Wireless Communication Technologies

16:332:559 (Advanced Topics in Communications)
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This lecture note is about multiuser CDMA systems and we study in detail about the performance of such systems.

1.1 Multiuser BPSK CDMA System

Let's assume that we have a CDMA system with K users indexed as j =1,2…K and each user transmits BPSK modulated signal waveforms simultaneously. Then the transmitted signal of one such user j can be represented as,

$$S_j(t) = \sqrt{2P_j}\, c_j(t) b_j(t) \cos(\omega_c t + \theta_j) \tag{1}$$

where,

$$c_j(t) = \sum_{n=-\infty}^{n=\infty} c_j^{(n)} p_{T_c}(t - nT_c)$$

$$c_j^{(n)} \in \{-1,+1\}$$

$$b_j(t) = \sum_{n=-\infty}^{n=\infty} b_j^{(n)} p_T(t - nT)$$

$$b_j^{(n)} \in \{-1,+1\}$$

$T = NT_c$

N = Processing Gain

The received signal at the receiver can be represented as,

$$r(t) = \sum_{j=1}^{k} \sqrt{2P_j}\, c_j(t-\tau_j) b_j(t-\tau_j) \cos(\omega_c t + \phi_j) + \eta(t)$$
$$\phi_j = \theta_j - \omega_c \tau_j \tag{2}$$

where,

t_j = relative time offset

ϕ_j = phase offset

$\eta(t) \approx N(0, \frac{N_o}{2})$

In general signal from the various users are received at various propagation delays, $\tau_j \in [0, T]$ and $\phi_j \in [0, 2p]$

Receiver in the multiuser scenario can be modeled as figure below:

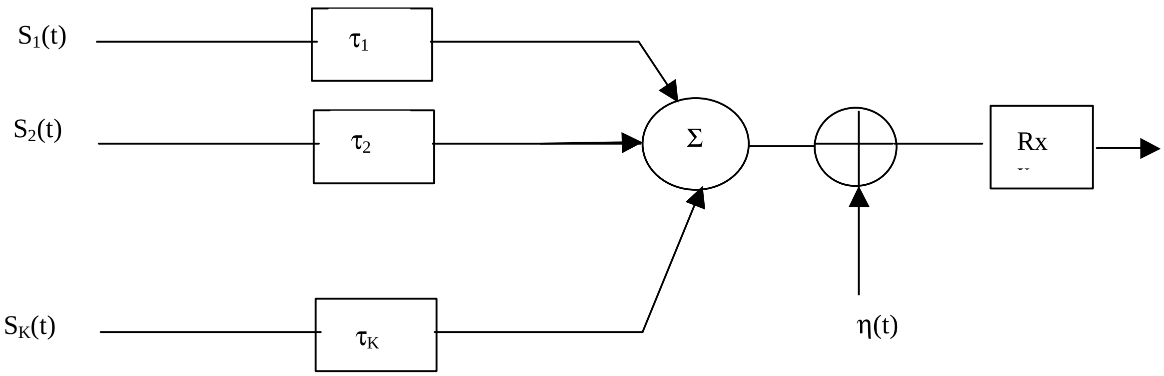
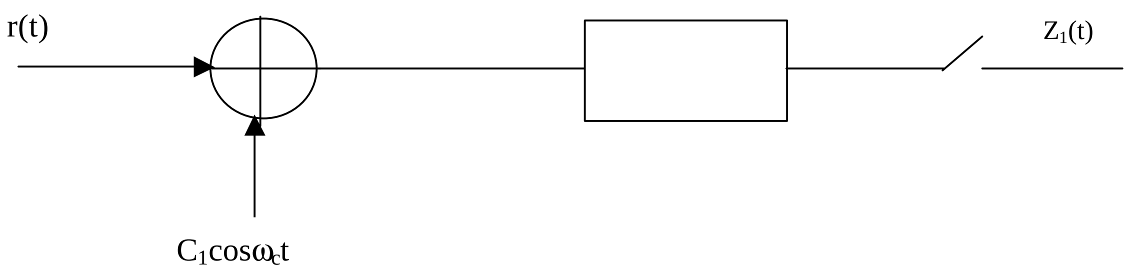


Figure 1: Received signal in multiuser scenario

1.1.2 Single User Reception

Focusing on user 1, we try to look at the demodulation of bit stream of user 1. Assume the receiver is synchronized to user 1. So, without loss of generality we can say, $\tau_1 = 0$ and $\phi_1 = 0$.
We will consider demodulation of a single bit of user 1, which is bit $b_1(0)$.
The matched filter receiver matched to user 1 is illustrated in the following figure,



