Course: Wireless communication technologies

16.332.559(Advanced topics in communication engineering)
Lectures 19 (04/08/2002) and 20 (04/10/2002)
Instructor: Dr. Narayan B. Mandayam
Summary by: Pavan C. Kaivaram (kpavan@ece.rutgers.edu)

Multi-user detection

Multi-user detection in a CDMA was pioneered by Sergio Verdu. The recent ideas in multiuser detection are an outcome of Verdu's Ph.D. thesis in 1984(3). Multi-user detection is necessary in all cellular and mobile communications. Multi-user detection describes various techniques used to reduce interference and design of practical receivers to obtain near-optimal detection of signals.

Matched filter in a CDMA Channel:
Consider a CDMA system with synchronous users. The received signal due to k users is given as:

$$y(t) = \sum_{k=1}^{K} A_k b_k s_k(t) + \sigma n(t)$$

$s_k(t)$ is the signature waveform assigned to user k. We also assume that $s_k(t)$ is normalized such that:

$$\|s_k\|^2 = \int_0^T s_k^2(t)dt = 1$$

A_k is the received amplitude from the k^{th} user and A_k^2 is the energy, and $A_k \succ 0$.
$n(t)$ is the additive white Gaussian noise, with unit power spectral density.
$b_k \in \{-1,+1\}$ is the bit transmitted by the k^{th} user.
T is the bit period.

The output of the k^{th} matched filter receiver y_k is given by:

$$y_k = \int_0^T y(t)s_k(t)dt = A_k b_k + \sum_{j \neq k} A_j b_j \rho_{jk} + n_k$$

$n_k = \int_0^T \sigma n(t)s_k(t)dt \approx N(0,\sigma^2)$ is the noise component in the direction of the k^{th} signature waveform.
ρ_{jk} is the cross correlation between the signature waveforms of j^{th} and k^{th} users.

Orthogonal signature waveforms:

For orthogonal signature waveforms, we have $\rho_{jk} = 0$ and hence, the probability of the k^{th} user error $P_k^c(\sigma)$ is given by:

$$P_k^c(\sigma) = Q(A_k / \sigma)$$

The superscript in $P_k^c(\sigma)$ denotes the probability of error when a conventional receiver is used. The probability of error is same as the single-user case.

Two users using non-orthogonal signature waveforms:

If the users are not orthogonal, the statistics are not Gaussian anymore. When the number of users is two and the signature waveforms not orthogonal, the system can be represented as:

$k = 2$, $\rho_{12} = \rho$ and the received signal by the matched filter at user one is given by:

$$y_1 = A_1 b_1 + A_2 b_2 \rho + n_1$$

The probability of error by user one is given by: $P_1^c(\sigma) = prob(b_1 \neq \hat{b}_1)$.

The decision rule for the conventional receiver is: If $y_1 \succ 0$, $b_1 = +1$; else $b_1 = -1$;

$$\therefore P_1^c(\sigma) = P\left[b_1 \neq \hat{b}_1\right] = P[b_1 = +1]P[y_1 \prec 0 / b_1 = +1] + P[b_1 = -1]P[y_1 \succ 0 / b_1 = -1]$$

Note y_1 is not Gaussian. So conditioning on b_2 we obtain:

$$P[y_1 \succ 0 / b_1 = -1] = P[y_1 \succ 0 / b_1 = -1; b_2 = -1]P[b_2 = -1] + P[y_1 \succ 0 / b_1 = -1; b_2 = +1]P[b_2 = +1]$$
$$= P[n_1 \succ A_1 - A_2\rho]P[b_2 = +1] + P[n_1 \succ A_1 + A_2\rho]P[b_2 = -1]$$

Assuming that bits are equally likely and noting that,

$$P[y_1 \succ 0 / b_1 = -1] = P[y_1 \prec 0 / b_1 = +1],$$

probability of error at user one is given by:

$$P_1^c(\sigma) = \frac{1}{2}Q\big((A_1 - A_2|\rho|)/\sigma\big) + \frac{1}{2}Q\big((A_1 + A_2|\rho|)/\sigma\big) \quad \text{.................(*)}$$

Interchanging the roles of user one and two gives us:

$$P_2^c(\sigma) = \frac{1}{2}Q\big((A_2 - A_1|\rho|)/\sigma\big) + \frac{1}{2}Q\big((A_2 + A_1|\rho|)/\sigma\big)$$

Let us now focus on to user one and the same arguments hold true for user two. Since, $Q(x)$ is a monotonically decreasing function, we can bound $P_1^c(\sigma)$ as,

$$P_1^c(\sigma) \leq Q\big((A_1 - A_2|\rho|)/\sigma\big)$$

This bound is smaller than 0.5, provided $\frac{A_2}{A_1} \prec \frac{1}{|\rho|}$, i.e. the interferer is not dominant.

