Wireless Communication Technologies

ECE559 (Advanced Topics in Communications Engineering)
Lecture 7 (February 13, 2002)
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Effect of Co-channel Interference

Multiple Ricean or Rayleigh Interferers

In microcellular environments, the received signal (at the mobile) often consists of a desired direct line of sight (LOS) component, accompanied by a diffuse component. In this case the envelope of the received signal experiences Ricean fading. In the same environment the co-channel signals can be assumed to be Rayleigh faded because a direct line of sight between the co-channel cells is not likely to exist and the propagation path lengths are much longer. The probability of co-channel interference (also called as outage probability) is derived for the case of only fading.

s_0 : instantaneous signal power of the desired base station

s_k : instantaneous interfering signal power of the k^{th} base station ($k = 1,2,......., N_I$)

N_I : number of co-channel base stations

$\lambda = \dfrac{s_0}{\sum_{k=1}^{N_I} s_k}$ is the signal to interference ratio (SIR)

λ_{th} : receiver threshold

$b_k = \dfrac{\Omega_0}{(K+1)\Omega_k}$ where K is the Ricean factor

$\Omega_k = E[s_k]$ is the expected value of s_k

$$\lambda_k = \frac{1}{\lambda_{th}\Omega_k}$$

$$\Lambda = \prod_{k=1}^{N_I} \lambda_k$$

s_0 has non-central chi-square distribution with two degrees of freedom.

$$P_{S_0}(x) = \frac{(K+1)}{\Omega_0}\exp\left\{-K - \frac{(K+1)x}{\Omega_0}\right\} I_0\left(2\sqrt{\frac{K(K+1)x}{\Omega_0}}\right) \quad x \geq 0$$

s_k has exponential distribution

$$P_{S_k}(x) = \frac{1}{\Omega_k}\exp\left\{-\frac{x}{\Omega_k}\right\} \qquad x \ge 0$$

$P_{out} = P(\lambda < \lambda_{th})$ is the probability of co-channel interference (outage probability)

Independent co-channel interferers

$$P_{out} = P(s_0 < \lambda_{th}\sum_{k=1}^{N_I} s_k) = 1 - P(\lambda_{th}\sum_{k=1}^{N_I} s_k \le s_0)$$

let $y = \lambda_{th}\sum_{k=1}^{N_I} s_k = \sum_{k=1}^{N_I} y_k$

$$P_Y(y) = \sum_{k=1}^{N_I}\prod_{j=1, j\ne k}^{N_I}\frac{\Lambda}{(\lambda_j - \lambda_k)}\exp(-\lambda_k y) \qquad y \ge 0$$

$$P(y \le s_0) = \int_0^{\infty}\left(\int_0^{s_0} P_Y(y)dy\right)P_{S_0}(s_0)ds_0$$

$$P(y \le s_0) = \sum_{k=1}^{N_I}\left(1 - \frac{\lambda_{th}}{\lambda_{th} + b_k}\exp\left\{-\frac{Kb_k}{\lambda_{th} + b_k}\right\}\right)\prod_{j=1, j\ne k}^{N_I}\frac{b_j}{b_j - b_k}$$

$$P_{out} = 1 - \sum_{k=1}^{N_I}\left(1 - \frac{\lambda_{th}}{\lambda_{th} + b_k}\exp\left\{-\frac{Kb_k}{\lambda_{th} + b_k}\right\}\right)\prod_{j=1, j\ne k}^{N_I}\frac{b_j}{b_j - b_k}$$

When $N_I = 1$, P_{out} is given by

$$P_{out} = \frac{\lambda_{th}}{\lambda_{th} + b_1}\exp\left\{-\frac{Kb_1}{\lambda_{th} + b_1}\right\}$$

For $\lambda_{th} = 10.0dB$ the probability of co-channel interference with a single interferer when the desired signal is Ricean faded with different Rice factors and the interfering signal is Rayleigh faded is shown in **Figure 1**.

Independent, identically distributed co-channel interferers

In this case $P_Y(y)$ has the Gamma density function.

$$P_Y(y) = \frac{y^{N_I - 1}}{(\lambda_{th}\Omega_1)^{N_I}(N_I - 1)!}\exp\left\{-\frac{y}{\lambda_{th}\Omega_1}\right\} \qquad y \ge 0$$

The probability of co-channel interference is derived as

$$P_{out} = \frac{\lambda_{th}}{\lambda_{th}+b_1}\exp\left\{-\frac{Kb_1}{\lambda_{th}+b_1}\right\}\sum_{k=0}^{N_I-1}\left(\frac{b_1}{\lambda_{th}+b_1}\right)^k \sum_{m=0}^{k} {}^kC_m \frac{1}{m!}\left(\frac{K\lambda_{th}}{\lambda_{th}+b_1}\right)^m$$

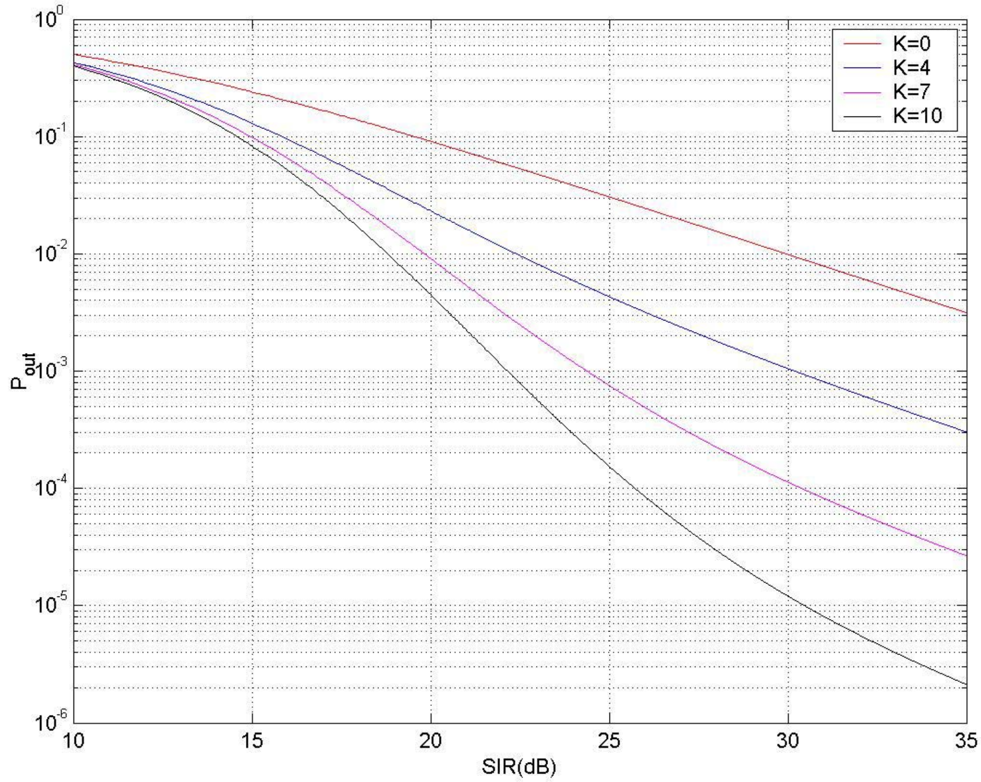

Figure 1

Modulated Signals and their Power Spectral Density (PSD)

Modulation is the process of transmitting the message in the amplitude, frequency, phase (or a combination of these) of the radio carrier, in either digital or analog form. Digital modulation offers many advantages over analog modulation. Some advantages include greater noise immunity, use of error control codes which detect and/or correct transmission errors, greater security, easier multiplexing of various forms of data (e.g video, data and voice) and sophisticated signal processing techniques like equalization to improve the overall communication link performance. It is desirable to use bandwidth and power resources most efficiently.