The output of the matched filter corresponding to user 1 is given as,

$$z_1 = \int_0^T c_1(t)\cos(\omega_c t) r(t) dt \tag{3}$$

$$z_1 = \sum_{j=1}^{K} \sqrt{2P_j} \int_0^T c_1(t) c_j(t - t_j) b_j(t - t_j) \cos(w_c t + f_j) \cos(w_c t) dt + h_1$$

where,

$$h_1 = \int_0^T h(t) c_1(t) \cos(w_c t) dt$$

Neglecting all double frequency terms,

$$z_1 = \sqrt{\frac{P_1}{2}} T b_1^{(0)} + \sum_{j=2}^{K} \sqrt{\frac{P_j}{2}} \left[b_j^{(-1)} R_{j,1}(t_j) + b_j^{(0)} \hat{R}_{j,1}(t_j) \right] \cos f_j + h_1 \qquad (4)$$

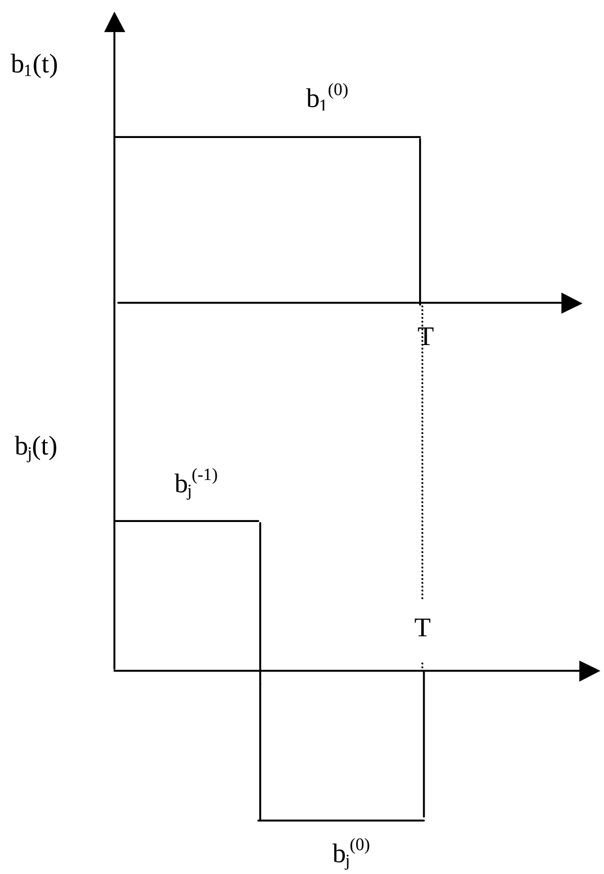
We introduce continuous time partial cross-correlation functions,

$$R_{j,1}(t) = \int_0^t c_1(t) c_j(t - t) dt$$

$$\hat{R}_{j,1}(t) = \int_t^T c_1(t) c_j(t - t) dt \qquad (5)$$

$$0 \le t \le T$$

Thus, in one bit interval of the desired user i.e. user 1, there would be effectively two bits of every interferer j, by assuming that the signals of the interferers are delayed by not more than one bit interval. This is illustrated in the following figure:



Case of Synchronous users

For the case of synchronous system, $\tau_j = 0$ and $\phi_j = 0$ for all j. So there is only one term from each interferer i.e. one bit of j interferes with one bit of the desired user. Then from eq. (4) we have,

$$z_1 = \sqrt{\frac{P_1}{2}} T b_1^{(0)} + \sum_{j=2}^{K} \sqrt{\frac{P_j}{2}} b_j^{(0)} R_{j,1} + h_1 \qquad (6)$$

where,

$$R_{j,1} = \underline{c}_j^T \underline{c}_1 \frac{T}{N}$$

c_j = Spreading code vector of length N for user j
c_1 = Spreading code vector of length N for user 1

$$\eta_1 \approx N\left(0, \frac{N_0 T}{4}\right)$$

(a) When code sequence are Orthogonal

$$\underline{c}_j^T \underline{c}_1 = d_{j1}$$

Therefore the matched filter output is,

$$z_1 = \sqrt{\frac{P_1}{2}} T b_1^{(0)} + h_1 \qquad (7)$$

The probability of error is given as,

$$P_{b,1} = Q\left(\sqrt{\frac{2E_{b,1}}{N_0}}\right)$$

Thus performance is same as single user channel

(b) When users use random codes

The error probabilities are calculated in two steps, noting that $\underline{c}_j$, j=1,2….K are random binary vectors.

Step 1: Fix the codes and bits of other users. Then calculate the conditional probability of error for synchronous users with fixed data bits and conditioned on $R_{j,1}$ and $b_j(0)$.
This conditional Probability of error is represented as,
$P_{b,1}(\{R_{j,1}\},\{b_j(0)\})$ (8)

Step 2 : Find the average Probability of error $\overline{P}_{b,1}$, by averaging over $\{R_{j,1}\}$ and $\{b_j^{(0)}\}$.