As $\sigma \to 0$, the equation (*), is dominated by smallest argument and hence the upper bound, is a good approximation for all but low SNRs.

$\therefore$ BER of conventional receiver behaves like a single-user system with reduced SNR i.e. $\left(\frac{A_2 - A_1|\rho|}{\sigma}\right)^2$.

However if the relative amplitude of the interferer is stronger i.e. $\frac{A_2}{A_1} \succ \frac{1}{|\rho|}$ (*referred from here on as condition (**)),* then the conventional receiver exhibits highly anomalous behavior, referred to as the *near-far problem*.
To show the anomalous behavior, we see that BER is not monotonic in σ, a property which is usually expected of any detector.
From equation (*) $\lim_{\sigma \to \infty} P_1^c(\sigma) = 0.5$ and $\lim_{\sigma \to 0} P_1^c(\sigma) = 0.5$, which shows that BER is not monotonic. This anomalous behavior can be attributed to the fact that, the polarity of the output for matched filter for user one is governed by the sign of b_2 rather than b_1. (as seen from condition (**)

Infact $\sigma \succ 0$ is actually better than $\sigma = 0$ for detection of b_1, in the sense that, $P_1^c(\sigma) < 0.5$. Further, we can show that the optimum noise variance that minimizes BER, under condition (**) is given by:

$$\sigma_{opt}^2 = \frac{A_1 A_2 |\rho|}{\tanh^{-1}\left(\frac{A_1}{A_2 |\rho|} \right)}$$

When $A_1 = A_2|\rho|$, with a probability of 0.5, signal of user two exactly cancels signal of user one, which leaves a zero-mean Gaussian noise and with a probability of 0.5, signal of user two doubles the contribution of desired signal at the matched filter out put. So the expression for probability of error of user one is given by:

$$P_1^c(\sigma) = \frac{1}{4} + \frac{1}{2} Q\left(\frac{2A_1}{\sigma} \right)$$

Figure1. shows the probability of error $P_1^c(\sigma)$ for user one versus normalized amplitude of user one A_1/σ plotted at , for different ratios A_2 / A_1 at $\rho = 0.2$. MATLAB code for figure1 is included in Appendix1.

Observations:
We can see form the curve that BER degrades quite rapidly as $A_2 \Uparrow$ (increases).
When $A_2 / A_1 = 6, \rho = 0.2,$ the condition (**) hold and hence the probability of error doesn't go to zero as $\sigma \to 0$. The near far effect can thus be observed.

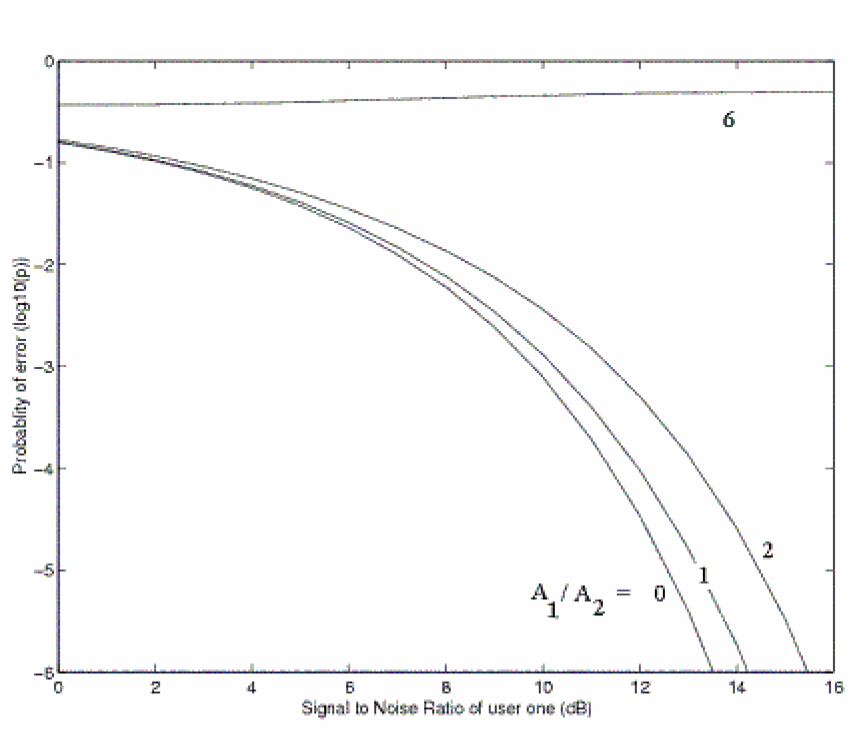

Figure1. $\log_{10}(P_1^c(\sigma))$ *vs.* $(A_1/\sigma)^2$ *for* $A_2/A_1 \in \{0,1,2,6\}$ *and* $\rho = 0.2$

Power Trade-off Regions: Power trade-off regions plotted for a given BER(p*) are defined as the region of permissible normalized powers for which the BER is always less the than the specified BER = p*, for all users. The power trade off region for $BER = 3*10^{-5}$ and different $\rho \in \{0, 0.1, 0.3, 0.5\}$ are plotted in Figure2.

