,1} = Q(\sqrt{\frac{2E_b}{N_0}(\frac{1}{1+\frac{2E_b}{N_0}}\frac{K-1}{3N})}) \tag{55}$$

$$\approx Q(\sqrt{\frac{3N}{K-1}}) \tag{56}$$

It is better than $Q(\sqrt{\frac{N}{K-1}})$the synchronous case.

Problems with analytical expressions derived here. When K is not large or users have disparate powers, the Gaussian approximation above is not appropriate. Fortunately, there are other methods to solve the question.

B. Improved Gaussian Approximation (IGA)

In the situation where the Gaussian Approximation is not appropriate, we can utilize a more in-depth analysis called improved Gaussian Approximation (IGA), which defines the interference terms I_k conditioned on the particular operating condition of each user. Thus each I_k becomes Gaussian for large K.

Let ψ as the variance of the multiple access interference for a specific operating condition,

$$\psi = Var[I_1(\underline{\phi}, \underline{\tau}, \underline{P}, \underline{c_1}, ..., \underline{c_K})|\underline{\phi}, \underline{\tau}, \underline{P}, \underline{c_1}, ..., \underline{c_K}] \tag{57}$$

$$= Var[I_1(\cdot)|\underline{\phi}, \{\triangle_k\}, \{P_k\}, B] \tag{58}$$

$$P_{b,1} = E[Q(\sqrt{\frac{PT^2}{2\psi}})]$$

$$= \int_0^\infty Q(\sqrt{\frac{PT^2}{2\psi}})f_\psi(\psi)d\psi \tag{59}$$

It is possible to show that

$$\psi = \sum_{k=2}^{K} T_c^2 P_k \cos^2 \phi_k (\frac{N}{2} + (2B+1)[(\frac{\triangle_k}{T_c})^2 - \frac{\triangle_k}{T_c}]) \tag{60}$$

$$= \sum_{k=2}^{K} z_k \tag{61}$$

where

$$z_k = \frac{T_c^2}{2} P_k u_k v_k \tag{62}$$

$$u_k = (1 + \cos(2\phi_k)) \tag{63}$$

$$v_k = \frac{N}{2} + (2B+1)((\frac{\triangle_k}{T_c})^2 - \frac{\triangle_k}{T_c}) \tag{64}$$

Using this method, we can approximate the probability of error for the case in which the power levels of the interfering users are constant but unequal and the case in which the power levels of the interfering users are independent and identically distributed random variables.

C. Simple Improved Gaussian Approximation (SIGA)

A simple improved Gaussian approximation (SIGA) method is based on the Taylor series for a continues function $f(x)$ as

$$f(x) = f(u) + (x-u)f'(u) + \frac{1}{2}(x-u)^2 f''(u) + ... \tag{65}$$

If x is a random variable and $E[x] = u$,, then

$$E[f(x)] = f(u) + \frac{\sigma^2}{2} f''(u) + ... \tag{66}$$

If derivatives are expressed as difference

$$f(x) = f(u) + (x-u)[\frac{f(u+h) - f(u-h)}{2h}] + \frac{1}{2}(x-u)^2(\frac{f(u+h) - f(u) + f(u-h)}{h^2}) + ... \tag{67}$$

If we neglect the higher order terms,

$$E[f(x)] \doteq f(u) + \frac{\sigma^2}{2}[\frac{f(u+h) - 2f(u) + f(u-h)}{h^2}] \tag{68}$$

It was shown that $h = \sqrt{3}\sigma$ is a good choice for generality of approximation, which yields,

$$E[f(x)] = \frac{2}{3} f(u) + \frac{1}{6} f(u + \sqrt{3}\sigma) + \frac{1}{6} f(u - \sqrt{3}\sigma) \tag{69}$$

Therefore, we can get the average probability of error by

$$P_e = E[Q(\sqrt{\frac{PT_b^2}{2\psi}})] \tag{70}$$

$$\approx \frac{2}{3}Q(\sqrt{\frac{PT_b^2}{2\mu_\psi}}) + \frac{1}{6}Q(\sqrt{\frac{PT_b^2}{2(\mu_\psi + \sqrt{3}\sigma_\psi)}}) + \frac{1}{6}Q(\sqrt{\frac{PT_b^2}{2(\mu_\psi - \sqrt{3}\sigma_\psi)}}) \tag{71}$$