For fixed $\{R_{j,1}\}$ and $\{b_j^{(0)}\}$,

$$P_{b,1}(\{R_{j,1}\},\{b_j(0)\}) = \Pr\{ z_1 < 0 \mid b_1^{(0)} = 1\}$$

$$= Q\left(\frac{\sqrt{\frac{P_1}{2}}T + \sum_{j=2}^{K}\sqrt{\frac{P_j}{2}}b_j^{(0)}R_{j,1}}{\sqrt{\frac{N_0 T}{4}}}\right)$$

$$= Q\left(\sqrt{\frac{2E_{b,1}}{N_0}} + \sum_{j=2}^{K}\sqrt{\frac{2E_{b,j}}{N_0}}b_j^{(0)}\frac{R_{j,1}}{T}\right)$$

The average probability of error is,

$$\overline{P}_{b,1} = E_{\{b_j^0\}\{R_{j,1}\}}Q\left(\sqrt{\frac{2E_{b,1}}{N_0}} + \sum_{j=2}^{K}\sqrt{\frac{2E_{b,j}}{N_0}}b_j^{(0)}\frac{R_{j,1}}{T}\right) \tag{9}$$

This average probability of error is difficult to calculate.

1.1.3 <u>Using Gaussian approximation for the Interference</u>

Assume P_j=P for all j and the no. of user K is very large.

The interference term for user 1 is given as,

$$I_1 = \sum_{j=2}^{K}\sqrt{\frac{P}{2}}b_j^{(0)}R_{j,1}$$

We approximate I_1 to be Gaussian. We proceed to find the mean and variance of I_1 as follows,

$$E[I_1] = E\left[\sum_{j=2}^{K}\sqrt{\frac{P}{2}}b_j^{(0)}R_{j,1}\right] = \sum_{j=2}^{K}\sqrt{\frac{P}{2}}E\left[b_j^{(0)}\right]E\left[R_{j,1}\right] \tag{10}$$

b_j and $R_{j,1}$ is independent. Also,

$E\left[b_j^{(0)}\right] = 0$

Therefore $E\,[I_1] = 0$

$$Var[I_1] = \frac{P}{2}\sum_{j=2}^{K} E\left[\left(b_j^{(0)}\right)^2\right] Var\left[R_{j,1}\right] \qquad (11)$$

Now, $E\left[\left(b_j^{(0)}\right)^2\right] = 1$

$$R_{j,1} = \underline{c}_j^T \underline{c}_1 \frac{T}{N}$$

$$Var\left[R_{j,1}\right] = \left(\frac{T}{N}\right)^2 E\left[\left(\underline{c}_j^T \underline{c}_1\right)^2\right] = \left(\frac{T}{N}\right)^2 N = \frac{T^2}{N}$$

Therefore $Var[I_1] = \frac{P}{2}\frac{T^2}{N}(K-1)$

The output of the matched filter z_1 can be written as,

$$z_1 = \sqrt{\frac{P}{2}}Tb_1^{(0)} + I_1 + h_1$$

Therefore $$\overline{P}_{b,1} = Q\left(\frac{\sqrt{\frac{P}{2}}T}{\sqrt{\frac{N_0 T}{4} + \frac{PT^2}{2N}(K-1)}}\right)$$

$$\approx Q\left(\frac{\sqrt{\frac{P}{2}}T}{\sqrt{\frac{PT^2}{2N}(K-1)}}\right)$$

For large K

$$\approx Q\left(\sqrt{\frac{N}{K-1}}\right) \qquad (12)$$

1.1.4 <u>Case of Asynchronous Users</u>

For asynchronous user case, we have non- zero τ_j and ϕ_j. Each bit of the desired user is affected by two bits from the interferer. Therefore at the receiver each interferer has two terms. Recall that the matched filter output is given as:

$$Z_1 = \sqrt{\frac{P_1}{2}}Tb_1^{(0)} + \sum_{j=2}^{K}\sqrt{\frac{P_j}{2}}TI_{j,1}\left(\underline{b}_j, t_j, f_j\right) + h_1$$

where $I_{j,1}$ is the interference of user j to user 1, and given by

$$I_{j,1}\left(\underline{b}_j, t_j, f_j\right) = \frac{1}{T}\left[b_j^{(-1)} R_{j,1}\left(t_j\right) + b_j^{(0)} \hat{R}_{j,1}\left(t_j\right)\right]\cos\left(f_j\right)$$

$$\underline{b}_j = \left(b_j^{(-1)}, b_j^{(0)}\right)$$

Assume that P_j er controlled. The matched filter output can then be written as,

$$Z_1 = \sqrt{\frac{P}{2}}T\left[b_1^{(0)} + I_1\left(\underline{b}, \underline{t}, \underline{f}\right)\right] + h_1 \tag{13}$$

where the resultant interference $I_1\left(\underline{b}, \underline{t}, \underline{f}\right) = \sum_{j=2}^{K} I_{j,1}\left(\underline{b}_j, t_j, f_j\right)$