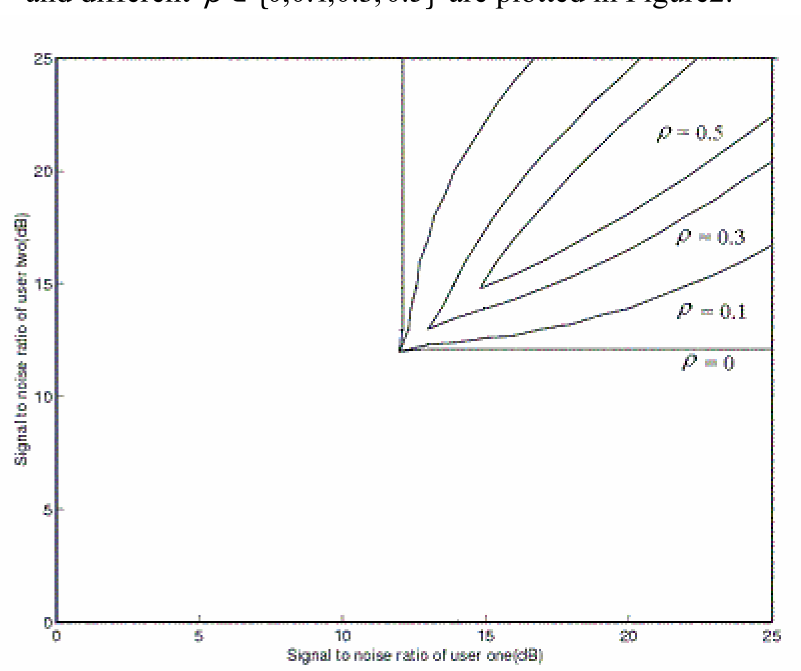

Figure2. Regions $(A_2/\sigma)^2$ *vs.* $(A_1/\sigma)^2$ *for* $\rho \in \{0, 0.1, 0.3, 0.5\}$, $BER = 3*10^{-5}$

The MATLAB code for figure 2 is included in Appendix 2.

Observations:

For both the users parameterized by ρ, $Q^{-1}(3*10^{-5}) = 12dB.$ So we can observer that for zero cross-correlation, we all SNRs greater than 12dB as permissible powers. We also observe that as ρ increases, even if both amplitudes are identical, necessary energy increases rapidly. The sensitivity to imbalances in received energy increases with ρ. In mobile systems, received amplitude may vary over a wide range and hence strict power control is necessary.

We also observe that lower the cross-correlation, the better we perform and hence, to reduce cross correlation, we need to use complex codes such as Walsh codes or long random codes need to be used in signature sequences. Also channel coding is necessary which is useful when errors are independent.

To visualize the operation of the conventional detector in signal space diagram, let now look at $y_1 - y_2$ plane.