Several factors influence the choice of a digital modulation scheme. A desirable modulation scheme provides

- Low bit error rates (BER) at low received signal to noise ratios (SNR)
- Easy and cost effective to implement

- Occupies minimum bandwidth
- Performs well in multipath fading environments

Existing modulation schemes do not simultaneously satisfy all of these requirements. Some modulation schemes are better in terms of the bit-error rate performance, while others are better in terms of bandwidth efficiency. Depending on the application, trade-off is made when selecting a digital modulation. The performance of a modulation scheme is often measured in terms of its *power efficiency* η_p and *bandwidth efficiency* η_B.

Power Efficiency

It is a measure of tradeoff between bit-error rate (BER) achieved by a modulation scheme and the signal power required to achieve that. η_p is formally defined as the ratio of signal energy per bit E_b, to noise power spectral density N_0, to achieve a certain probability of error P_e.

$$\eta_p = \frac{E_b}{N_0} \quad \text{to achieve, say } P_e = 10^{-6}$$

Bandwidth Efficiency

This describes the ability of a modulation scheme to accommodate data within a limited bandwidth. In general, increasing the data rate implies increasing the bandwidth of the signal. Some modulation formats have a better trade-off than others. Bandwidth efficiency reflects how efficiently the allocated bandwidth is utilized and is defined as the ratio of the data rate R to the bandwidth B occupied by the modulated RF signal.

$$\eta_B = \frac{R}{B} bps / Hz$$

A modulation scheme with a greater value of η_B will transmit more data in a given spectrum allocation. But there is a fundamental upper bound on achievable bandwidth efficiency given by Shannon's channel capacity formula

$\eta_{B\max} = \frac{C}{B} = \log_2\left(1+\frac{S}{N}\right)$ where C is the channel capacity (in bps) and $\frac{S}{N}$ is the signal to noise ratio.

In the design of a digital communication system, there is a trade off between bandwidth efficiency and power efficiency. For example introducing channel coding increases the power efficiency and decreases the bandwidth efficiency where as higher level modulation schemes (M-ary keying) decrease power efficiency and increase bandwidth efficiency.

Definition of Bandwidth (B)

Figure 2 shows the plot of power spectral density $S(f)$ of a modulated signal. There are different definitions for the bandwidth of a modulated signal given as:

- Absolute Bandwidth**:** The range of frequencies for which $S(f) \neq 0$
- Null-Null Bandwidth**:** Width of the main lobe of $S(f)$
- Half Power (3-dB) Bandwidth**:** Interval between frequencies at which the psd has dropped to half power
- FCC Defined Bandwidth**:** FCC defines bandwidth as that band which leaves exactly 0.5% above the band and 0.5% below the band. That is 99% of signal power is contained within occupied bandwidth.

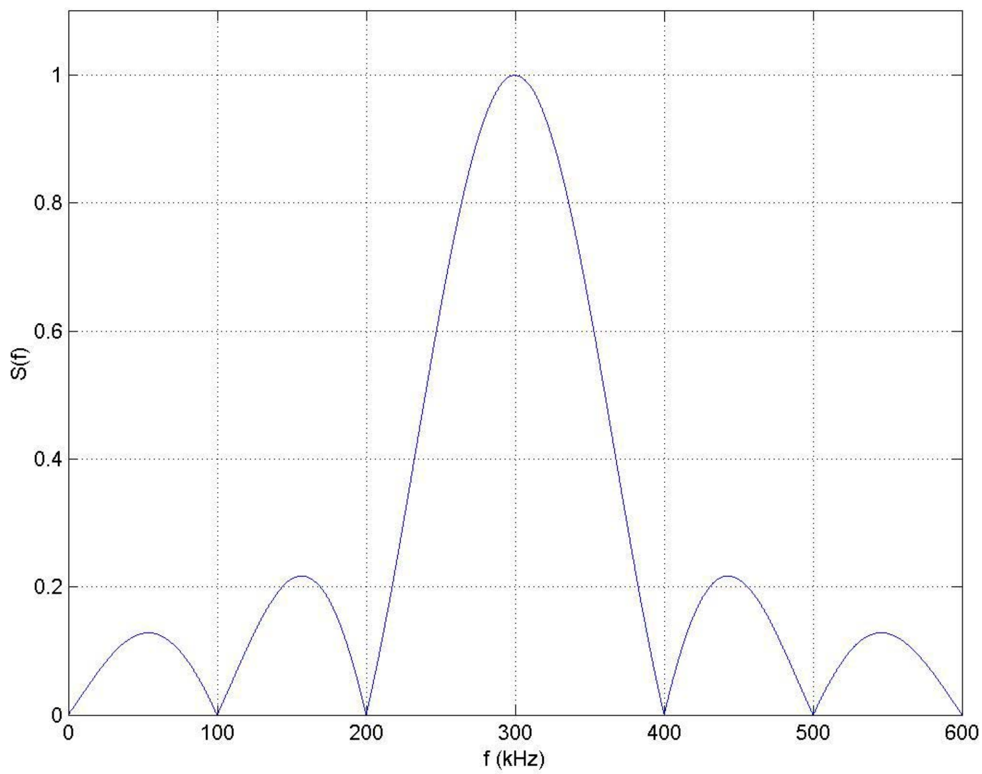

Figure 2

Standard Representation of Digitally modulated signal

For any digital modulation scheme, $v(t)$ and $s(t)$ can be written in the *standard form* as

$$v(t) = A\sum_k b(t - kT, X_k)$$
$$s(t) = \mathrm{Re}\{v(t)\exp(j(2\pi f_c t))\}$$

A is the carrier amplitude
$X_k = (x_k, x_{k-1}, \ldots\ldots, x_{k-L})$ is the source symbol sequence
L is the memory length
T is the symbol duration
$b(t, X_i)$ is an *equivalent shaping function* of duration T

The band-pass waveform $s(t)$ can be represented in the *quadrature form*

$$s(t) = v_I(t)\cos(2\pi f_c t) - v_Q(t)\sin(2\pi f_c t) \text{ where}$$

$$v(t) = v_I(t) + jv_Q(t)$$

The waveforms $v_I(t)$ and $v_Q(t)$ are known as the in-phase and quadrature components, respectively of $v(t)$.

