# Probability of Error, Digital Signaling on a Fading Channel And Equalization Schemes for ISI

Wireless Communications Technologies Spring 2005 Lectures 11&12 R1

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**Abstract –** This document discusses the probability of error for non-coherent FSK and DPSK. Digital signaling on a frequency selective channel will be reviewed as well as equalization schemes to eliminate ISI.

## 1.0 Probability of Error for non-coherent BFSK

Recall from the last lecture that we have Binary FSK orthogonal modulation. When a "1" is selected then $S_1(t)$ is transmitted and when a "0" is selected then $S_2(t)$is transmitted. $S_1(t)$ and $S_2(t)$ are orthogonal. The received signal is $g_1(t)$ if $S_1(t)$ was transmitted and $g_2(t)$ if $S_2(t)$ is transmitted. The received signals $g_1(t)$ and $g_2(t)$ are also orthogonal. The receiver structure is shown in figure 1.

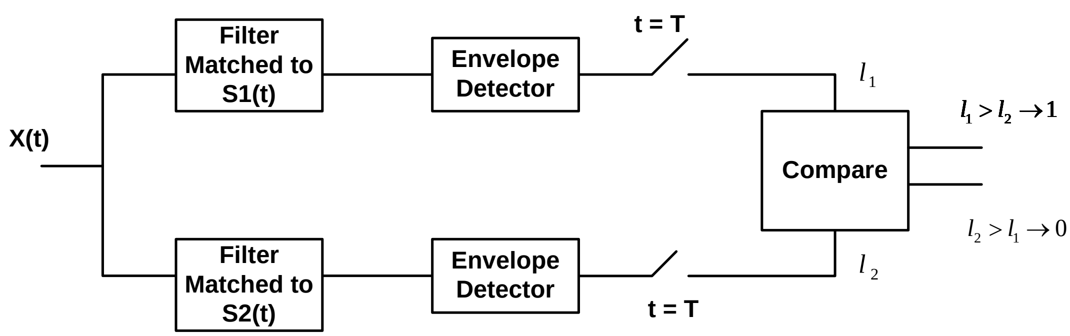


**Figure 1. Non-coherent Receiver for BFSK**

Let $S_1(t)$ be transmitted, then an error occurs if $l_2 > l_1$.

$$l_2 = \sqrt{x_{I_2} + x_{Q_2}}$$

Where $X_{I2}$ and $X_{Q2}$ are both Gaussian with zero mean and psd = $N_0/2$.

$$f_{l_2}(l_2) = \begin{cases} \frac{2l_2}{N_0}\exp\left(-\frac{l_2^{\,2}}{N_0}\right) & , \; l_2 \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$\text{Probability of error } P_{e_1} = \Pr\{l_2 > l_1\} = \int_0^{\infty} \Pr\{l_2 > \lambda_1 | l_1\} f_{l_1}(l_1)\, dl_1$$

$$\Pr\{l_2 > \lambda_1 | l_1\} = \int_{l_1}^{\infty} f_{l_2}(l_2)\, dl_2 = \exp\left(-\frac{l_1^2}{N_0}\right)$$

where $l_1 = \sqrt{x_{I_1} + x_{Q_1}}$ , $x_{I_1} \sim N\left(\sqrt{E}, \frac{N_0}{2}\right)$ and $x_{Q_1} \sim N\left(0, \frac{N_0}{2}\right)$

$$f_{x_I}(x_{I_1}) = \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{\left(x_{I_1} - \sqrt{E}\right)^2}{N_0}\right\}$$

$$f_{x_Q}(x_{Q_1}) = \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{x_{Q_1}{}^2}{N_0}\right\}$$

$$\text{Probability of error } P_{e_1} = \iint p\left(\text{error} \,|\, x_{I_1} x_{Q_1}\right) f_{x_{I_1} x_{Q_1}}\, dx_{I_1}\, dx_{Q_1}$$

$$P_{e_1} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{x_{Q_1}{}^2 + x_{I_1}{}^2}{N_0}\right\} \exp\left\{-\frac{\left(x_{I_1} - \sqrt{E}\right)^2}{N_0}\right\} \exp\left\{-\frac{x_{Q_1}{}^2}{N_0}\right\} dx_{I_1}\, dx_{Q_1}$$