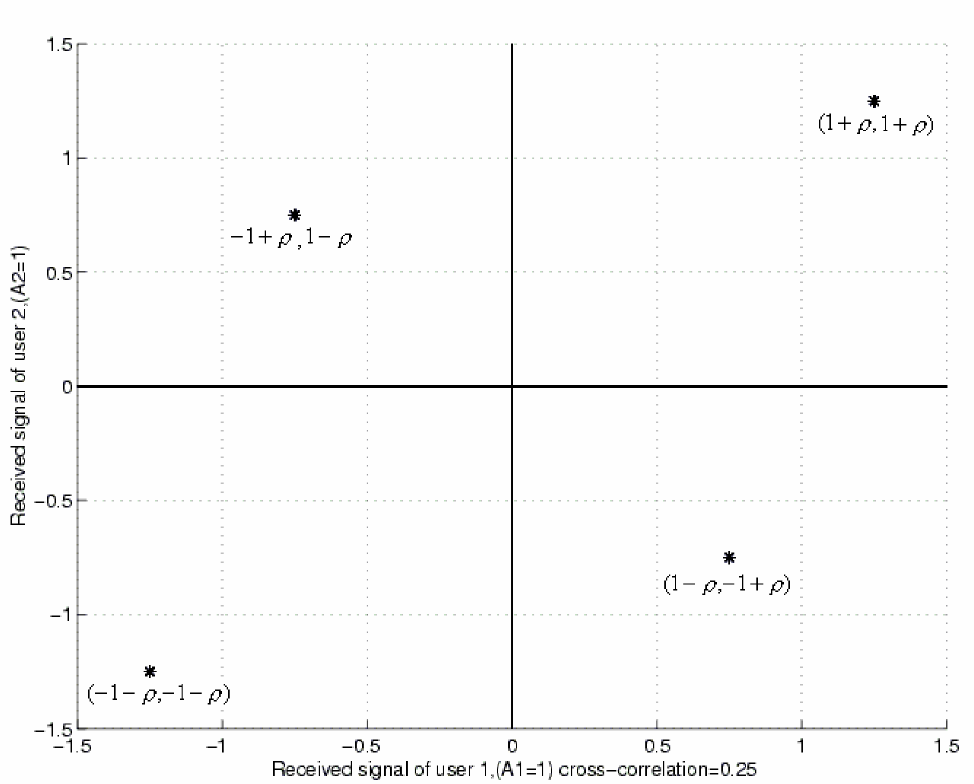

Figure3. Signal space representation of mean of received signals in $y_1 - y_2$ plane.

For the two-user system when bits (b_1, b_2) are transmitted, the output vector conditioned on (b_1, b_2) is Gaussian with:

Mean = $\begin{pmatrix} A_1 b_1 + A_2 b_2 \rho \\ A_2 b_2 + A_1 b_1 \rho \end{pmatrix}$ and covariance matrix, $\mathrm{cov}(y_1, y_2) = \sigma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$.

Received vector is a sum of the transmitted vector (mean) and the zero mean Gaussian noise vector, with covariance matrix as above.

If we keep decision regions fixed the transmitted vector changes according to amplitudes and cross correlations and therefore may lead to degradation in performance and anomalous behavior.

Generalization to the k-user system:

Based on the knowledge obtained from the two-user case, we know try to extend the arguments to K-user system and can obtain, the probability of error expression for user k as:

$$P_k^c(\sigma) = P[b_k = +1]P[y_k < 0 / b_k = +1] + P[b_k = -1]P[y_k > 0 / b_k = -1]$$
$$= \frac{1}{2^{k-1}} \sum_{\varepsilon_1 \in \{-1,+1\}} \cdots\cdots \sum_{\varepsilon_j \in \{-1,+1\}, j \neq k} \cdots\cdots \sum_{\varepsilon_k \in \{-1,+1\}} Q\left[\frac{A_k}{\sigma} + \sum_{j \neq k} \varepsilon_j \frac{A_j}{\sigma} \rho_{jk}\right] \ldots.(***)$$

Observations:

We observe that the bit error rate depends on shape of signature waveforms only through cross correlations. BER depends only on received amplitudes and noise level σ only through the ratio $\frac{A_k}{\sigma}$. The decisions are invariant under the scaling of received signal.

We can now upper bound equation $(***)$ as

$$P_k^c(\sigma) \leq Q\left(\frac{A_k}{\sigma} - \sum_{j \neq k} \frac{A_j}{\sigma} |\rho_{jk}|\right) \ldots..(****)$$

To note the near far behavior observe that $(***) \to 0$, as $\sigma \to 0$, iff

$$A_k \succ \sum_{j \neq k} A_j |\rho_{jk}| \ldots\ldots.\oplus$$

We refer to the condition above as eye-open condition i.e. error free decisions are possible in absence of background noise. When eye is open note that $(****) \to 0$, as $\sigma \to 0$,

Also note that computation of (***) grows exponentially in number of users. It is very tempting to use Gaussian approximation, i.e. replace the binomial random variable, $\sum_{j \neq k} A_j b_j \rho_{jk}$ with Gaussian random variable with same variance. The expression for probability of error of k^{th} user is given by,

$$\tilde{P}_k^c(\sigma) = Q\left(\frac{A_k}{\sqrt{\sigma^2 + \sum_{j \neq k} A_j^2 \rho_{jk}^2}}\right) \ldots..(*G)$$

The Gaussian approximation is good for low SNR, but highly unreliable at high SNR. Gaussian approximation states that the probability of error goes to a non-zero limit for high SNR which is not the same as $\lim_{\sigma \to 0} P_k^c(\sigma)$. We can easily see that:

$$\lim_{\sigma \to 0} P_k^c(\sigma) = f\left(K, \rho, \{A_j\}\right)$$

Consider the plot shown in Figure4, where the BER of one user is plotted against the SNR, when ten users are present cross-correlation $\rho_{kl} = 0.08$ and equal energy. The MATLAB code for Figure4, Figure5 and Figure6 is shown in Appendix 1. The different plots shown below can be obtained by changing the parameters corresponding to number of users, cross-correlation and the range of x-axis and y-axis.