In *envelope phase* form

$$s(t) = a(t)\cos(2\pi f_c t + \phi(t)) \text{ where}$$

$$a(t) = \sqrt{(v_I(t))^2 + (v_Q(t))^2} \text{ and}$$

$$\phi(t) = \tan^{-1}\left(\frac{v_Q(t)}{v_I(t)}\right)$$

Power spectral density evaluation for band-pass signals

A modulated bandpass signal can be written in the form

$$s(t) = \mathrm{Re}\{v(t)\exp(j(2\pi f_c t + \phi_T))\}$$

ϕ_T is a random phase uniformly distributed over $[-\pi, \pi]$

$$s(t) = \frac{1}{2}\{v(t)\exp(j(2\pi f_c t + \phi_T)) + v^*(t)\exp(-j(2\pi f_c t + \phi_T))\}$$

The auto-correlation function of $s(t)$ is given by

$$\Phi_{ss}(\tau) = E[s(t)s(t+\tau)]$$

$$\Phi_{ss}(\tau) = \frac{1}{4}E\left[v(t)v(t+\tau)\exp\left(j(4\pi f_c t + 2\pi f_c \tau + 2\phi_T)\right)\right] + \frac{1}{4}E\left[v(t)v^*(t+\tau)\exp(-j2\pi f_c \tau)\right] +$$
$$\frac{1}{4}E\left[v^*(t)v^*(t+\tau)\exp\left(-j(4\pi f_c t + 2\pi f_c \tau + 2\phi_T)\right)\right] + \frac{1}{4}E\left[v^*(t)v(t+\tau)\exp(j2\pi f_c \tau)\right]$$

The complex base band signal is $v(t)$ is independent of the random phase ϕ_T. $E[\ ]$ denotes the ensemble average. Therefore

$$\Phi_{ss}(\tau) = \frac{1}{2}\Phi_{vv}^*(\tau)\exp(-j2\pi f_c \tau) + \frac{1}{2}\Phi_{vv}(\tau)\exp(j2\pi f_c \tau)$$

The power spectral density $S_{ss}(f)$ is the Fourier transform of $\Phi_{ss}(\tau)$.

$$S_{ss}(f) = \frac{1}{2}S_{vv}(f - f_c) + \frac{1}{2}S_{vv}^*(-f - f_c)$$

Since $\Phi_{vv}(\tau) = \Phi_{vv}^*(-\tau)$, the above equation reduces to

$$S_{ss}(f) = \frac{1}{2}S_{vv}(f - f_c) + \frac{1}{2}S_{vv}(f + f_c)$$

The above expression shows that the psd of the band-pass waveform $s(t)$ is completely determined by the psd of its complex envelope $v(t)$.

$\Phi_{vv}(t+\tau, t)$ for the standard form representation of $v(t)$ is given by

$$\Phi_{vv}(t+\tau, t) = \frac{1}{2}E[v(t+\tau)v^*(t)]$$

$$\Phi_{vv}(t+\tau, t) = \frac{A^2}{2}\sum_i \sum_k E[b(t+\tau - iT, X_i)b^*(t - kT, X_k)]$$

Under the assumption that the source sequence is a stationary random process we can write the above equation as

$$\Phi_{vv}(t+T+\tau, t+T) = \frac{A^2}{2}\sum_i \sum_k E[b(t+T+\tau - iT, X_i)b^*(t+T - kT, X_k)]$$

$$\Phi_{vv}(t+T+\tau,t+T)=\frac{A^2}{2}\sum_{i'}\sum_{k'}E[b(t+\tau-i'T,X_{i'})b^*(t-k'T,X_{k'})]$$

$$=\Phi_{vv}(t+\tau,t)$$

The autocorrelation function $\Phi_{vv}(t+\tau,t)$ of $v(t)$ is periodic in t with period T. That is $v(t)$ is a *cyclostationary random process*. The autocorrelation $\Phi_{vv}(\tau)$ can be obtained by taking the time average of $\Phi_{vv}(t+\tau,t)$, given by

$$\Phi_{vv}(\tau)=\frac{A^2}{2}\sum_{i}\sum_{k}\frac{1}{T}\int_0^T E[b(t+\tau-iT,X_i)b^*(t-kT,X_k)]dt$$

$$=\frac{A^2}{2T}\sum_{i}\sum_{k}\int_{-kT}^{-(k-1)T} E[b(z+\tau-(i-k)T,X_i)b^*(z,X_k)]dz$$

$$\Phi_{vv}(\tau)=\frac{A^2}{2T}\sum_{m}\sum_{k}\int_{-kT}^{-(k-1)T} E[b(z+\tau-mT,X_{m+k})b^*(z,X_k)]dz$$

$$=\frac{A^2}{2T}\sum_{m}\sum_{k}\int_{-kT}^{-(k-1)T} E[b(z+\tau-mT,X_m)b^*(z,X_0)]dz$$

$$=\frac{A^2}{2T}\sum_{m}\int_{-\infty}^{\infty} E[b(z+\tau-mT,X_m)b^*(z,X_0)]dz$$

The power spectral density of $v(t)$ is given by

$$S_{vv}(f)=\frac{A^2}{2T}\int_{-\infty}^{\infty}\left(\sum_{m}\int_{-\infty}^{\infty} E[b(z+\tau-mT,X_m)b^*(z,X_0)]dz\right)\exp(-j2\pi f\tau)d\tau$$

$$=\frac{A^2}{2T}\sum_{m}E\left[\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} b(z+\tau-mT,X_m)b^*(z,X_0)dz\right)\exp(-j2\pi f\tau)d\tau\right]$$

$$=\frac{A^2}{2T}\sum_{m}E\left[\left(\int_{-\infty}^{\infty} b(z+\tau-mT,X_m)\exp(-j2\pi f(z+\tau-mT)d\tau\right)\left(\int_{-\infty}^{\infty} b^*(z,X_0)\exp(j2\pi fz)dz\right)\exp(-j2\pi fmT)\right]$$

$$=\frac{A^2}{2T}\sum_{m}E\left[\left(\int_{-\infty}^{\infty} b(u,X_m)\exp(-j2\pi fu)du\right)\left(\int_{-\infty}^{\infty} b^*(z,X_0)\exp(j2\pi fz)dz\right)\right]\exp(-j2\pi fmT)$$

$$S_{vv}(f) = \frac{A^2}{2T} \sum_m E\left[B(f, X_m) B^*(f, X_0)\right] \exp(-j2\pi f m T)$$

The above equation shows that $S_{vv}(f)$ depends on the correlation properties of the source sequence and the form of the equivalent shaping function $b(t, X_m)$. $B(f, X_m)$ is the Fourier transform of $b(t, X_m)$.