$\underline{b} = \left(\underline{b}_2, \underline{b}_3, \ldots. \underline{b}_K\right)$, $\underline{t} = \left(t_2, t_3, \ldots. t_K\right)$, $f = \left(f_2, f_3, \ldots\ldots f_K\right)$

Phase and Time as assumed to be acquired for user 1, i.e. the Rx is synchronized in time and phase w.r.t user 1.

For fixed $\underline{b}$, $\underline{\tau}$, $\underline{\phi}$ the conditional Probability of Error when -1 is transmitted is,

$$P_{b,1} = P_r\left\{Z_1 > 0 \mid b_1^{(0)} = -1\right\}$$

$$= P_r\left\{h_1 + \sqrt{\frac{P_1}{2}}T\left(-1 + I_1\left(\underline{b}, \underline{t}, \underline{f}\right)\right) > 0\right\}$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\left(1 - I_1\left(\underline{b}, \underline{t}, \underline{f}\right)\right)\right) \tag{14a}$$

Average probability of error is,

$$\overline{P}_{b,1} = E_{\underline{b}, \underline{t}, \underline{f}}\left[P_{b,1}\right]$$

Suppose $b_1^{(0)} = +1$, the error probability is given as,

$$P'_{b,1} = P_r\left\{Z_1 < 0 \mid b_1^{(0)} = +1\right\}$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\left(1 + I_1\left(\underline{b}, \underline{t}, \underline{f}\right)\right)\right) \tag{14b}$$

$$\overline{P'}_{b,1} = E_{\underline{b}, \underline{t}, \underline{f}}\left[P'_{b,1}\right]$$

As seen from eq.(14a) & (14b) the error probabilities are not symmetric as the interference is not symmetric.
It is not possible to get a closed form for the average probability of error. It is very difficult to compute exact error probability however good bounds and approximations exist. ("Error probability for DS/SS multiple access comm. Part-I Upper and lover bound IEEE TranCom, May 1982").

1.1.5 Error Probability for Gaussian Approximation

Model interference $I_1(\underline{b},\underline{\tau},\underline{\phi})$, as a Gaussian random variable.
The other assumptions are:

- b_j^m and b_j^n are independent for all $i \neq j$ & $m \neq n$
- τ_i, τ_j are independent for all $i \neq j$ & uniform over $[0,T]$
- ϕ_i, ϕ_j are independent for all $i \neq j$ & uniform over $[0,2\pi]$

We now proceed to find the mean and variance of $I_1(\underline{b},\underline{\tau},\underline{\phi})$.

$E[I_{j,1}(\underline{b}_j,\tau_j,\phi_j)] = 0$ for all j (because bits are independent w.r.t codes)

$$\sigma_{j,1}^2 = Var[I_{j,1}(\underline{b}_j,\tau_j,\phi_j)]$$
$$= \frac{1}{T^2} E[\cos^2 \phi_j] E_{\tau_j}\left[R_{j,1}^2(\tau_j) + \hat{R}_{j,1}^2(\tau_j)\right]$$
$$= \frac{1}{2T^2}\left[\frac{1}{T}\int_0^T \left[R_{j,1}^2(\tau) + \hat{R}_{j,1}^2(\tau)\right] d\tau\right]$$
$$= \frac{1}{2T^3}\int_0^T \left[R_{j,1}^2(\tau) + \hat{R}_{j,1}^2(\tau)\right] d\tau \tag{15}$$

From eq.(13) we have,

$$Z_1 = \sqrt{\frac{P}{2}} T b_1^{(0)} + \sqrt{\frac{P}{2}} T I_1(\underline{b},\underline{\tau},\underline{\phi}) + \eta_1$$

Therefore for $b_1^{(0)}$=-1 we have,

$$E[Z_1 \mid b_1^{(0)} = -1] = -\sqrt{\frac{P}{2}} T + \sqrt{\frac{P}{2}} T E[I_1(\underline{b},\underline{\tau},\underline{\phi})] + E[\eta_1] = -\sqrt{\frac{P}{2}} T + 0$$

$$\therefore E[Z_1 \mid b_1^{(0)} = -1] = -\sqrt{\frac{P}{2}} T$$

$$Var[Z_1 \mid b_1^{(0)} = -1] = \frac{PT^2}{2} Var[I_1(\underline{b},\underline{\tau},\underline{\phi})] + Var[\eta_1] = \frac{PT^2}{2}\sum_{j=2}^{K} \sigma_{j,1}^2 + \frac{N_0 T}{4}$$