Figure5 and Figure6 are plotted with BER on the y-axis and SNR on the x-axis for fourteen users with equal powers and cross correlations. The second plot is specifically included to show the asymptotic behavior of exact probability of error under eye-closed condition. The asymptotic probability of error is 1/8192 in the case when $K = 14, \rho = 0.08, A_k = A_j$.

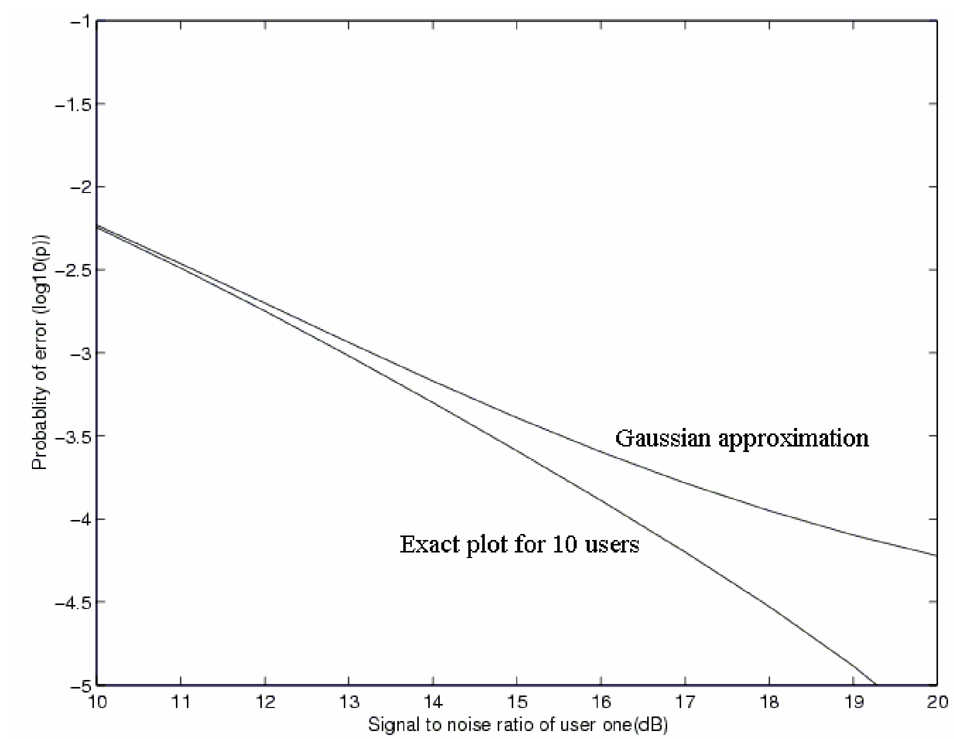

Figure4. Probability of error vs. Signal to noise ratio for $K = 10, \rho = 0.08, A_k = A_j \forall j$

(Eye-open condition)