References

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ECE559 Wireless Communication Technologies

Instructor: Prof. Narayan B Mandayam
Lecture 8 (February 18,2002)
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1. Representation of Modulation Formats

1.1 Amplitude Shift Keying (ASK):

ASK is a linear modulation scheme with non constant envelope. ASK has the advantage of being more spectrally efficient than other modulation schemes having constant envelope. However amplitude non linearity degrades the
performance of ASK. The complex envelope can be represented as follows,

$$v(t) = A\sum_n x_n h_a(t - nT) \quad (1)$$

where $\{x_n\}$ = complex source symbol sequence = $\{x_n^{\ I} + jx_n^{\ Q}\}$ (2)

$h_a(t)$ = amplitude shaping pulse

Expressed in the standard form, $v(t) = A\sum_k b(t - kT, \underline{x}_k)$ (3)

$b(t, \underline{x}_k) = x_k h_a(t)$ (memory less modulation)

ASK band-pass signal has the following quadrature representation,

$s(t) = \mathrm{Re}\{v(t)e^{j2\pi f_c t}\} = v^I(t)\cos(2\pi f_c t) - v^Q(t)\sin(2\pi f_c t)$ where (4)

$v(t) = v_I(t) + jv_Q(t)$

$v_I(t)$ and $v_Q(t)$ are the in-phase and quadrature phase components of the complex low-pass equivalent signal.

Also, $s(t) = \mathrm{Re}\{A.\sum_n (x_n^{\ I} + jx_n^{\ Q})h_a(t - nT).(\cos(2\Pi fct + j\sin 2\Pi fct)\}$

$$= A.\sum_n (x_n^{\ I} h_a(t - nT)\cos(2\Pi fct) - x_n^{\ Q} h_a(t - nT)\sin(2\Pi fct)) \quad (5)$$

and the envelope phase representation is,

$$= A\sum_n |x_n| h_a(t - nT)\cos(2\Pi fct + \theta_n)$$

where $|x_n| = \sqrt{(x_n^{\ I})^2 + (x_n^{\ Q})^2}$ and $\theta_n = \tan^{-1}(\frac{x_n^{\ Q}}{x_n^{\ I}})$. (6)

Thus note that both amplitude and phase of the ASK signal depend on the complex symbol. However, in the simplest form of ASK which is the familiar PAM(Pulse Amplitude Modulation), only the signal amplitude is varied according to the source symbol. This is accomplished by making $x_n = x_n^{\ I}$ real.On-Off Keying(OOK) is a special case of M-PAM with M=2 and $x_n = \{0,1\}$

The basis functions are : $\phi_1(t) = \sqrt{\frac{2}{T}} h_a(t)\cos(2\pi f_c t) \quad 0 \le t \le T \quad$; 0 otherwise (7)

$$\phi_2(t)=\sqrt{\frac{2}{T}}h_a(t)\sin(2\pi f_c t) \qquad 0\le t\le T \qquad ;\ 0 \text{ otherwise} \tag{8}$$

Using these basis functions, we can represent $s_n(t)$ as follows:

$$s_n(t)=\sqrt{E_A}x_n^I\phi_1(t)+\sqrt{E_A}x_n^Q\phi_2(t), \tag{9}$$

where $E_A=\frac{A^2T}{2}$. The symbol energy is $E_m=E_A|x_m|^2$.

A popular form of ASK is M-ary QAM, where the source symbols $x_n=x_n^I+jx_n^Q$ are chosen from an M-ary constellation such that

$$x_n^I, x_n^Q \in \{\pm1,\pm3,.........\pm(N-1)\},\ N=\sqrt{M} \tag{10}$$

The signal space diagram for 16-QAM would look like Fig 1 below.

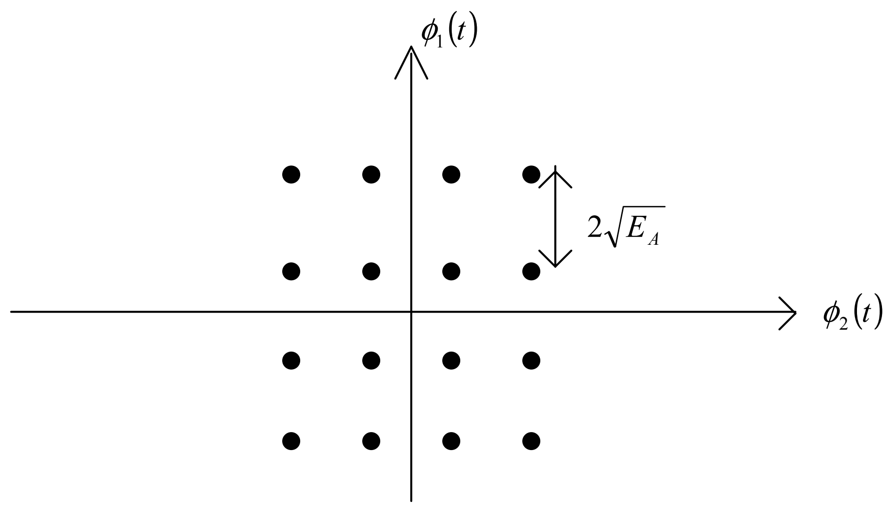

Fig 1: Signal constellation for 16-QAM

<u>PSD of ASK</u>

Since $b(t,\underline{x}_k)=x_k h_a(t),\ B(f,\underline{x}_k)=x_k H_a(f)$. (11)

From (16),

$$S_{vv}(f)=\frac{A^2}{T}\sum_m E[B(f,\underline{x}_m)B^*(f,\underline{x}_0)]=\frac{A^2}{T}\sum_m E\left[x_m x_0^*|H_a(f)|^2\right] \tag{12}$$

$$=\frac{A^2}{T}|H_a(f)|^2\sum_m E[x_m x_0] \tag{13}$$

If the source symbols are zero mean and uncorrelated, we get

$$S_{vv}(f)=\frac{A^2}{T}|H_a(f)|^2\sigma_x^2 \tag{14}$$

where $\sigma_x^2=E[x^2]$ (since mean in zero) is the variance of the source symbols.

The amplitude shaping pulse $h_a(t)$ is very often chosen to be the square root raised cosine pulse

$$h_a(t) = 4\beta \frac{\cos\left[\frac{(1+\beta)\pi t}{T}\right] + \sin\left[\frac{(1-\beta)\pi t}{T}\right]\left[\frac{4\beta t}{T}\right]^{-1}}{\pi\sqrt{T}\left[1 - \frac{16\beta^2 t^2}{T^2}\right]} \tag{15}$$

where β is the roll-off factor.

The corresponding Fourier Transform, $H_a(f)$, is

$$H_a(f) = \begin{cases} \sqrt{T} & 0 \le |f| \le \frac{(1-\beta)}{2T} \\ \sqrt{\frac{T\left[1 - \sin\left(\frac{\pi T}{\beta}\left(f - \frac{1}{2T}\right)\right)\right]}{2}} & \frac{(1-\beta)}{2T} \le |f| \le \frac{(1+\beta)}{2T} \end{cases} \tag{16}$$

In practice, the pulse $h_a(t)$ is truncated to length τ yielding the new pulse

$$h_a^{'}(t) = h_a(t) rect(t/\tau) \tag{17}$$

The corresponding Fourier Transform becomes

$$H_a^{'}(f) = H_a(f) \otimes \tau \sin c(f\tau) \tag{18}$$

Pulse truncation leads to sidelobe regeneration. The PSD of ASK is plotted below.