Rewriting $x_{Q_1}{}^2 + x_{I_1}{}^2 + \left(x_{I_1} - \sqrt{E}\right)^2 + x_{Q_1} = 2\left(x_{I_1} - \sqrt{E}\right)^2 + 2x_{Q_1}{}^2 + \frac{E}{2}$

$$P_{e_1} = \frac{1}{\pi N_0} \exp\left(-\frac{E}{2N_0}\right) \int_{-\infty}^{\infty} \exp\left(-\frac{2}{N_0}\left(x_{I_1} - \sqrt{E}\right)^2\right) dx_{I_1} \int_{-\infty}^{\infty} \exp\left\{-\frac{x_{Q_1}{}^2}{N_0}\right\} dx_{Q_1}$$

$$P_{e_1} = \frac{1}{2} \exp\left(-\frac{E}{2N_0}\right)$$

**2.0 Probability of error for non-coherent M-ary FSK**

In general for M-ary FSK with non-coherent detection:

$$P_e = \frac{1}{2(m-1)} \sum_{i=2}^{M} (-1)^i \exp\left\{\frac{-(i-1) k E_b}{i N_0}\right\}$$

Where $k = \log_2 M$

A non-coherent receiver for M-ary FSK is shown in Figure 2. As with the binary FSK receiver a comparison of the output of the envelope detectors selects the appropriate branch.

Figure 3 illustrates the probability of bit error for various values of M. As M increases the bandwidth efficiency decreases but the power efficiency increases. More constellations require more chunks of orthogonal spaces.