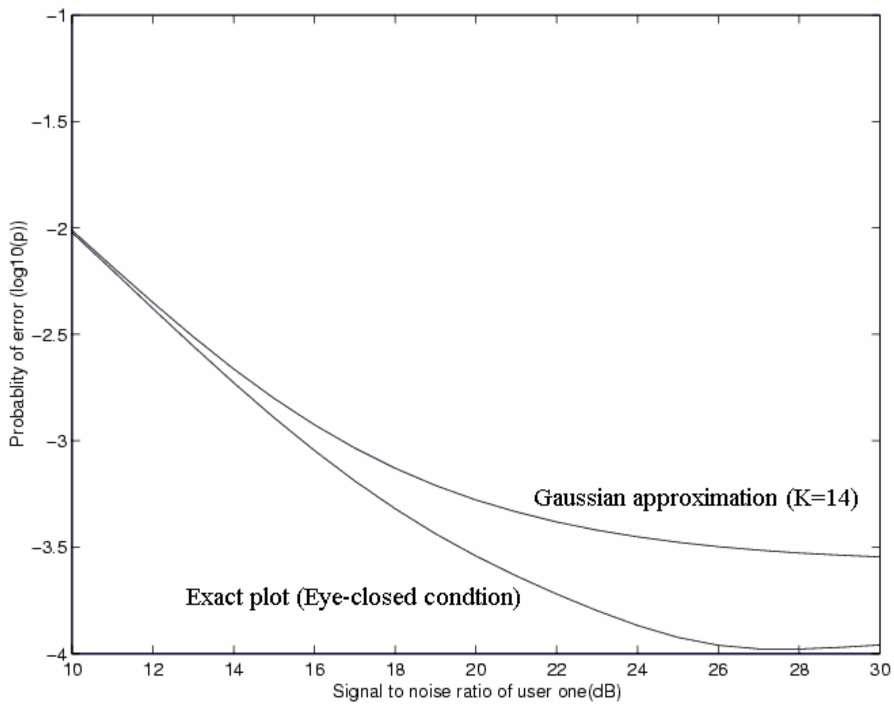

Figure5. Probability of error vs. Signal to Noise ratio for $K = 14, \rho = 0.08, A_k = A_j \forall j$ *(Eye-closed condition)*

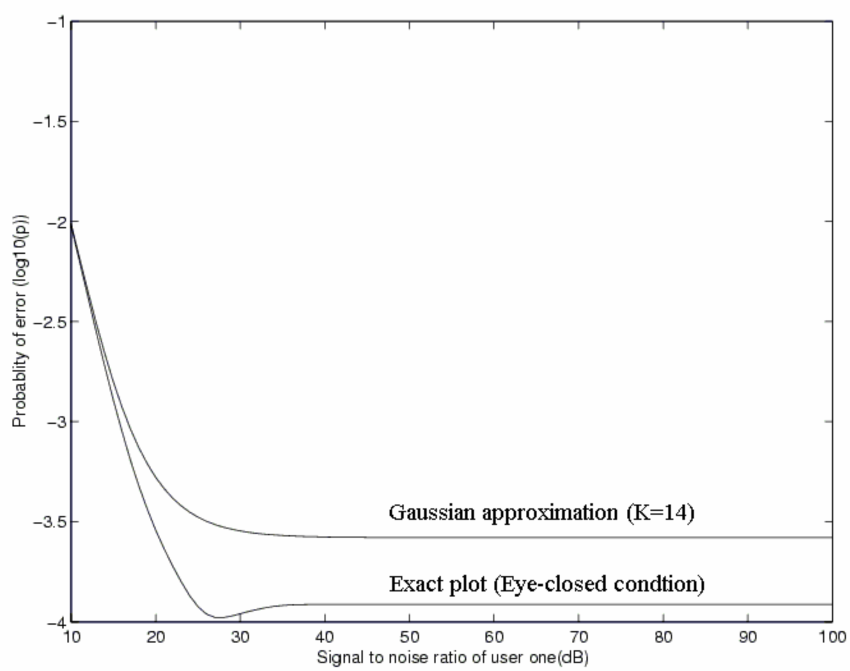

Figure6. Asymptotic behavior of exact probability of error for $K = 14, \rho = 0.08, A_k = A_j \forall j$ *(Eye-closed condition)*

$\therefore$ We can observe that in the limit as $\sigma \to 0$ (***), (*G) behave very differently. For e.g. (*G) has a non-zero limit, even if eye-open condition is satisfied. This difference can be attributed to error in replacing binomial random variable with Gaussian random variable and the error is greatest in the tails, which determine the BER (when background noise is dominant).

When the users are not synchronized, each bit of k^{th} user is affected by, 2k-2 interfering bits and the expression for probability of error of k^{th} user is given by:

$$P_k^c(\sigma) = \frac{1}{4^{k-1}} \sum_{(e_1,d_1)\in\{-1,+1\}^2} \cdots\cdots \sum_{(e_j,d_j)\in\{-1,+1\}^2, j\neq k} \cdots\cdots \sum_{(e_K,d_K)\in\{-1,+1\}^2} Q\left(\frac{A_k}{\sigma} + \sum_{j\neq k} \frac{A_j}{\sigma}\left(e_j \rho_{jk} + d_j \rho_{jk}\right)\right)$$

where $\rho_{kl}(\tau) = \int_{\tau}^{T} s_k(t) s_l(t-\tau)dt$ and $\rho_{lk}(\tau) = \int_{0}^{\tau} s_k(\tau) s_l(t+T-\tau)dt$, $\tau \in [0,T]$

In the eye open condition the asynchronous environment is thus given by,

$$A_k > \sum_{j\neq k} A_j \left(\left|\rho_{jk}\right| + \left|\rho_{kj}\right|\right)$$

Multi-user Efficiency and related measures:

While BER is a main performance measure in most communication systems, there are several performance measures derived from it. These can be useful in design analysis and understanding various detectors. SIR is a good measure. Outage is another measure which is defined as the probability of SIR going below some threshold. In the absence of interferers, SIR = A_k^2 / σ^2 and single user performance is achieved. i.e. probability of error is given by, $P_k^c(\sigma) = Q(A_k/\sigma)$. In a multi-user environment the Signal to interference ratio is given by, SIR = $A_k^2 \Big/ \left(\sigma^2 + \sum_{j\neq k} A_j^2 \rho_{jk}^2\right)$.