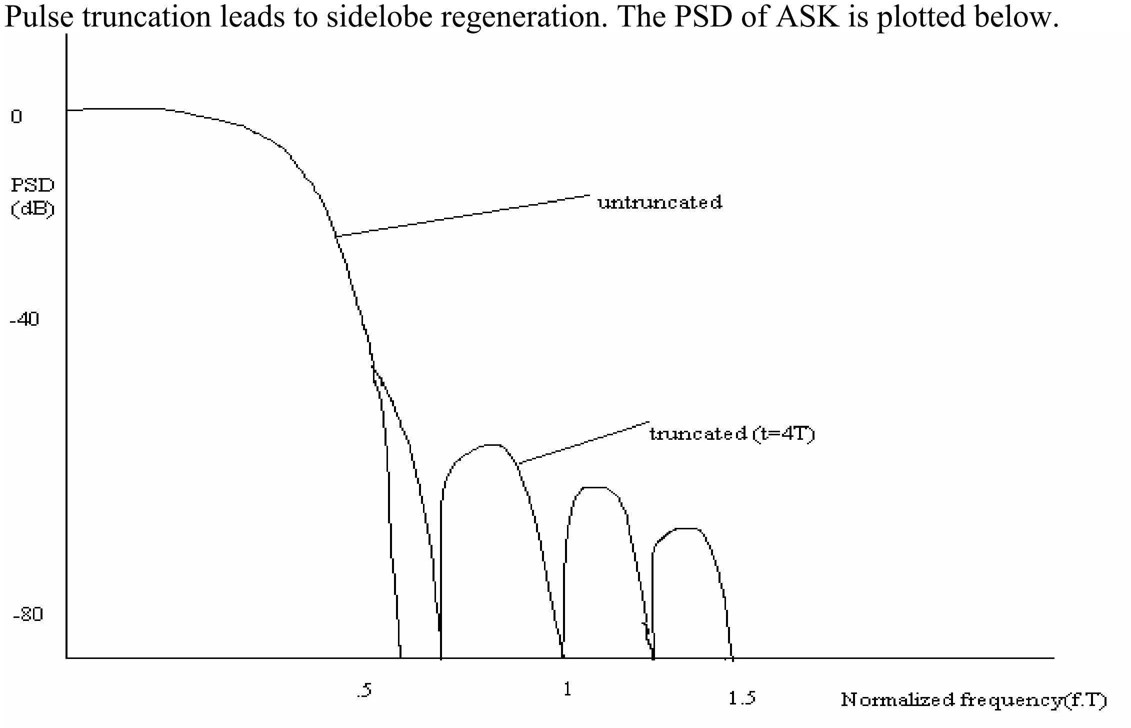

Fig2: PSD of ASK using square root raised cosine pulse shape with $\beta = 0.5$

1.2 Orthogonal Frequency Division Multiplexing (OFDM)

Multipath causes considerable InterSymbol Interference (ISI) in single carrier systems, especially at high bit rates. OFDM (a multicarrier system) is a block modulation scheme designed to combat the effect of multipath frequency selective fading. In an OFDM system, a block of N serial source symbols (each of duration Ts) is converted into a block of N parallel modulated symbols(each of duration $T = NTs$).Typically, N is chosen such that $NTs >> \tau$,where τ is the rms delay spread. Seen in the time domain, this N fold increase in the symbol duration, reduces the effect of multi-path delay spread. Correspondingly, in the frequency domain, the wideband frequency selective channel is decomposed into N narrowband channels, such that each parallel channel encounters almost flat fading. Each of the N source symbols is transmitted in parallel by employing N orthogonal subcarriers. Such a scheme has practical advantages, because it may reduce or even eliminate the need for equalization. OFDM is being used in Digital Audio Broadcasting (DAB) and Digital Video Broadcasting(DVB) in Europe and in the IEEE 802.11a WLAN standard. It is being considered for use in broadband communications for 4th generation wireless systems.

The complex envelope of an OFDM signal is described by

$$v(t) = A\sum_{k}\sum_{n=0}^{N-1} x_{k,n}\phi_n(t-kT) \qquad (19)$$

where $\phi_n(t)$ are the orthonormal waveforms given by

$$\phi_n(t) = h_a(t)\exp\left\{\frac{j2\pi\left(n-\frac{N-1}{2}\right)t}{T}\right\},\ 0 \le t \le T,\ 0 \le n \le (N-1) \qquad (20)$$

If $h_a(t) = u_T(t)$(rectangular pulse from 0 to T), $H_a(f)$ is a sinc function with nulls at multiples of $1/T$. Thus, the N subcarriers which are placed at intervals of $1/T$ overlap (in the time domain), but are orthogonal (in the frequency domain) as the peak of each subcarrier coincides with the nulls of all the other subcarriers. Thus, $\{\phi_n(t)\}_{n=0}^{N-1}$ are orthogonal.

At time epoch k , N source symbols are transmitted using N distinct subcarriers. Since OFDM is typically used for high data rates, $x_{k,n}$ are usually chosen from a QAM constellation. It is to be noted that OFDM is just a multiplexing technique and does not specify a modulation scheme and any modulation scheme can be used. Infact, by knowing (estimating) the channel, different modulation schemes can be employed in the different subcarriers (adaptive modulation) to give improved performance.

The OFDM signal can also be expressed in the standard form as

$$v(t) = A\sum_{k} b(t-kT, \underline{x}_k) \qquad (21)$$

where $b(t,\underline{x}_k) = h_a(t)\sum_{n=0}^{N-1} x_{k,n} \exp\left\{\frac{j2\pi\left(n-\frac{N-1}{2}\right)t}{T}\right\}$, $0 \le t \le T$

$\underline{x}_k = (\underline{x}_{k,1}, \underline{x}_{k,2}, \ldots\ldots\ldots, \underline{x}_{k,N-1})$ is source symbol at time epoch k

$h_a(t) = u_T(t) \equiv$ rectangular pulse from 0 to T

From the notation it is clear that N source symbols are transmitted in parallel.

Basis functions:

$$\phi_n(t) = h_a(t)\exp\left\{\frac{j2\pi\left(n-\frac{N-1}{2}\right)t}{T}\right\} \quad , \ n = 0,1,\ldots\ldots\ldots,(N-1) \ ; \quad 0 \le t \le T \tag{22}$$

One of the major advantages of OFDM is that the modulation can be performed in the discrete domain using an Inverse Discrete Fourier Transform (IDFT) or the more computationally efficient Inverse Fast Fourier Transform (IFFT). To illustrate this, consider $k=0$, ignore the frequency offset term $\exp\left\{\frac{-j2\pi(N-1)t}{T}\right\}$ and let $h_a(t) = u_T(t)$.

Then,

$$v(t) = A\sum_{n=0}^{N-1} x_{0,n} \exp\left\{\frac{j2\pi nt}{NTs}\right\}, 0 \le t \le NTs, \tag{23}$$

by letting $T = NTs$.