Therefore the average probability of error is given as,

$$\overline{P}_{b,1} = Q\left(\frac{\sqrt{\frac{P}{2}}T}{\sqrt{\frac{N_0 T}{4} + \frac{PT^2}{2}\sum_{j=2}^{K} s_{j,1}^2}} \right)$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}\left(\frac{1}{1 + \frac{2E_b}{N_0}\sum_{j=2}^{K} s_{j,1}^2} \right)} \right) \qquad (16)$$

As seen, the performance depends critically on $\sigma_{j,1}^2$ which in turn depend on cross correlation properties of chip waveform or code sequence. Performance can be made better if better code sequences are designed with good cross correlation properties $\sigma_{j,1}^2$.

M-sequence and gold sequence are two types of code with fairly desirable cross correlation properties (refer to “Error probability for DS/SS multiple access communications, Geranitis and Pursley, IEEE TranCom, May 1982”)

2.1 Probability of Error for asynchronous users

This section contains mathematical analysis, which provides expressions for determining the average bit error rate for users in a single channel, CDMA system. (refer to the paper "Performance evaluation for Phase Coded Spread Spectrum Communication - Part II, Code Sequence analysis" by Pursley, Sarvate & Stark, IEEE TCOM Aug 1977)

2.1.1 Gaussian Approximations

As seen from eq. (3), the contribution from k[th] interfering user, to user 1 is,

$$I_{k,1} = \int_0^{T_b} \sqrt{2P_k}\, b_k(t-t_k) c_k(t-t_k) c_1(t) \cos(w_c t + f_k) \cos w_c t dt \qquad (17)$$

It is useful to simplify this equation in order to examine the effects of multiple access interference on the average bit error probability for a single user. The relationship between $b_k(t-t_k)$, $c_k(t-t_k)$ and $c_1(t)$ is illustrated in Fig.1

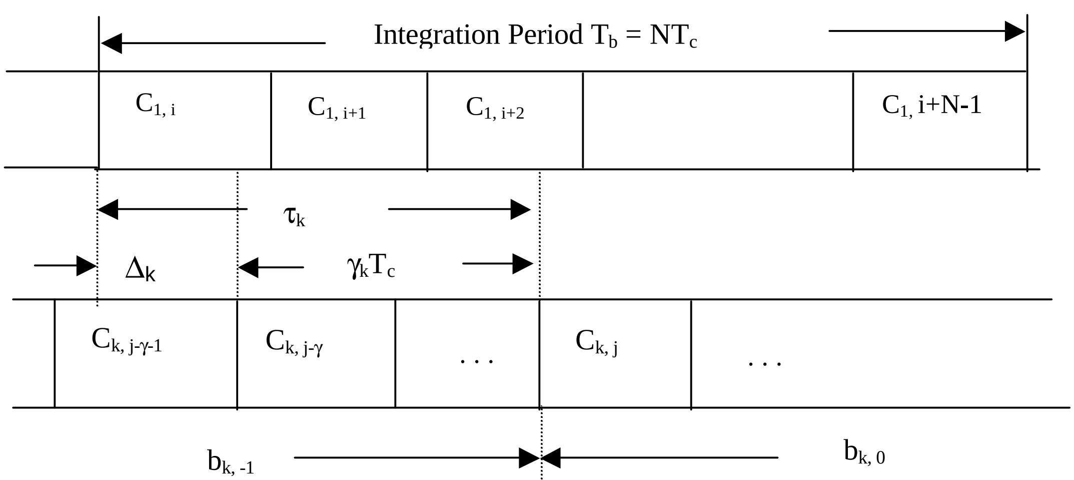


Figure 2: Timing of the local PN sequence for user 1 and user k

The quantities γ_k and Δ_k in Fig. 1 are defined from the delay of user k relative to user 1, τ_k, such that,

$\tau_k = \gamma_k T_c + \Delta_k \qquad 0 <= \Delta_k < T_c$

It is possible to show that, the contribution from k[th] interfering user to user 1 is given as,

$$I_{k,1} = T_c \sqrt{\frac{P_k}{2}} \cos(j_k) \left\{ X_k + (1 - 2\frac{\Delta_k}{T_c})Y_k + (1 - \frac{\Delta_k}{T_c})U_k + (\frac{\Delta_k}{T_c})V_k \right\} \qquad (18)$$