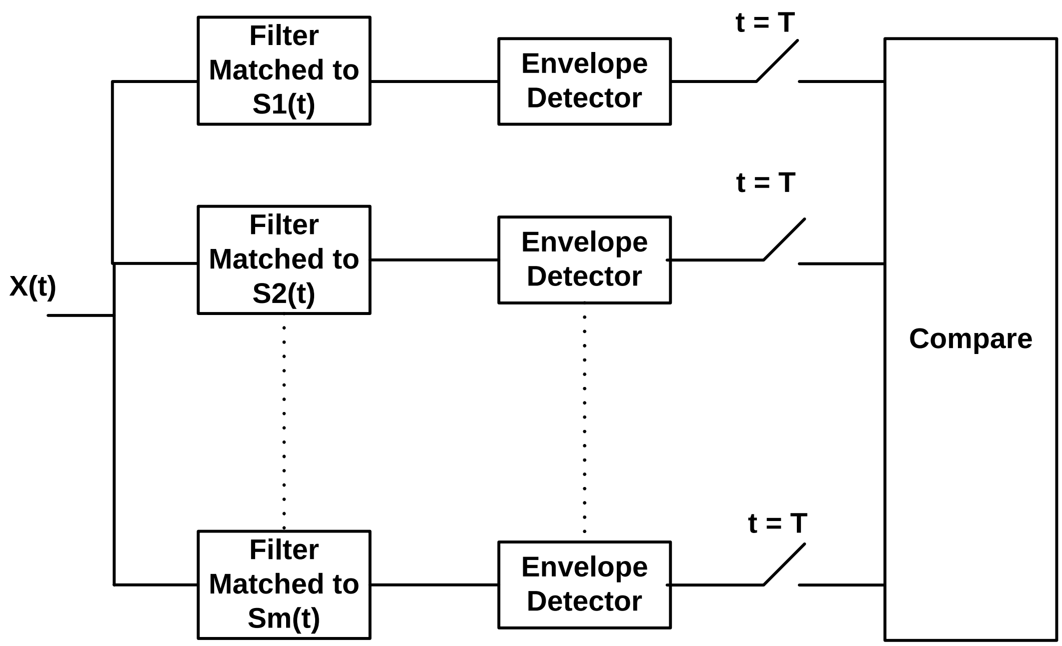


**Figure 2. Non-coherent Receiver for M-ary FSK**

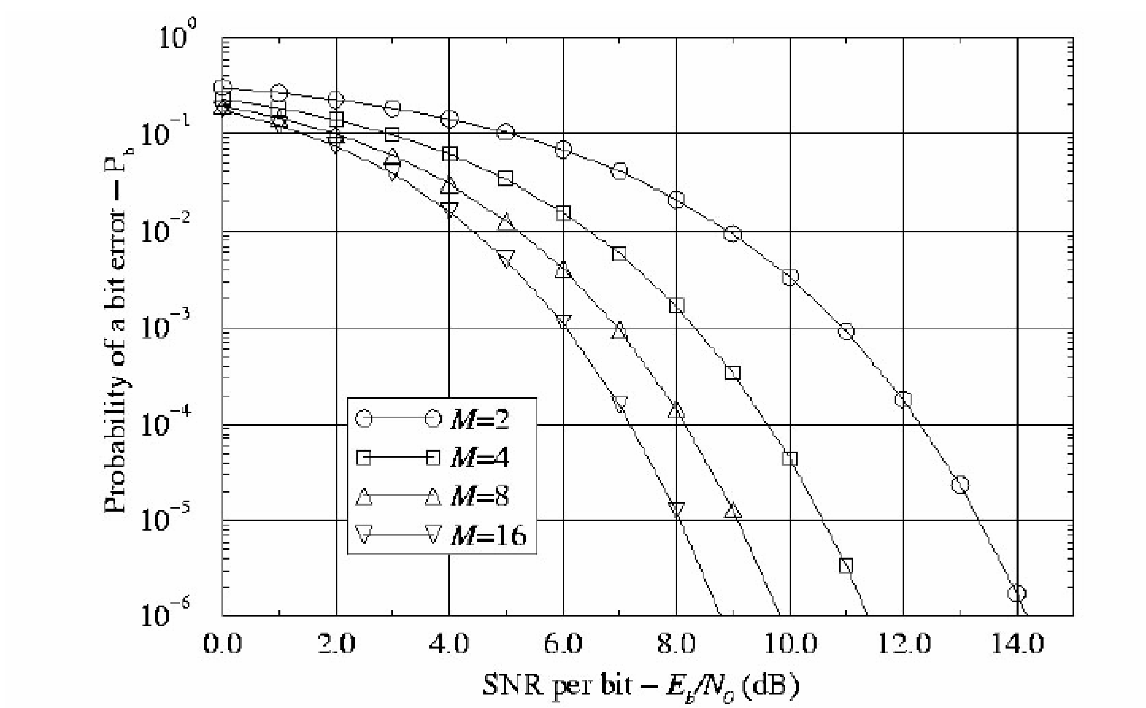

**Figure 3. Non-coherent M-ary FSK BER**

## 3.0 Differential Phase Shift Keying

If information is keyed onto the phase, PSK, it cannot be detected by non-coherent methods. Use phase difference between two consecutive waveforms to carry the information.

Assumption: It works provided the unknown phase introduced by the channel varies slowly (i.e. slow enough to be considered constant over 2 bit intervals). Consider input sequence $\{m_k\}$. Generate differentially encoded sequence $\{d_k\}$ from $\{m_k\}$ as follows:

1. Sum $d_{k-1}$ and $m_k$ Modulo 2.
2. Set $d_k$ to be the complement of the result from 1 above.
3. Use $d_k$ to phase shift a carrier as follows:

$d_{k=}1 \Rightarrow \theta = 0$

$d_{k=0} \Rightarrow \theta = \pi$

Example: if

$$\begin{aligned} \{m_k\} &\rightarrow 10010011 \\ \{d_{k-1}\} &\rightarrow 11011011 \\ \{d_k\} &\rightarrow 10110111 \\ \{\theta\} &\rightarrow 0\pi00\pi000 \end{aligned}$$

Observe: Symbol $d_k$ is unchanged from previous symbol if incoming symbol is 1. Symbol $d_k$ is toggled from previous symbol if incoming symbol is 0.