If we sample $v(t)$ at instants kTs (satisfying Nyquist criterion), we get

$$X_{o,k} = v(kTs) = A\sum_{n=0}^{N-1} x_{0,n} \exp\left\{\frac{2j\pi kn}{N}\right\}, k = 0,1,2,\ldots\ldots(N-1) \tag{24}$$

It is easily seen that $\{X_{o,k}\}_{k=0}^{N-1}$ is the IFFT of the block $A\underline{x}_0$

Thus, the OFDM transmitter will look like Fig 3 below.

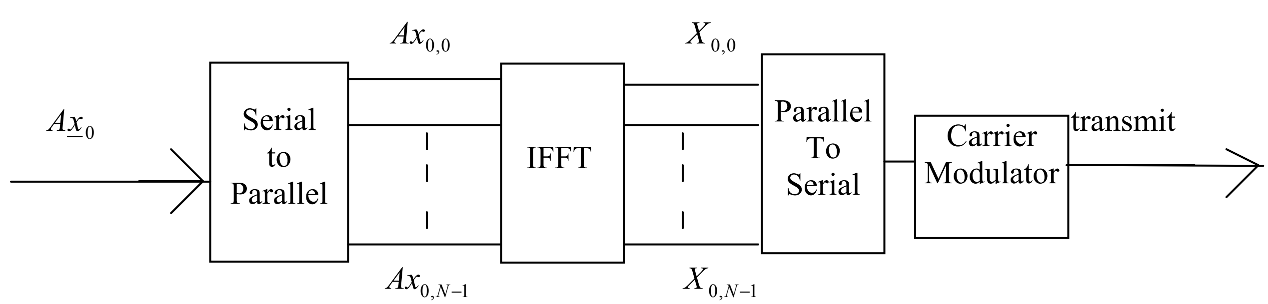

Fig 3: OFDM Transmitter

The receiver would similarly involve DFT of FFT after getting the signal to baseband. Though ISI is substantially reduced, it can be almost completely eliminated by using a guard band at the beginning of the OFDM symbol. The length of the guard band is obviously chosen such that it is greater than the rms delay spread. It is to be noted that increasing the symbol duration to counter multipath makes an OFDM system more prone to fast fading.

PSD of OFDM

The PSD of OFDM can be obtained by treating OFDM as independent modulation on orthogonal subcarriers that are separated in frequency by $1/T$. For a signal constellation with zero mean and with amplitude shaping function $h_a(t)$, the PSD of the complex envelope is

$$S_{vv}(f)=\frac{A^2}{T}\sigma_x^2\sum_{n=0}^{N-1}\left|H_a\left(f-\frac{1}{T}\left(n-\frac{N-1}{2}\right)\right)\right|^2, \text{ where} \quad (25)$$

$$\sigma_x^2=\frac{1}{2}E\left[\left|x_{k,n}\right|^2\right]$$

(Since $b(t,\underline{x}_k)_n = h_a(t)x_{k,n}\exp\left\{\dfrac{j2\pi\left(n-\frac{N-1}{2}\right)t}{T}\right\}$ for the n^{th} subcarrier, (26)

$$B(f,\underline{x}_k)_n=\left|H_a(f)\right|\otimes x_{k,n}\delta\left(f-\left(\frac{n-\frac{N-1}{2}}{T}\right)\right)=x_{k,n}\left|H_a\left(f-\frac{1}{T}\left(n-\frac{N-1}{2}\right)\right)\right| \quad (27)$$

If $h_a(t)=u_T(t)$ is used, $\left|H(f)=T\sin c(fT)\right|$, where $\sin c(x)=\sin(\pi x)/(\pi x)$

The PSD for this case is plotted for 3 different values of N in Fig 4 below

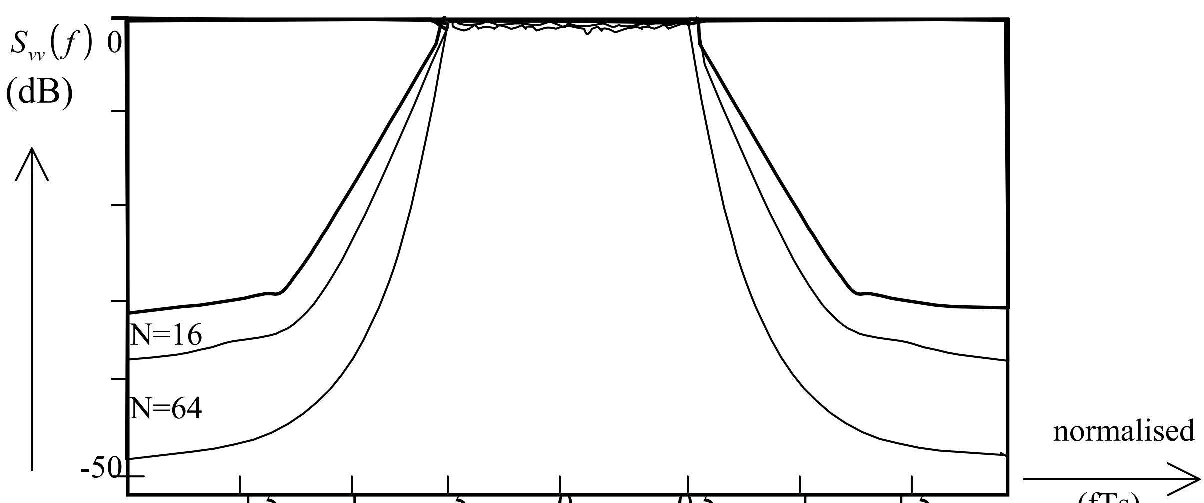

Fig 4: PSD of OFDM with different number of subcarriers

It is observed that for large values of N, the PSD becomes more flat in the $N/T=1/Ts$ bandwidth region. Therefore, as the block length N becomes large, the spectral efficiency approaches that of single carrier modulation with ideal Nyquist filtering. Further improvement in PSD can be obtained by using a square root raised cosine pulse, but it destroys the subcarrier orthogonality, which leads to degraded error rate performance.

1.3 Phase Shift Keying (PSK)

We consider the general case of M-ary PSK.

$$v(t) = A\sum_k b(t - kT, \underline{x}_k)$$

(28)

where $b(t, \underline{x}_k) = h_a(t)\exp\left\{\frac{j\pi x_k h_p(t)}{M}\right\}$

$\underline{x}_k = x_k$ (no memory) $\in \{\pm 1, \pm 3, ..., (\pm M - 1)\}$

$h_a(t)$ is the amplitude shaping function

$h_p(t)$ is the phase shaping function

M is the size of the constellation

The symbol energy $E_k = \frac{A^2 T}{2}$ is the same for every symbol and so is the amplitude. From the notation, it is clear that the information is contained in the phase, hence the name.