Where X_k, Y_k, U_k, V_k are random variables having distribution conditioned on A & B which are given as,

$$p_{X_k} = \begin{pmatrix} A \\ \frac{l+A}{2} \end{pmatrix} 2^{-A} \qquad l = -A, -A+2, \ldots, A-2, A$$

$$p_{Y_k}(l) = \begin{pmatrix} B \\ \frac{l+B}{2} \end{pmatrix} 2^{-B} \qquad l = -B, -B+2, \ldots, B-2, B$$

$$p_{U_k}(l) = \frac{1}{2} \qquad l = -1, +1$$

$$p_{V_k}(l) = \frac{1}{2} \qquad l = -1, +1$$

A and B are defined as follows:

A = Number of integers in [0, N - 2] for which $(c_{1,\, l+i})(c_{1,\, l+i+1}) = 1$ i.e. ***A,*** measures for a given code the no. of successive **no transitions** from +1 to -1 in the code c_1

B = Number of integers in [0,N - 2] for which $(c_{1,\, l+l})(c_{1,\, l+l+1}) = -1$ i.e. ***B,*** measures for a given code the no. of successive transitions from +1 to -1 in the code c_1.

Note: A + B = N - 1 since sets *A* and *B are disjoint and span the set of total possible signature sequences of length N in which there are a total of N - 1 possible chip level transitions*

The details of these derivations are in the following papers:
[1] Lehnart & Pursley :" Error Probability for binary DS-SS communication with Random Signature Sequence" IEEE TCOM Jan 1987
[2] Morrow & Lehnart : "Bit to bit error dependence in slotted DS/SSMa packet systems with Random Signature Sequence" IEEE TCOM Oct 1989

For random signature sequence statistics of A and B are known. We have seen the variance of the j^{th} interference term is given as,

$$\sigma_{j,1}^2 = \mathrm{Var}\,[\, I_{j,1}(b_j, \tau_j, \phi_j) \,]$$

$$= \frac{1}{T^2} E\left[\cos^2 j_j\right] E\left[\left(b_j^{(-1)} R_{j,1}(t_j) + b_j^{(0)} \hat{R}_{j,1}(t_j)\right)^2\right] \tag{19}$$

For random signature sequences it is possible to model,

$$\left(b_j^{(-1)} R_{j,1}(t_j) + b_j^{(0)} \hat{R}_{j,1}(t_j)\right) = \sum_{l=1}^{N} X_l$$

where $\{X_l\}_{l=1}^{N}$ are i.i.d random variables with distribution given as,

$$X_l = \begin{cases} Uniform\left(-\frac{T}{N}, \frac{T}{N}\right) \; with \;\; probability 1/2 \\ Bernoulli\left(-\frac{T}{N}, \frac{T}{N}, \frac{1}{2}\right) \;\; with \;\; probability 1/2 \end{cases}$$

Now,

$$E[X_l] = 0$$

$$Var[X_l] = \frac{1}{2}\left\{\frac{1}{3}\left(\frac{T}{N}\right)^2\right\} + \frac{1}{2}\left(\frac{T}{N}\right)^2$$

$$= \frac{2}{3}\left(\frac{T}{N}\right)^2$$

Therefore, $E\left[X_l^2\right] = \frac{2}{3}\left(\frac{T}{N}\right)^2$

$$E\left[\cos^2 f_j\right] = \frac{1}{2}$$

Therefore the variance of the jth interferer as given in eq.(19) can be written as,

$$\sigma_{j,1}^2 = \frac{1}{T^2}\frac{1}{2} N \frac{2}{3}\left(\frac{T}{N}\right)^2 = \frac{1}{3N} \tag{20}$$

Therefore the Average Probability of Error is,

$$\overline{P}_{b,1} = Q\left(\sqrt{\frac{2E_b}{N_o}\frac{1}{1+\frac{2E_b}{N_o}\sum_{j=2}^{K} s_{j,1}^{2}}}\right)$$

$$= Q\left(\sqrt{\frac{2E_b}{N_o}\frac{1}{1+\frac{2E_b}{N_o}\frac{K-1}{3N}}}\right) \quad (21)$$

For large K, the average probability of error for ***asynchronous*** users is,

$$\overline{P}_{b,1} = Q\left(\sqrt{\frac{3N}{K-1}}\right) \quad (22a)$$

For large K, the average probability of error for ***synchronous*** users is,

$$\overline{P}_{b,1} = Q\left(\sqrt{\frac{N}{K-1}}\right) \quad (22b)$$

Thus we see Probability of Error is less for asynchronous users than for synchronous users which is due to the better averaging effect due to offset of bits.

The expressions in this section are valid only if the number of users K is large. Furthermore, depending on the distribution of the power levels for the K-1 interfering users, even when K is large, if the interferer power levels are not equal or constant the Central Limit Theorem does not hold and the multiple access interference contribution $I_{j,1}$ cannot be modeled as a Gaussian Random Variable.