Therefore the presence of interference increases BER. It is interesting to quantify multi-user error probability relative to single-user BER.

Effective energy of user k $e_k(\sigma)$, is defined as the energy that user k would require to achieve a bit error rate equal to $P_k^c(\sigma)$ in a single-user, Gaussian channel with same back ground noise. $P_k(\sigma) = Q\left(\frac{\sqrt{e_k(\sigma)}}{\sigma}\right)$

Since multi-user error probability is lower bounded by the single user error probability, we have $P_k(\sigma) > Q(A_k/\sigma)$, and hence $e_k(\sigma) \prec A_k^2$. Effective energy is upper bounded by actual energy.

If we normalize the effective energy with noise, we obtain

$$\frac{e_k(\sigma)}{\sigma^2} \geq \left(Q^{-1}\left(P_k(\sigma)\right)\right)^2.$$

The power trade off regions can then be characterized in terms of effective energies as follows. The power trade off regions for a given permissible BER p (same for all users), are a set of SNRs $\left\{\frac{A_1^2}{\sigma^2}, \frac{A_2^2}{\sigma^2}, \ldots\ldots, \frac{A_K^2}{\sigma^2}\right\}$ such that $\max_k P_k(\sigma) \le p$, or equivalently,

$$\min_k \frac{e_k(\sigma)}{\sigma^2} \ge \left(Q^{-1}(p)\right)^2$$

Multi-user efficiency: Multi-user efficiency is defined as the ratio of effective and actual energies $e_k(\sigma)/A_k^2$, which quantifies the performance degradation due to existence of other users in the channel.

Asymptotic multi-user efficiency is defined as: $\eta_k = \lim_{\sigma \to 0} \frac{e_k(\sigma)}{A_k^2}$

An alternate and more formal definition of asymptotic multi-user efficiency is given as:

$$\eta_k = \sup_r \left\{ 0 \le r \le 1, \lim_{\sigma \to 0} \frac{P_k(\sigma)}{Q\left(\frac{\sqrt{r} A_k}{\sigma}\right)} = 0 \right\}$$

Therefore we can see that, when "eye is closed", (i.e. BER doesn't vanish as $\sigma \to 0$) the asymptotic multi-user efficiency is zero.
If $\eta_k \succ 0$, then BER goes to zero in the limit as $\sigma \to 0$, moreover vanishes exponentially. Typically the asymptotic multi-user efficiency is very close to multi-user efficiency, except in the low SNR regime.

Asymptotic multi-user efficiency η_k depends on receiver and cross-correlations.

Near-far resistance: The near-far resistance is defined as multi-user efficiency minimized over received energies of all the other users and is denoted by $\bar{\eta}_k$ is given by:

$$\bar{\eta}_k = \inf_{\substack{A_j > 0 \\ j \ne k}} \eta_k \, .$$

$\bar{\eta}_k$ depends on the receiver and cross-correlations.

It is sometimes easier to compute multi-user efficiency and near-far resistance than computing probability of error.

References:

1. Narayan Mandayam, *"Lectures 19 and 20, Advanced Topics in Communication, Spring 2002, Rutgers University"*. 2002.

2. S. Verdu, “Optimum Sequence Detection of Asynchronous Multiple-Access Communications”, *Abs. 1983 Int. symp. Information Theory*, *St. Jovite, Canada, p. 80, sept. 1983.*
3. S. Verdu, “Optimum Multi-User Signal Detection*”, Ph.D. Thesis, Dept. of Electrical and Computer Engineering, University of Illinois, Aug. 1984.*
4. S. Verdu, “Multi-user detection”, *Cambridge University Press, 1998.*

Appendix 1

MATLAB code for Figure1:

```
ro=0.2;
range = [0 16];
yr = [-6 0];
k=[0 1 2 6];
for na1=range(1):range(2)
   nna1(na1-range(1)+1) = 10^(na1/10);
end;
[k1 k2]=size(k);
plt = zeros(range(2)-range(1),k2);
for i=1:k2
   for j=1:range(2)-range(1)+1
      a1=nna1(j);
      a2=k(i)*nna1(j);
      if (a1>a2*ro)
         plt(j,i)=0.25*erfc(sqrt(a1-a2*ro)/sqrt(2))+0.25*erfc(sqrt(a1+a2*ro)/sqrt(2));
      end;
      if(a1<= a2*ro)
         plt(j,i)=0.5*(1-erfc(sqrt(a2*ro-a1)/sqrt(2))/2)+0.25*erfc(sqrt(a1+a2*ro)/sqrt(2));
      end;
   end;
end;
hold on;
n=range(1):range(2);
for i=1:k2;
   plot(n,log10(plt(:,i)));
end;
axis([range yr]);
xlabel('Signal to Noise Ratio of user one(dB)');
ylabel('Probablity of error (log10(p))');
hold off;
clear all;
```