## 4.0 $\frac{\pi}{4}$ - DQPSK

Exploit the above observation to derive the probability of error. Let the DPSK signal in $0 \le t \le T_b$ be $\sqrt{\frac{E_b}{2T_b}}\cos(2\pi f_c t)$. If the next bit (in the interval $T_b \le t \le 2T_b$) is 1, then the phase is unchanged. If the next bit is 0, then the phase is shifted by $\pi$.

$$S_1(t) = \begin{cases} \sqrt{\dfrac{E_b}{2T_b}} \cos(2\pi f_c t) \ , & 0 \le t \le \mathrm{T_b} \\ \sqrt{\dfrac{E_b}{2T_b}} \cos(2\pi f_c t) \ , & \mathrm{T_b} \le t \le 2\mathrm{T_b} \end{cases}$$

$$S_2(t) = \begin{cases} \sqrt{\dfrac{E_b}{2T_b}} \cos(2\pi f_c t) \ , & 0 \le t \le \mathrm{T_b} \\ \sqrt{\dfrac{E_b}{2T_b}} \cos(2\pi f_c t + \pi) \ , & \mathrm{T_b} \le t \le 2\mathrm{T_b} \end{cases}$$

It is possible to consider $S_1(t)$ and $S_2(t)$ as similar to non-coherent orthogonal modulation over a 2 bit interval, $0 \le t \le 2T_b$.

$$P_e = \frac{1}{2} \exp\left(-\frac{E}{N_0}\right)$$

Figure 4 compares the BER for DPSK and FSK. Observe that DPSK BER is approximately 3 dB better than non-coherent FSK.

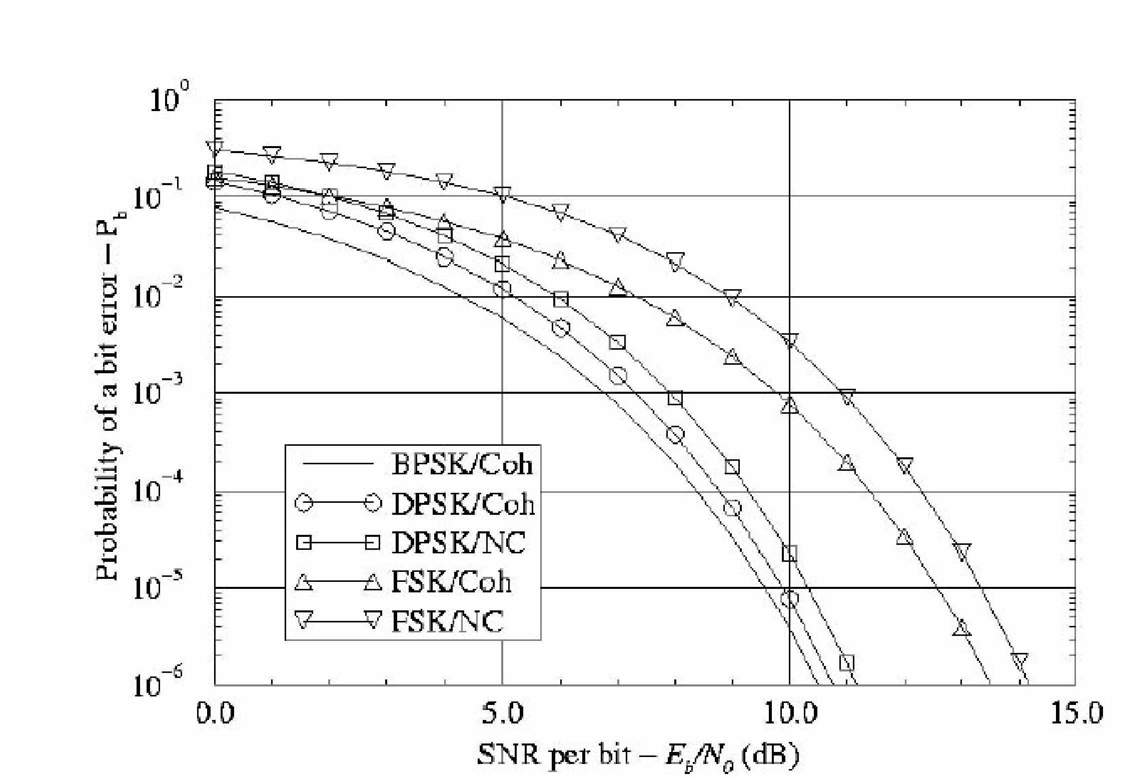


Figure 4 BER Comparisons of DPSK and FSK

## 5.0 Digital Signaling on Frequency Selective Fading Channels

Modeling: Recall any modulated signal is given as $v(t) = A\sum_k b\left(t - kT, □_k\right)$. We restrict ourselves to linear modulation.

