Wireless Communications Technologies
Course No: 16:332:546

Homework 1

1. (a) The characteristic function of $X_{i,n}$ is

$$M_{X_{i,n}}(u) = E[\exp(juX_{i,n})] = (1 - \lambda/n) + \exp(ju)\lambda/n = 1 + \frac{\lambda}{n} \exp(ju - 1)$$

Therefore,

$$M_{Y_n}(u) = \{1 + \frac{\lambda}{n} \exp(ju - 1)\}^n$$

(b) \(\lim_{n \to \infty} M_{Y_n}(u) = \exp(\lambda(\exp(ju) - 1))\),

which implies \(\lim_{n \to \infty} Y_n\) is a Poisson random variable with mean and variance \(\lambda\). (This follows from the fact that the mapping from a characteristic function to a distribution is 1:1)

2. \(E[Y_t] = \mu D\)

By definition \(\phi_Y(t, s) = E[(X_{t+D} - X_t)(X_{s+D} - X_s)]\). Therefore,

$$\phi_Y(t, s) = \sigma^2[\min(t + D, s + D) - \min(t + D, s) - \min(t, s + D) + \min(t, s)] + \mu^2[(t + D)(s + D) - (t + D)s - t(s + D) + ts]$$

There are 2 cases:

Case 1: \(|t - s| \leq D\)

$$\phi_Y(t, s) = \sigma^2[D - |t - s|] + \mu^2D^2$$

Case 2: \(|t - s| > D\)

$$\phi_Y(t, s) = \mu^2D^2$$

From Case 1 and Case 2, it is clear that \(Y_t\) is wide-sense-stationary. Further, since \(Y_t\) is Gaussian, it is strictly stationary!

3. The analog signal is sampled at \(f_s = 8\text{ KHz}\). Each sample is quantized with \(L = 64\) levels of representation. Therefore the number of bits \(R\) required to represent each sample is

$$R = \log_2 L = 6\text{ bits}$$

The total bit rate after sampling and quantization is \(f_s \times R\text{ Kbps}\).

The minimum transmission bandwidth required \(W\) is given as \(W = \frac{1}{2T}\), where \(T\) is the symbol duration of the \(M\)-ary PAM system.

(a) \(M = 2\)

For \(M = 2\) amplitude levels, each pulse can represent \(\log_2 M = \log_2 2 = 1\) bit. Therefore,

$$T = \frac{1}{f_s R} \log_2 M = \frac{1}{f_s R}$$
⇒ \( W = f_sR/2 = 48/2 \text{ KHz} = 24 \text{ KHz} \)

(b) \( M = 4 \)

For \( M = 4 \) amplitude levels, each pulse can represent \( \log_2 M = \log_2 4 = 2 \) bits. Therefore,

\[
T = \frac{1}{f_s R} \log_2 M = \frac{1}{f_s R} \times 2
\]

⇒ \( W = f_s R/4 = 48/4 \text{ KHz} = 12 \text{ KHz} \)

4. A bit 1 is represented by a pulse of height \( A \) for a duration of 1 second and a bit 0 is represented by sending no pulse for a duration of 1 second. The signals are transmitted over a AWGN channel with zero mean and power spectral density \( 1/2 \). Let \( y \) denote the output of the integrator in Figure 1.

![Figure 1: Receiver for the PCM System with On-off Keying](image)

(a) For equiprobable bit-transmission, \( p_0 = p_1 = 1/2 \). To find the optimum threshold \( \lambda \) that minimizes the probability of error, we need to solve the following equation

\[
\frac{p_0}{p_1} = 1 = \frac{f_Y(\lambda_{opt}|1)}{f_Y(\lambda_{opt}|0)} \tag{1}
\]

Let us first find the density functions \( f_Y(y|1) \) and \( f_Y(y|0) \)

When a 1 is transmitted

\[
Y = A + \int_0^1 w(t)dt
\]

It follows that \( y \) is a Gaussian random variable with \( E[Y|1] = A \), and variance

\[
\sigma^2_{Y|1} = E[\int_0^1 \int_0^1 w(t)w(u)dtdu] = \int_0^1 \int_0^1 \frac{1}{2} \delta(t-u)dtdu = \frac{1}{2}
\]