MATLAB code for figure2:

```
ro = [0 0.1 0.3 0.5];
[k1 k2]=size(ro);
ber = 3*10^-5;
del = 1;
```

```
range=[0 25];
nincr = 10;
for na1=range(1):range(2)
   nna1(na1-range(1)+1) = 10^(na1/10);
end;
for na2=range(1)*nincr:range(2)*nincr
   nna2(na2-range(1)*nincr+1) = 10^(na2/(10*nincr));
end;
for i=1:k2
   for j=1:range(2)-range(1)+1
      a1 = nna1(j);
      min=1;
      for k=1:(j-1)*nincr+1
         a2=nna2(k);
         if (a2>a1*ro(i))
          plt(k,j)=0.25*erfc(sqrt(a2-a1*ro(i))/sqrt(2))+0.25*erfc(sqrt(a2+a1*ro(i))/sqrt(2));
         end;
         if(a2<= a1*ro(i))
            plt(k,j)=0.5*(1-erfc(sqrt(a1*ro(i)-
            a2)/sqrt(2))/2)+0.25*erfc(sqrt(a2+a1*ro(i))/sqrt(2));
         end;
         if (abs(plt(k,j)/ber-1)<del)
             if(abs(plt(k,j)-ber)<min)
                min = abs(plt(k,j)-ber);
                abra(i,j)=k;
             end;
         end;
      end;
   end;
end;
for i=1:k2
   bool=0;
   for j=1:range(2)-range(1)+1
      if((abra(i,j)>0)&(bool==0))
         bool=1;
         tt(i)=j;
      end;
   end;
end;
hold on;
for i=1:k2
   n=tt(i):range(2);
   abn=zeros(range(2)-tt(i)+1,1);
   for j=tt(i):range(2);
      abn(j-tt(i)+1,1)=abra(i,j);
   end;
```

```
   plot(n,(abn-1)/nincr);
   plot((abn-1)/nincr,n);
end;
hold off;
axis([range range]);
xlabel('Signal to noise ratio of user one(dB)');
ylabel('Signal to noise ratio of user two(dB)');
clear all;
```

MATLAB code for figure3:

```
ro = 0.25
A=[1+ro -1+ro -1-ro 1-ro;1+ro 1-ro -1-ro -1+ro]
hold on;
for i=1:4
   plot(A(1,i),A(2,i),'*')
end;
axis([-1-2*ro 1+2*ro -1-2*ro 1+2*ro]);
xlabel('Received signal of user 1,(A1=1) cross-correlation=0.25')
ylabel('Received signal of user 2,(A2=1)')
grid on;
line([-1-2*ro;1+2*ro],[0;0]);
line([0;0],[-1-2*ro;1+2*ro]);
hold off;
clear all;
```

MATLAB for figure 4,5,6:

```
 ro=0.08;
range=[10 100];
nou = 14;
yr = [-4,-1];
for i=range(1):range(2);
   nna1(i-range(1)+1)=sqrt(10^(i/10));
end;
for k=1:(range(2)-range(1)+1)
   sum1(k)=0;
   for i=0:nou-1
      sum=1+(ro*i-ro*(nou-1-i));
      if(sum>=0)

sum1(k)=sum1(k)+0.5*nchoosek(nou-1,i)*erfc(sum*nna1(k)/sqrt(2));
      end;
      if(sum<0)

sum1(k)=sum1(k)+nchoosek(nou-1,i)*(1-0.5*erfc(-sum*nna1(k)/sqrt(2)));
      end;
   end;
```

```
    sum1(k)=sum1(k)/2^(nou-1);
end;
xr =range(1):range(2);
plot(xr,log10(sum1));
hold on;
for i=range(1):range(2)

temp(i-range(1)+1)=0.5*erfc(nna1(i-range(1)+1)/(sqrt(2)*sqrt(1+(nou-1)*ro^2*nna1(i-
range(1)+1)^2)));
end;
plot(xr, log10(temp));
axis([range yr]);
xlabel('Signal to noise ratio of user one(dB)');
ylabel('Probablity of error (log10(p))');
hold off;
clear all;
```