$$b\left(t, □_k\right) = x_k h_a\left(t\right) \qquad \left\{x_k\right\} \rightarrow \text{complex symbol sequence}$$

amplitude sampling pulse

The above signal is transmitted through a channel c(t) that results in the received signal w(t).

$$w(t) = \sum_{k=0}^{\infty} x_k h\left(t - kT\right) + z(t)$$

zero-mean AWGN

$$h(t) = \int_{-\infty}^{\infty} h_a(\tau) c(t-\tau) d\tau \ .$$

For causal channels $\rightarrow h(t) = \int_0^{\infty} h_a(\tau) c(t-\tau) d\tau \quad , t \geq 0$.

We also assume h(t) = 0 for $t \leq 0$ and h(t) = 0 for $t \geq LT$. L is some real positive integer.

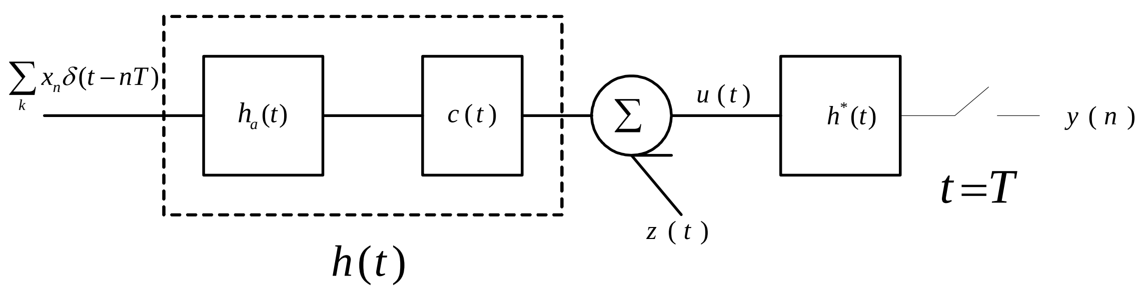


**Figure 5. Matched Filter Receiver in AWGN Channel**

If we know h(t), we can implement a matched filter as above.

$$y(t) = \sum_{k=-\infty}^{\infty} x_k f(t-kT) + v(t)$$

Filter noise

$$f(t) = \int_{-\infty}^{\infty} h^*(\tau) h(\tau + t) d\tau .$$

$f(t)$ is the overall pulse response and it accounts for transient filter, channel and receive filters.

$$\begin{aligned} y_n = y(nt) &= \sum_{k=-\infty}^{\infty} x_k f(nT - kT) + v(nT) \\ &= \sum_k x_k f_{n-k} + v_n \\ &= x_n f_0 + \sum_{\substack{k=-\infty \\ k \neq n}}^{\infty} x_k f_{n-2} + v_n \end{aligned}$$

called intersymbol interference(ISI)

ISI is caused by the channel so if we can eliminate it then we can treat the channel as AWGN channel. Therefore to achieve the same performance as on a AWGN channel, we require $\sum_{\substack{k=-\infty \\ k \neq n}}^{\infty} x_k f_{n-2} = 0$. In order to do this we must make $f_{n-2} = 0$. Alternately,

$$f_k = \delta_{k_0} f_0 \text{ where } \delta_{ij} = \left\{ \begin{matrix} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{matrix} \right\}.$$

Equally in the frequency domain

$$\frac{1}{T} \sum_{n=-\infty}^{\infty} F\left( f + \frac{n}{T} \right) = f_0 \quad , F(f) = \mathrm{F}\{ f(t) \}$$

It can be shown that f(t) can be any function that has equally spaced zero crossings. What is the optimum receiver? We must express $w(t) = \lim_{n \to \infty} \sum w_n \Phi_n(t)$ using the Karhunen-Loeve expansion.

$$h_{nk} = \int_0^T h(t - kT) \Phi_n^*(t) dt$$

$$z_n = \int_0^T z(t) \Phi_n^*(t) dt$$

$\underline{w} = (w_1, w_2, \ldots w_n)$ is a multivariate Gaussian distribution because z(t) is Gaussian.