Therefore Gaussian approximation is not appropriate in the following cases (1) Number of users K is not large (2) Interferers have disparate power levels.
In these situations a more in-depth analysis is required which we shall study in the next section.

2.1.2 Improved Gaussian Approximation (IGA)

This analysis defines the interference terms I_j, conditioned on the particular operating condition of each user. When this is done, each I_j becomes Gaussian for large K.
Define ψ as the variance of the multiple access interference for a specific operating condition i.e. ψ is the conditional variance of I_j.

$$\psi = Var\left[I_1\left(\underline{b}, \underline{t}, \underline{f}, c_1, ..., c_K, \underline{p}\right) \mid \underline{f}, \underline{t}, \underline{p}, c_1, ..., c_K\right]$$

$$= Var\left[I_1(.) \mid \underline{f}, \{\Delta_k\}, \{p_k\}, B\right]$$

Here p_k = Power of kth user

Δ_k = Offset of kth user

If the distribution of ψ is known the bit error rate may be found by averaging over all possible values of ψ.

$$P_{b,1} = E\left[Q\left(\sqrt{\frac{P_1 T_b^2}{2y}}\right)\right]$$

$$= \int_0^{\infty} Q\left(\sqrt{\frac{P_1 T_b^2}{2y}}\right) f_y(.)dy \qquad (23)$$

It is possible to show that,

$$\psi = \sum_{k=2}^{K} T_c^2 P_k \cos^2 f_k \left(\frac{N}{2} + (2B+1)\left(\left(\frac{\Delta_k}{T_c}\right)^2 - \frac{\Delta_k}{T_c}\right)\right) = \sum_{k=2}^{K} Z_k \qquad (24)$$

where,

$$Z_k = \frac{T_c^2}{2} P_k U_k V_k$$

$$U_k = 1 + \cos(2\phi_k)$$

$$V_k = \frac{N}{2} + (2B+1)\left(\left(\frac{\Delta_k}{T_c}\right)^2 - \frac{\Delta_k}{T_c}\right)$$

Closed form expression for f_u (.) and $f_{v \mid B}$(.) can be found so the distribution of Z_k can be determined.

PDF of ψ can be obtained by K-1 fold convolution of Z_k,

$$f_y(.) = f_{z_2} * \ldots * f_{z_k}$$

This equation may be used in eq. (23) to determine the average bit error probability. This technique has been shown (refer to paper by Morrow and Lehnert - TCOM Oct 1989) to be accurate for a very small number of interfering users for the case of perfect power control.

This results in an Improved Gaussian Approximation.

2.1.3 Simple Improved Gaussian Approximation (SIGA)

The expressions presented in the previous section are complicated and require significant computational time to evaluate. A simplified expression for IGA, is given by Holtzman in the paper "A simple accurate method to calculate SSMA error probabilities", IEEE, TCOM Mar 1992.
The simplified bit error probability expressions are based on the fact that a continuous function f(x) may be expressed as,

$$f(x) = f(m) + (x-m) f'(m) + \frac{1}{2}(x-m)^2 f''(m) + \ldots \tag{25}$$

If *x* is a random variable and μ is the mean of *x* i.e. E[*x*] = μ
Therefore,

$$\begin{aligned} E[f(x)] &= E[f(m)] + 0 + \frac{1}{2} s^2 f''(m) + \ldots \\ &= f(m) + \frac{1}{2} s^2 f''(m) \end{aligned}$$

If derivatives are expressed in terms of finite differences,

$$f(x) = f(m) + (x-m)\left\{\frac{f(m+h) - f(m-h)}{2h}\right\} + \frac{1}{2}(x-m)^2\left\{\frac{f(m+h) - 2f(m) + f(m-h)}{h^2}\right\}$$

Neglecting higher order terms,

$$E[f(x)] = f(m) + \frac{s^2}{2}\left\{\frac{f(m+h) - 2f(m) + f(m-h)}{h^2}\right\} \tag{26}$$

In [Hol92] it is suggested that an appropriate choice for h is $\sqrt{3}s$ which yields,

$$E[f(x)] = \frac{2}{3} f(m) + \frac{1}{6} f(m + 3s) + \frac{1}{6} f(m - 3s) \tag{27}$$

If we consider $Q(.) = f(x)$, then using eq. (27) , the average probability of error is given by,

$$\begin{aligned} \overline{P}_{b,1} &= E\left[Q\left(\sqrt{\frac{P_1 T_b^2}{2y}}\right)\right] \\ &= \frac{2}{3} Q\left(\sqrt{\frac{P_1 T_b^2}{2m_y}}\right) + \frac{1}{6} Q\left(\sqrt{\frac{P_1 T_b^2}{2\left(m_y + \sqrt{3}s_y\right)}}\right) + \frac{1}{6} Q\left(\sqrt{\frac{P_1 T_b^2}{2\left(m_y - \sqrt{3}s_y\right)}}\right) \end{aligned} \tag{28}$$

μ_ψ = Mean of variance of Multiple Access Interference (MAI) conditioned on parameters

σ_ψ^2 = Variance of ψ

Fig. 3 and Fig.4 illustrate the average bit error rates for single-cell CDMA systems using the different analytical methods discussed in this lecture (i.e. GA, IGA and SIGA). For results shown the processing gain is N=31.