Often, the phase shaping pulse $h_p(t) = u_T(t)$, while the amplitude shaping pulse $h_a(t)$ is usually the square root raised cosine pulse, given as

$$h_a(t) = 4\beta \frac{\cos\left[\frac{(1+\beta)\pi t}{T}\right] + \sin\left[\frac{(1-\beta)\pi t}{T}\right]\left[\frac{4\beta t}{T}\right]^{-1}}{\pi\sqrt{T}\left[1 - \frac{16\beta^2 t^2}{T^2}\right]} \quad (29)$$

and the corresponding square root raised cosine spectrum is

$$H_a(f) = \begin{cases} \sqrt{T} & 0 \le |f| \le \frac{(1-\beta)}{2T} \\ \sqrt{\frac{T\left[1 - \sin\left(\frac{\pi T}{\beta}\left(f - \frac{1}{2T}\right)\right)\right]}{2}} & \frac{(1-\beta)}{2T} \le |f| \le \frac{(1+\beta)}{2T} \end{cases} \quad (30)$$

The signal space diagram of 8-ary PSK is shown in Fig 5 below

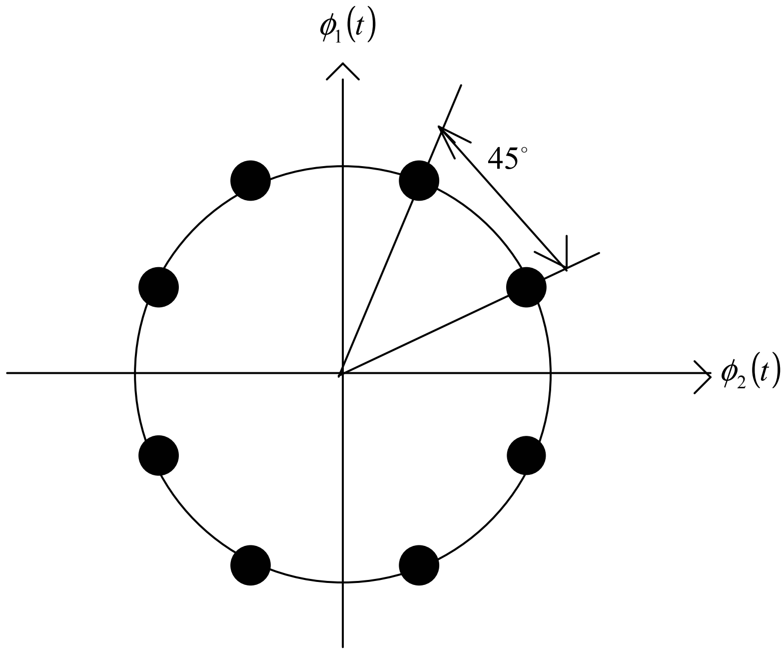

Fig 5: Signal space diagram of 8-ary PSK

PSD of PSK

It is assumed that the source symbols are uncorrelated and equally likely and defined by the set

$$x_n \in \{2i-1-M; i=1,2,........,M\} \tag{31}$$

Now,

$$b(t,\underline{x}_k) = h_a(t)\exp\left\{\frac{j\pi x_k h_p(t)}{M}\right\} \qquad \text{from} \qquad (28)$$

Taking expectation , we get

$$E[b(t,\underline{x}_k)] = h_a(t)\sin f\left(\frac{\pi h_p(t)}{M}\right) \tag{32}$$

where

$$\sin f(x) = \frac{\sin(Mx)}{M\sin(x)}$$

Recalling equation (15),

$$S_{vv}(f) = \frac{A^2}{T}\sum_m E\left[\int_{-\infty}^{\infty} b(\tau',\underline{x}_m)e^{-j2\pi f(\tau')}d\tau' \int_{-\infty}^{\infty} b^*(z,\underline{x}_0)e^{j2\pi fz}dz\right]e^{-j2\pi fmT}$$

$$= \frac{A^2}{T} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[b(t,\underline{x}_0)b^*(z,\underline{x}_0)\right]e^{-j2\pi f(z-\tau)}dzd\tau \text{(uncorrelated symbols)} \quad (33)$$

$$= \frac{A^2}{T} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[\frac{j\pi\left(h_p(\tau)-h_p(z)\right)x_0}{M}\right]h_p(\tau)h_a(z)e^{-j2\pi f(z-\tau)}dzd\tau \quad (34)$$

$$= \frac{A^2}{T} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sin(\pi\left[h_p(\tau)-h_p(z)\right]}{M\sin\left(\frac{\pi\left(h_p(\tau)-h_p(z)\right)}{M}\right)}h_a(\tau)h_a(z)e^{-j2\pi f(z-\tau)}dzd\tau \quad (35)$$

If the phase shaping function is $h_p(t)=u_T(t)$, we get

$$S_{vv}(f) = \frac{A^2}{T} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_a(\tau)h_a(z)e^{-j2\pi f(z-\tau)}dzd\tau = \frac{A^2}{T}\left|H_a(f)\right|^2 \quad (36)$$

on taking the limit in (60).
If the amplitude shaping function is $h_a(t)=u_T(t)$, we get

$$S_{vv}(f) = \frac{A^2}{T}\left[\frac{T\sin(\pi fT)}{(\pi fT)}\right]^2 = E\left[\frac{\sin(\pi fT)}{(\pi fT)}\right]^2 \text{,where } E=\frac{A^2T}{2} \quad (37)$$

To make a fair comparison on BW efficiency with different M, the frequency variable must be normalized by the bit interval T_b. For M-ary signaling,

$$T = T_b \log_2 M \quad (38)$$

Thus,

$$S_{vv}(f) = \frac{A^2T_b\log_2 M}{2}\left[\frac{\sin(\pi fT_b\log_2 M)}{\pi fT_b\log_2 M}\right]^2 \quad (39)$$