$$p(\underline{w}|\underline{x},H)=\prod_{n=1}^{N}\frac{1}{\pi N_0}\exp\left\{-\frac{1}{N_0}\left|w_n-\sum_{k=-\infty}^{\infty}x_k h_{nk}\right|^2\right\}$$

where $H=\left[\underline{h}_1,\underline{h}_2,\underline{h}_3,\ ...\ ,\underline{h}_n\right]$ and $\underline{h}_n=(....,h_{n-3},h_{n-2},h_{n-1},h_0)$

Therefore the optimum receiver is the Maximum Likelihood receiver, for the case of AWGN it reduces to

$$\arg\left\{\max_{\underline{x}}\ \ u(\underline{x})=-\sum_{n=1}^{N}\left|w_n-\sum_{k=-\infty}^{\infty}x_k h_{nk}\right|^2\right\}$$

The bottom line is:

1. We need knowledge of $\{f_n\}$ to understand the channel response and knowledge of $\{c_n\}$ in order to perfectly eliminate ISI. Therefore we need to estimate the channel in order to equalize it.

2. An additional problem results as well by observing the following:

$$y(t)=\sum_{k=-\infty}^{\infty}x_k f(t-kT)+v(t)$$

where the noise function is

$$v(t)=\int_{-\infty}^{\infty}h^*(\tau)z(t-\tau)d\tau$$

is still Gaussian but is not white. Therefore the noise samples at the output are correlated. In this case we can obtain a discrete time white noise model as follows.

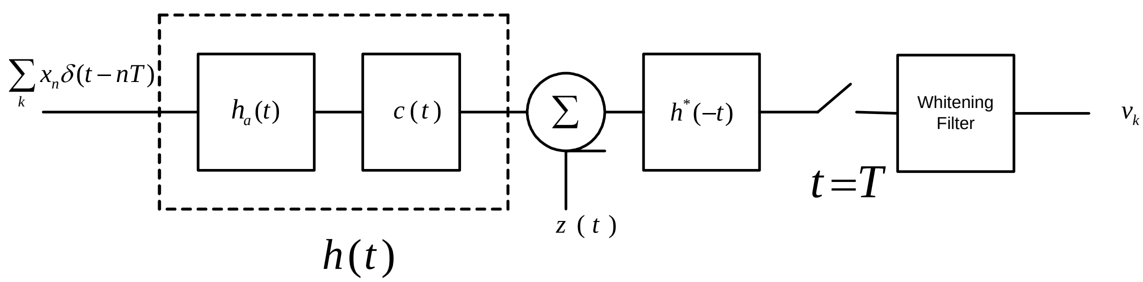


**Figure 6. Whitening Filter with Matched Filter Receiver**

The output of the whitening filter $v_k$ is given by

$$v_k=\sum_{n=0}^{L}g_n x_{k-n}+\eta_k$$

where $g_n$ is the overall filter for the channel and the whitening filter and $\eta_k$ is the white noise process.

3. Another important consideration is the design of ISI equalizing filters is extremely sensitive to timing information. We can solve this sensitivity in two ways.
   a. Use pulse shaping. Raised cosine pulse shaping allows us to derive the length of the pulse by sampling at even points.
   b. Use fractional sampling. Sample the output y(t) at a rate higher than 2/T. Although we will still have correlated noise and will need a whitening filter the overall pulse shape will be less sensitive to timing errors.

**6.0 Equalization Schemes**

There are two types of equalization schemes.

1. Symbol by symbol equalizers that can be linear or non-linear.
2. Sequence estimation equalizers that are non-linear.

**6.1 Symbol-by-Symbol Equalizers**

We consider the discrete time white noise model shown in figure 7. Where $\{a_n\}$ is the input signal, L is the memory of the channel, $\eta_n$ is the additive noise and $\underline{g} = (g_0, g_{1,}.....,g_l)^T$ is the channel vector that describes the overall channel impulse response.