Therefore

\[
f_Y(y|1) = \frac{1}{\sqrt{\pi}} \exp(-y^2) \tag{2}
\]

Similarly, when a 0 is transmitted

\[
Y = 0 + \int_0^1 w(t)dt,
\]

\[
f_Y(y|0) = \frac{1}{\sqrt{\pi}} \exp(-y^2)
\]
and it follows that $y$ is a Gaussian random variable with $E[Y|0] = 0$, and variance

$$\sigma^2_{Y|0} = E[\int_0^1 \int_0^1 w(t)w(u)dtdu] = \int_0^1 \int_0^1 \frac{1}{2} \delta(t-u)dtdu = \frac{1}{2}$$

Therefore

$$f_Y(y|0) = \frac{1}{\sqrt{\pi}} \exp(-y^2)$$

(3)

Using equations (2) and (3) in equation (1), we get

$$1 = \frac{f_Y(\lambda_{opt}|1)}{f_Y(\lambda_{opt}|0)} = \frac{\exp(-(\lambda_{opt} - A)^2)}{\exp(-\lambda^2_{opt})}$$

Taking log on both sides and rearranging, we get

$$\lambda^2_{opt} = (\lambda_{opt} - A)^2$$

$$\Rightarrow \lambda_{opt} = A/2.$$  

I guess you could have guessed this answer knowing that either $A$ or $0$ was being transmitted with equal probability in AWGN of zero mean!

(b) Using the threshold in part (a), i.e., $\lambda = A/2$, we can evaluate the average probability of error for this receiver in terms of the the complementary error function erfc(x) as follows:

Consider a zero being transmitted, then the conditional probability of making an error is

$$P_{e0} = P(y > \frac{A}{2}|0) = \int_{A/2}^{\infty} \frac{1}{\sqrt{\pi}} \exp(-y^2) = \frac{1}{2} \text{erfc}(\frac{A}{2})$$

By symmetry it follows that $P_{e1} = P_{e0}$ ⇒

$$P_e = P_{e1} = P_{e0} = \frac{1}{2} \text{erfc}(\frac{A}{2})$$

5. The channel bandwidth is given to be $B = 60 \text{ KHz}$ and the bit rate is $R_b = 100 \text{ Kbps}$. The bit duration is therefore given as $T_b = 1/R_b = 10 \mu\text{sec}$. The signal bandwidth can be found as $W = \frac{1}{2T_b} = 50 \text{ kHz}$. Therefore, the raised cosine pulse should be designed such that its rolloff factor $\alpha$ satisfies

$$B = W(1 + \alpha)$$

$$\Rightarrow \alpha = 0.2$$

6. Consider a set of orthonormal basis functions $\{\phi_j(t)\}_{j=1}^N$. Let $w(t)$ be an AWGN process of zero mean and p.s.d. $\frac{N_0}{2}$. 

We need to show that the sequence \( \{w_j\}_{j=1}^N \) are i.i.d. Gaussian random variables, where

\[
  w_j = \int_0^T w(t)\phi_j(t)dt, \quad j = 1, \cdots, N.
\]

Since \( w(t) \) is a Gaussian process, it follows that \( w_j \) is a Gaussian random variable. Further, \( E[w_j] = 0 \), since \( w(t) \) is zero mean.

Consider the covariance function

\[
  Cov(w_j w_k) = E[w_j w_k] = E\left[\int_0^T w(t)\phi_j(t)dt \int_0^T w(t)\phi_k(t)dt\right]
\]

Rearranging the integrals \( \Rightarrow \)

\[
  Cov(w_j w_k) = E\left[\int_0^T \int_0^T w(t)\phi_j(t)w(u)\phi_k(u)dtdu\right]
\]

Taking the expectation inside the integral \( \Rightarrow \)

\[
  Cov(w_j w_k) = \int_0^T \int_0^T \phi_j(t)\phi_k(u)E[w(t)w(u)]dtdu
\]

But \( E[w(t)w(u)] = \frac{N_0}{2}\delta(t-u) \Rightarrow \)

\[
  Cov(w_j w_k) = \frac{N_0}{2} \int_0^T \phi_j(t)\phi_k(t)dt = 0
\]

\( \Rightarrow \) \( w_j \) and \( w_k \) are uncorrelated.