The figures below illustrate the average bit error rates (BER) for single-cell CDMA systems using the different analytical algorithms discussed above. For the figure shown, the processing gain, N, is 31 chips per bit. In Fig. (5), $K_2 = [K/2]$ users have power level $P_1/4$, while $K_1 = K - K_2 - 1$ have power level P_1. In Fig.(6), the interferer power levels obey a log-normal distribution with standard deviation of $5dB$ [7].

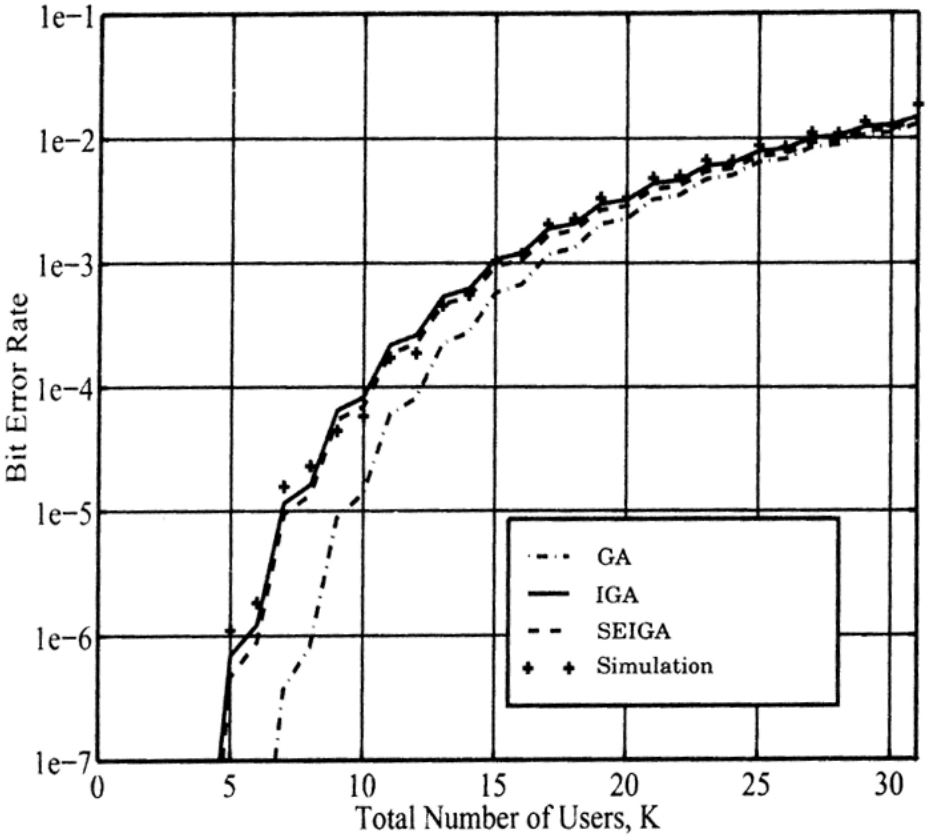

Fig. 5. BER for a desired user vs. K with fixed power levels for all K users. [7]

Common like single user detection Gaussian Interference. We can see that single user detection cannot work very well under the high rate services, because $N = \frac{W}{R} \rightarrow \frac{T_b}{T_c}$ decrease with R.