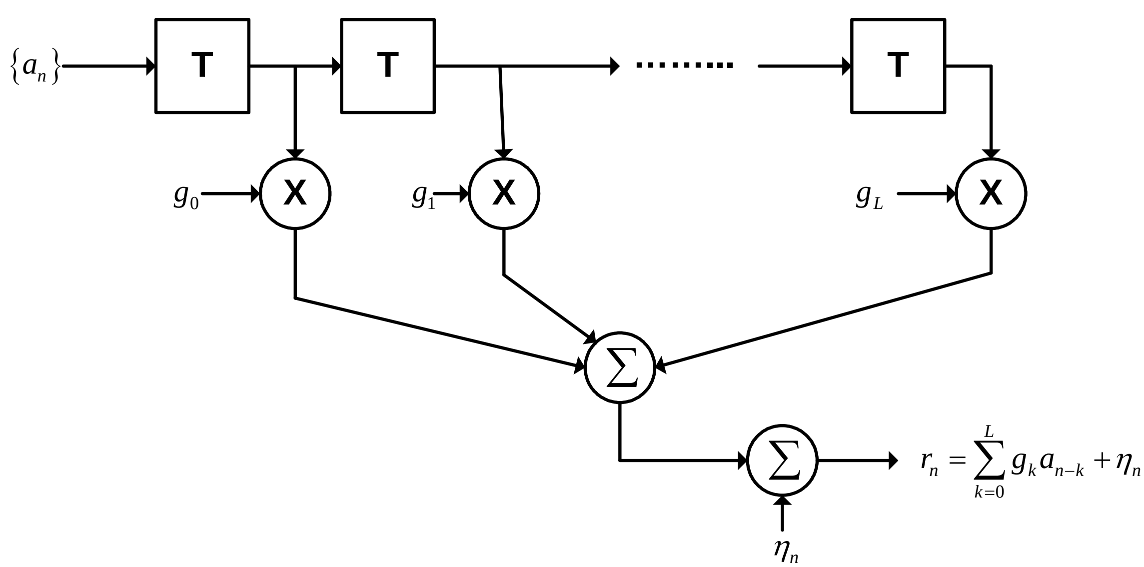


**Figure 7. Discrete Time White Noise Model**

Linear equalizers estimate the n^th transmitted symbol $\hat{a}_n$ as

$\hat{a}_n = \sum_{j=-M}^{M} c_j r_{n-j}$ Where $\{c_j\}_{-M}^{+M}$ is the equalizer filter of (2M+1) taps. M is a design choice and C is the chosen number of taps to estimate $\hat{a}_n$.

### 6.2 Zero Forcing Equalizer

The Zero Forcing Equalizer combines the channel and equalizer impulse response to force to zero at all but one of the taps in the TDL filter. The tap coefficients, **c**, are chosen to "zero-out" the ISI. Given the channel vector, $\underline{g}$, we can select the tap coefficients, **c**, to get the desired response $\underline{q} = (0,0,0,......,q_n,0,...)$. The Zero Forcing Equalizer zeros out all but the desired result $q_n$. The tap coefficients are calculated using the relationship $q_n = \underline{c}^T \underline{g_n}$. Zero forcing equalizers can perfectly eliminate ISI as $M \to \infty$ but it also enhances the noise. In practice, the receiver does not know the channel vector and finite length training sequences are used to choose the tap coefficients.

### 6.3 Minimum Mean Square Equalizer (MMSE)

The MMSE is another symbol-by-symbol equalizer but is superior to the zero forcing equalizer in performance. The MSME utilizes the mean square error criterion to adjust the tap coefficients. We define an estimation error:

$$\varepsilon_n = a_n - \hat{a}_n$$

Where $a_n$ is the symbol sent and $\hat{a}_n$ is the estimated symbol. The function to be minimized is:

$$\begin{aligned} J &= \min_{\underline{c}} \; E\left[\varepsilon_n{}^2\right] = \min_{\underline{c}} \; E\left[(a_n - \hat{a}_n)^2\right] \\ &= \min_{\underline{c}} \; E\left[(a_n - \sum_{j=-M}^{M} c_j r_{n-j})^2\right] \end{aligned}$$