When \( j = k \), \( Cov(w_j w_j) = Var(w_j) = \frac{N_0}{2} \Rightarrow \) the random variables \( w_j \) have the same variance as well.

Therefore, the sequence \( \{w_j\}_{j=1}^N \) are uncorrelated and identically distributed. Since they are Gaussian, it follows that they are also independent.

7. We first observe that the signals \( \{s_i(t)\} \) for \( i = 1, 2, 3 \) are linearly independent.

The energy of signal \( s_1(t) \) is given as

\[
  E_1 = \int_0^T s_1^2(t)dt = 4,
\]

where \( T = 3 \). Therefore, the first basis function is

\[
  \phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}
\]

Based on the definition of the coefficients as

\[
  s_{ij} = \int_0^T s_i(t)\phi_j(t)dt, \quad \text{(4)}
\]
we can find that $s_{21} = -4$.
Based on definition of the function $g_i(t)$ as

$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij} \phi_j(t),$$

we can evaluate $g_2(t)$ as

$$g_2(t) = \begin{cases} 
-4, & 1 \leq t \leq 2 \\
0, & \text{otherwise}
\end{cases}$$

The second basis function is now given as

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t) dt}} = \begin{cases} 
-1, & 1 \leq t \leq 2 \\
0, & \text{otherwise}
\end{cases}$$

Using equation (4), we can now compute

$$s_{31} = 3, \ s_{32} = -3$$

Using the above coefficients in (5), we get

$$g_3(t) = \begin{cases} 
3, & 2 \leq t \leq 3 \\
0, & \text{otherwise}
\end{cases}$$

Hence, the third basis function is given as

$$\phi_3(t) = \frac{g_3(t)}{\sqrt{\int_0^T g_3^2(t) dt}} = \begin{cases} 
1, & 2 \leq t \leq 3 \\
0, & \text{otherwise}
\end{cases}$$

We can write the signals in terms of the basis functions as

$$s_1(t) = 2\phi_1(t)$$
$$s_2(t) = -4\phi_1(t) + 4\phi_2(t)$$
$$s_3(t) = 3\phi_1(t) - 3\phi_2(t) + 3\phi_3(t)$$

8. Consider the set of signals $\{s_i(t)\}_{i=1}^{i=4}$, where the signal $s_i(t)$ is of the form

$$s_i(t) = \begin{cases} 
\sqrt{\frac{2E}{T}} \cos(2\pi \frac{t}{T} + i \frac{\pi}{4}), & 0 \leq t \leq T \\
0, & \text{otherwise}
\end{cases}$$

Observe that using the cosine formula $\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b)$, we can write each of the above signals as
Therefore each signal can be written as a weighted sum of the two functions \( \cos(2\pi t/T) \) and \( \sin(2\pi t/T) \). Do these two functions make an orthonormal basis?

They do if we choose \( \phi_1(t) = \sqrt{2/T} \cos(2\pi t/T) \) and \( \phi_2(t) = \sqrt{2/T} \sin(2\pi t/T) \), since we can easily verify that

\[
\int_0^T \phi_i(t)\phi_j(t)dt = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}
\]

Therefore, each of the signals can now be written as

\[
s_i(t) = \begin{cases} \sqrt{E} \cos(i\pi/4) \phi_1(t) - \sqrt{E} \sin(i\pi/4) \phi_2(t), & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}
\]

Therefore the coefficients in the expansion are

\[
s_{11} = \sqrt{E} \cos(\pi/4) = \sqrt{E}/2, \quad s_{12} = -\sqrt{E} \sin(\pi/4) = -\sqrt{E}/2
\]

\[
s_{21} = \sqrt{E} \cos(2\pi/4) = 0, \quad s_{22} = -\sqrt{E} \sin(2\pi/4) = -\sqrt{E}
\]

\[
s_{31} = \sqrt{E} \cos(3\pi/4) = -\sqrt{E}/2, \quad s_{32} = -\sqrt{E} \sin(3\pi/4) = -\sqrt{E}/2
\]

\[
s_{41} = \sqrt{E} \cos(4\pi/4) = -\sqrt{E}, \quad s_{42} = -\sqrt{E} \sin(4\pi/4) = 0
\]