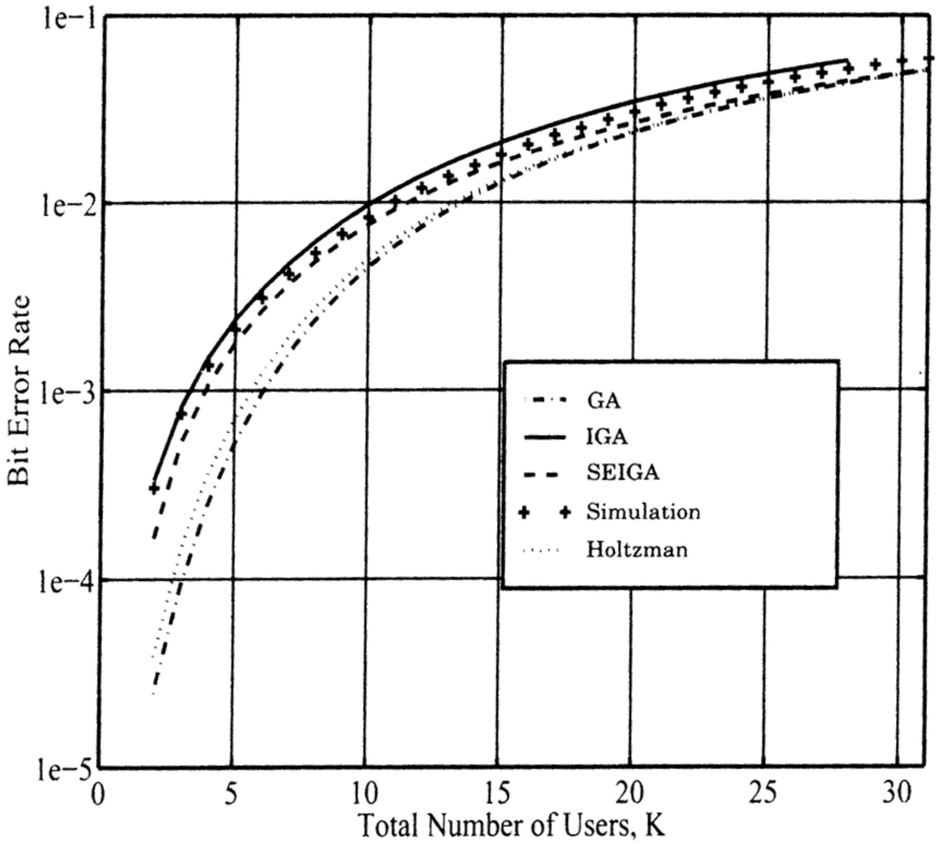

Fig. 6. BER vs. K with i.i.d power levels of the interference users. [7]

VI. Multiuser Detection

Interference signals in spread spectrum multiple access need not be treated as noise. If the spreading code of the interference signal is known, then that signal can be detected and subtracted out. Multiuser detection deals with the demodulation of mutually interfering digital signals, which exploits the structure of the multiaccess interference. This technique is applicable to multiaccess techniques such as asynchronous CDMA (where interuser interference occurs by design) and TDMA (where interuser interference occurs due to nonideal effects such as channel distortion and out-of-cell interference). Sergio Verdu's work in 1980s [8] pioneered the receiver design technology of multiuser detection.

A. *The Matched Filter in the CDMA Channel*

This section are focusing on the probability of error for synchronous uses. The received signal due to k users is

$$y(t) = \sum_{k=1}^{K} A_k b_k S_k(t) + \sigma n(t), \tag{72}$$

where $S_k(t)$ is the signature waveform assign to user k. Without loss of generality, $S_k(t)$ is normalized as

$$||S_k||^2 = \int_0^T S_k^2(t)dt = 1. \tag{73}$$

A_k is the received amplitude of the kth user and $A_k > 0$. Note that A_k^2 is the energy of the received signal due to the normalized $S_k(t)$. $b_k \in \{-1, 1\}$ is the bit transmitted by the kth user with period T. $n(t)$ is the white Gaussian noise with zero mean and unity power spectrum density.

The output of the match filter for the kth user is

$$y_k = \int_0^T y(t)S_k(t)dt \tag{74}$$

$$= A_k b_k + \sum_{j \neq k} A_j b_j \rho_{j,k} + n_k \tag{75}$$

where

$$\rho_{j,k} = \int_0^T S_j(t)S_k(t)dt \tag{76}$$

$$n_k = \sigma \int_0^T n(t)S_k(t)dt \tag{77}$$

Note that the noise component $n_k \sim N(0, \sigma^2)$.

If the signature are orthogonal, the cross correlation $\rho_{j,k} = 0$ so that the probability of error for the kth user is

$$\rho_k^c(\sigma) = Q\left(\frac{A_k}{\sigma}\right) \tag{78}$$

where the superscript c denotes the probability of error when a conventional receiver is used. The probability of error is the same as the one in the single user case.