The error is minimized by choosing $\{c_j\}$ so as to make the error sequence orthogonal to the signal sequence $r_{n-l}$, for $|l| \le M$, i.e. $E\left[\varepsilon_n r_{n-l}\right] = 0$, $|l| \le M$. The optimum $\underline{c}$ is obtained by solving the following set of equations.

$$\sum_{j=-M}^{M} c_j E\left[r_{n-j} r_{n-l}\right] = E\left[a_n r_{n-l}\right], \quad |l| \le M$$

Define the following functions.

$$\Gamma \triangleq \mathrm{E}\left[r_{n-j}r_{n-l}\right], \quad j \neq l, = M, ...., -M$$

and $$\underline{b} = \left(E\left[a_n r_{n+m}\right]E\left[a_n r_{n+m-1}\right]........E\left[a_n r_{n-m}\right]\right)$$

then $$\underline{c}_{opt} = \Gamma^{-1}\underline{b}$$

To implement the MSME we need to us adaptive algorithms because the channel response is not readily known. Typically, we use the steepest descent algorithm.

$$c_j(n+1) = c_j(n) - \frac{1}{2}\mu \frac{\partial E\left[\varepsilon^2(n)\right]}{\partial c_j}, \quad j = -M.......+M$$

Where $\mu$ is a positive number.

Note:

$$\frac{\partial E\left[\varepsilon^2(n)\right]}{\partial c_j} = -2E\left[\varepsilon(n)\frac{\partial \varepsilon(n)}{\partial c_j}\right] = -2E\left[\varepsilon(n)r_{n-j}\right] = -2R_{er}(j)$$

The term $R_{er}$ is the cross correlation between the input and the error. The equalizer taps can be obtained by implementing a stochastic gradient algorithm.

$$c_j(n+1) = c_j(n) - \frac{1}{2}\mu\varepsilon(n)r_{n-j}$$

To evaluate $\varepsilon(n)$ in this algorithm we use a training sequence.

**6.4 Decision Feedback Equalizer (DFE)**

A DFE is a nonlinear equalizer that consists of a feed forward section and a feedback section. DFEs are very effective in frequency selective channels because they mitigate the effects of noise enhancements that degrade the performance of linear equalizers. The DFE uses previous decisions to eliminate ISI caused by previously detected symbols on current symbols. The output of the equalizer can be expressed as:

$$\hat{a}_n = \underbrace{\sum_{j=-M_1}^{0} c_j r_{n-j}}_{feed\ forwad\ filter\ M_1+1\ taps} + \underbrace{\sum_{j=1}^{M_2} c_j \tilde{a}_{n-j}}_{feedback\ filter\ M_2\ taps}$$

Where $\tilde{a}_{n-1}, \tilde{a}_{n-2}, ....., \tilde{a}_{n-M_2}$ are earlier detected symbols. Both the feed forward loop and the feedback loop need adaptation to adjust the coefficients.

### 6.5 Maximum Likelihood Sequence Estimation (MLSE)

Recall the discrete-time white noise channel model.

$$r_n = \sum_{m=0}^{L} g_m a_{n-m} + \eta_n$$

Assume k symbols are transmitted over the channel. Then after receiving the sequence $\{r_n\}_{n=1}^{k}$, the ML receiver decides in favor of the sequence $\{a_n\}_{n=1}^{k}$ that maximizes the likelihood function

$$\log\left(r_k, r_{k-1,} ..., r_1 \middle| a_{k,} a_{k-1}, ..., a_1\right)$$

Since the noise samples are independent and $r_n$ depends only on the L most recent transmitted symbols.

$$\log\left(r_k, r_{k-1,} ..., r_1 \middle| a_{k,} a_{k-1}, ..., a_1\right) = \log\left(r_k \middle| a_{k,} a_{k-1}, ., a_{k-l}\right) + \underbrace{\log\left(r_{k-1}, r_{k-2,} .., r_1 \middle| a_{k-1,} a_{k-2}, .., a_1\right)}_{\text{This has already been calculated}}$$

where $a_{k-L} = 0$ for $k - L \leq 0$

Since the 2nd term has been calculated at the previous time (k-1), only the first term needs to be computed at time k for each incoming signal $r_k$. The Veterbi algorithm can be used to implement the MLSE equalizer. Adaptive MLSE is used in GSM.

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