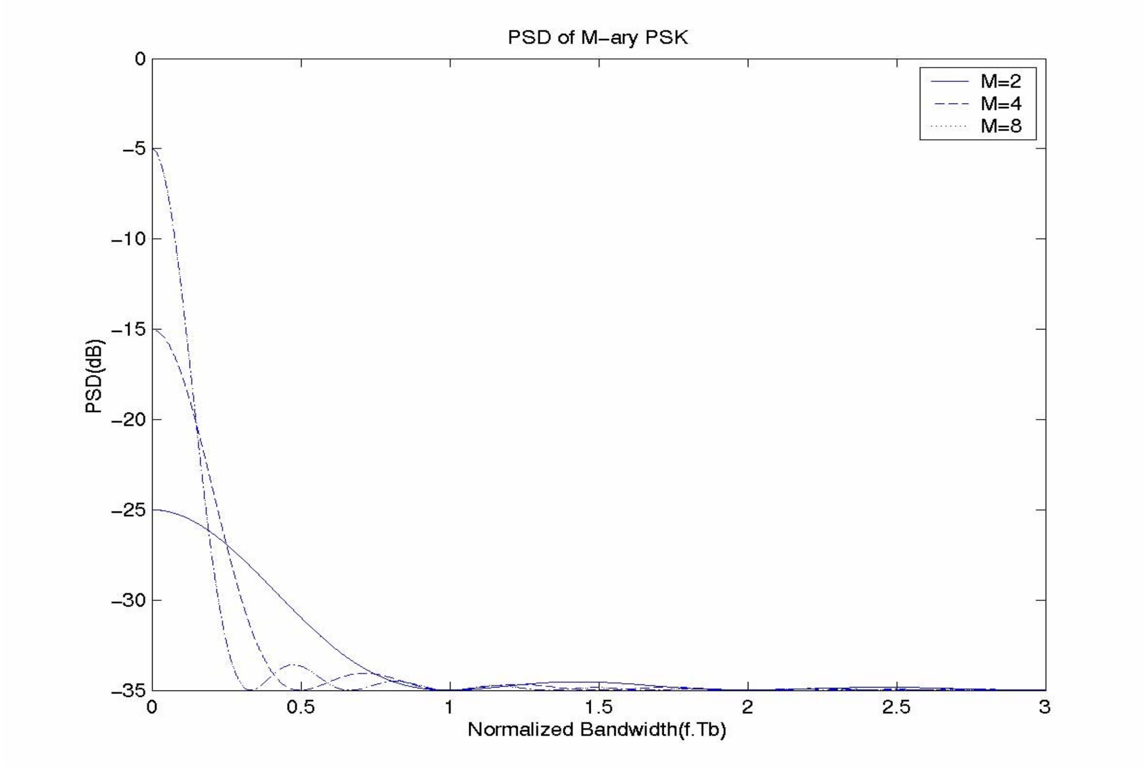

Fig 6: PSD of M-ary PSK for different M

Comparing the bandwidth efficiency for different M, we get the results, shown in Table 1 below, using the relation

$$\eta_\beta = \frac{R_b}{B}$$, where R_b is the bit rate and B is the null-to-null bandwidth

M	2	4	8	16	32	64
η_B	0.5	1	1.5	2	2.5	3

Table 1: Bandwidth efficiency depending on alphabet size

It is observed that with an increase in N, the bandwidth efficiency is improving.

References:

1. Gordon L. Stuber, Principles of Mobile Communication, 1996.
2. Simon Haykin ,Communication Systems, 1996.
3. John Proakis ,Digital Communications, 2000.
4. N.B Mandayam, Class notes,2002

Appendix: Signal Space Representation of Signals

This representation is used for time domain representation or characterization of the entire class of baseband waveforms. Any set of M finite energy signals $\{v_i(t)\}_{i=1}^{M}$ can be represented in terms of N orthonormal basis functions $\{\phi_n(t)\}_{n=1}^{N}$ where N<=M

Bases are orthonormal => $\int_0^T \phi_i(t)\phi_j^*(t)dt = \delta_{ij}$ where δ_{ij} is the Kronecker Delta Function.

Each of the M signals can be represented as

$$v_m(t) = \sum_{n=1}^{N} v_{mn}\phi_n(t);\ m = 1,2........,(M-1) \tag{1}$$

where $v_{mn} = \int_0^T v_m(t)\phi_n^*(t)dt$ is the projection of the signal $v_m(t)$ onto the basis function $\phi_n(t)$.

Thus, on fixing the orthonormal basis functions, a signal $v_m(t)$ can be represented in vector notation as follows:

$$v_m(t) \equiv \underline{v}_m = (v_{m1}, v_{m2},......., v_{mN}) \tag{2}$$

The bandpass waveforms $s_m(t)$ can also be represented in terms of a set of real orthonormal basis functions that are defined in the interval $[0,T]$. The corresponding vector notation would be:

$$s_m(t) \equiv \underline{s}_m = (s_{m1}, s_{m2},.........s_{mN^{*}}) \tag{3}$$

where we require N^* real orthonormal basis functions.

Signal Energy, Correlation and Euclidean Distance

Consider the bandpass waveform $s_m(t) = \mathrm{Re}\{v_m(t)e^{j2\pi f_c t}\}$ of finite duration $[0,T]$. The energy of the signal is

$$E_m = \int_0^T s_m^2(t)dt = \int_0^T \left(\sum_{n=1}^{N'} s_{mn}\phi_n(t)\right)^2 dt = \sum_{n=1}^{N'} s_{mn} = \|\underline{s}_m\|^2 \tag{4}$$

E_m can also be represented in terms of the signal energy of the low-pass equivalent as follows:

$$E_m = \int_0^T \left(\mathrm{Re}\{v_m(t)e^{j2\pi f_c t}\}^2\right)dt = \int_0^T \left(\frac{v_m(t)e^{j2\pi f_c t} + v_m^*(t)e^{-j2\pi f_c t}}{2}\right)^2 dt \tag{5}$$

$$= \frac{1}{2}\left(\int_0^T |v_m(t)|^2 dt + \int_0^T \left(\frac{\left(v_m(t)e^{j2\pi f_c t}\right)^2 + \left(v_m^*(t)e^{-j2\pi f_c t}\right)^2}{2}\right)dt\right) \tag{6}$$

$$= \frac{1}{2}\left(\int_0^T |v_m(t)|^2 dt + \int_0^T \mathrm{Re}\{v_m^2(t)e^{4j\pi f_c t}\}dt\right) \tag{7}$$

If $f_c T >> 1$ (symbol duration contains many cycles of the carrier), the second term $\rightarrow 0$.
Thus,

$$E_m = \frac{1}{2}\int_0^T |v_m(t)|^2 dt = \frac{1}{2}\|\underline{v}_m\|^2 \tag{8}$$

Also, the cross-correlation between $s_m(t)$ and $s_k(t)$ is

$$\rho_{km} = \frac{\int_0^T s_m(t)s_k(t)dt}{\sqrt{E_k E_m}} = \frac{\underline{s}_m^T \underline{s}_k}{\sqrt{E_k E_m}}$$

(9)

In terms of the low-pass equivalent,

$$\rho_{km} = \frac{\int_0^T \frac{\left(v_m(t)e^{j2\pi f_c t} + v_m^*(t)e^{-j2\pi f_c t}\right)\left(v_k(t)e^{j2\pi f_c t} + v_k^*(t)e^{-j2\pi f_c t}\right)}{4}dt}{\sqrt{E_k E_m}} \tag{10}$$

$$= \frac{\int_0^T \mathrm{Re}\left(v_m(t)v_k^*(t)\right)dt + \int_0^T \mathrm{Re}\left\{v_m(t)v_k(t)e^{j4\pi f_c t}\right\}dt}{\|\underline{v}_m\|\|\underline{v}_k\|} \tag{11}$$

$$= \mathrm{Re}\left\{\frac{\underline{v}^T{}_m v_k^*}{\|\underline{v}_m\|\|\underline{v}_k\|}\right\} \quad \text{, as the second term } \rightarrow 0 \text{ as } f_c T >> 1 \tag{12}$$

The Euclidean distance d_{ij} between two signals $s_i(t)$ and $s_j(t)$ is defined as

$$d_{ij}{}^2 = \left\|\underline{s}_i - \underline{s}_j\right\|^2 = E_i + E_j - 2\sqrt{E_i E_j}\rho_{ij} \tag{13}$$

$$= 2E\left(1 - \rho_{ij}\right) \text{ if } E_i = E, \forall i \tag{14}$$