If the signature are not orthogonal, the statistics are not Gaussian anymore. Consider $k = 2$ users. Let $\rho_{1,2} = \rho$ and the received signal by the matched filter for user 1 is then

$$y_1 = A_1 b_1 + A_2 b_2 \rho + n_1 \tag{79}$$

The probability of error of user 1 is

$$P_1^c(\sigma) = P\{\hat{b_1} \neq b_1\} \tag{80}$$

$$= P[b_1 = +1]P[y_1 < 0|b_1 = +1] + P[b_1 = -1]P[y_1 > 0|b_1 = -1] \tag{81}$$

according to the decision rule for the conventional matched filter.

Since y_1 is conditioned on b_2 as well, it is not Gaussian. Hence, we have

$$P[y_1 > 0|b_1 = -1] = P[y_1 > 0|b_1 = -1, b_2 = +1]P[b_2 = +1] \tag{82}$$

$$+ P[y_1 > 0|b_1 = -1, b_2 = -1]P[b_2 = -1] \tag{83}$$

$$= P[n_1 > A_1 - A_2\rho]P[b_2 = +1] + P[n_1 > A_1 + A_2\rho]P[b_2 = -1] \tag{84}$$

$$= \frac{1}{2}Q\left(\frac{A_1 - A_2\rho}{\sigma}\right) + \frac{1}{2}Q\left(\frac{A_1 + A_2\rho}{\sigma}\right) \tag{85}$$

$$= \frac{1}{2}Q\left(\frac{A_1 - A_2|\rho|}{\sigma}\right) + \frac{1}{2}Q\left(\frac{A_1 + A_2|\rho|}{\sigma}\right) \tag{86}$$

The first equality follows from the fact that b_1 and b_2 are independent.

By symmetry, $P[y_1 < 0|b_1 = +1] = P[y_1 > 0|b_1 = -1]$. Therefore, the probability of error of user 1 is

$$P_1^c(\sigma) = \frac{1}{2}Q\left(\frac{A_1 - A_2|\rho|}{\sigma}\right) + \frac{1}{2}Q\left(\frac{A_1 + A_2|\rho|}{\sigma}\right) \tag{87}$$

Interchanging the roles of user 1 and 2, we have

$$P_2^c(\sigma) = \frac{1}{2}Q\left(\frac{A_2 - A_1|\rho|}{\sigma}\right) + \frac{1}{2}Q\left(\frac{A_2 + A_1|\rho|}{\sigma}\right) \tag{88}$$

Let us consider user 1. Since $Q(x)$ is a monotonically decreasing function,

$$P_1^c(\sigma) \leq Q\left(\frac{A_1 - A_2|\rho|}{\sigma}\right) \tag{89}$$

This bound is smaller than $1/2$ provided $A_2/A_1 < 1/|\rho|$, i.e., the interferer is not dominant. Note as $\sigma \to 0$, (87) is dominated by the term with the smallest argument and hence the upper bound is an excellent approximation (for all but low SNRs). Therefore, BER of conventional receiver behaves like a single user system with a reduced SNR, $\left(\frac{A_1 - A_2|\rho|}{\sigma}\right)^2$.

However, if relative amplitude of interferer is stronger, i.e., $A_2/A_1 > 1/|\rho|$, then the conventional receiver exhibits highly anomalous behavior, as known as the *near-far* problem. For example, BER is not monotonic in σ anymore, a property which is usually expected of any detector. For equation (87), since $A_1 - A_2|\rho| < 0$, we have

$$\lim_{\sigma\to\infty} P_1^c(\sigma) = \frac{1}{2} \quad \text{and} \quad \lim_{\sigma\to 0} P_1^c(\sigma) = \frac{1}{2} \tag{90}$$

which shows that BER is not monotonic. This anomalous behavior follows from the fact that the polarity of the output of the match filter for user 1 is governed by b_2 other than by b_1. In fact, $\sigma > 0$ is actually good in detection in the sense that $P_1^c(\sigma) < 1/2$.

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