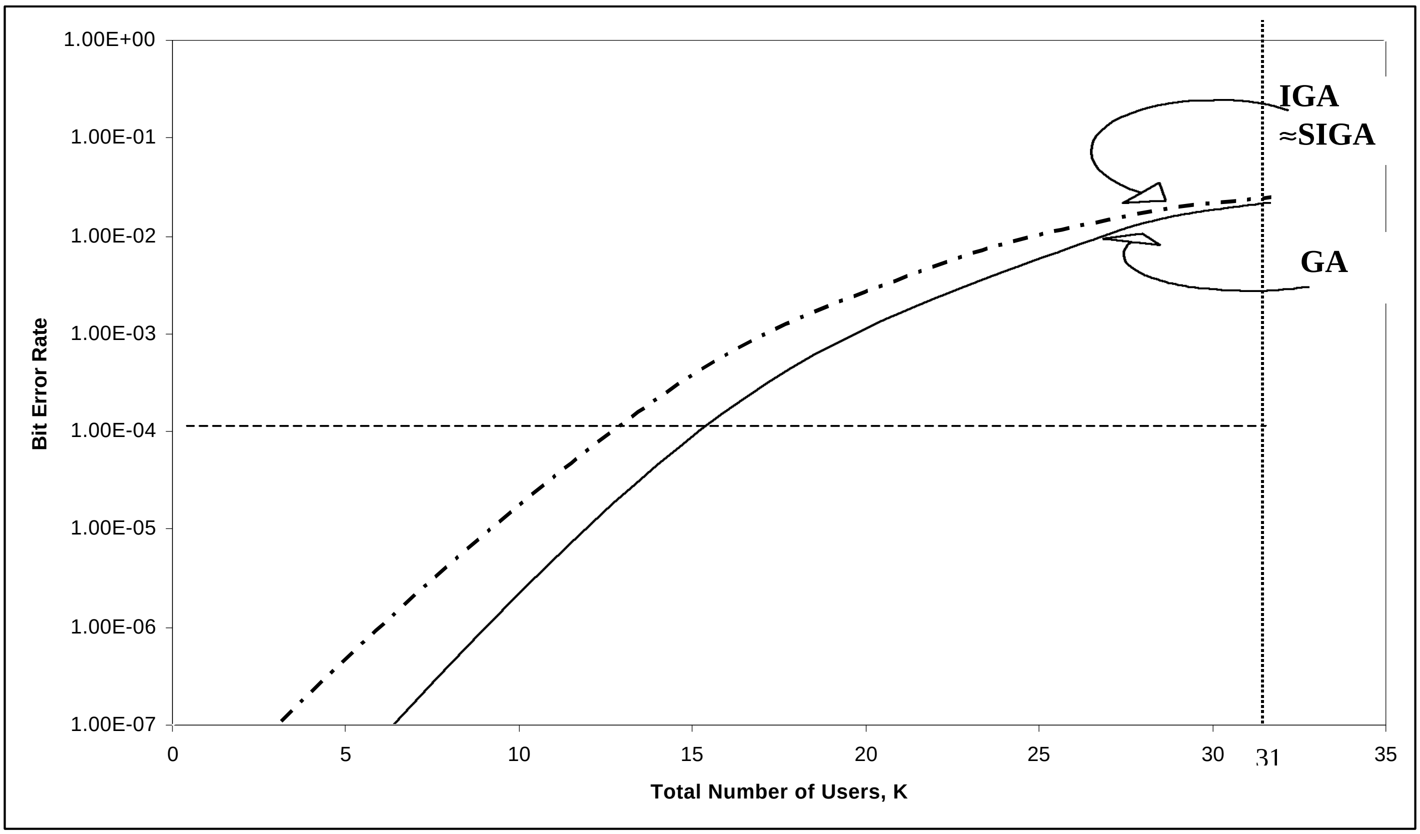


Figure 3: Comparison of BER for a desired user assuming that power level for all K users are fixed

In this comparison it is assumed that the power levels for all K users are fixed.

K_2 = [K/2] users have a power level of $P_1/4$. The remaining $K_1 = K - 1 - K_2$ users each have power level equal to that of the desired user, P_1.
Note that SIGA provides a very close match to IGA. For small number of users GA falls apart.

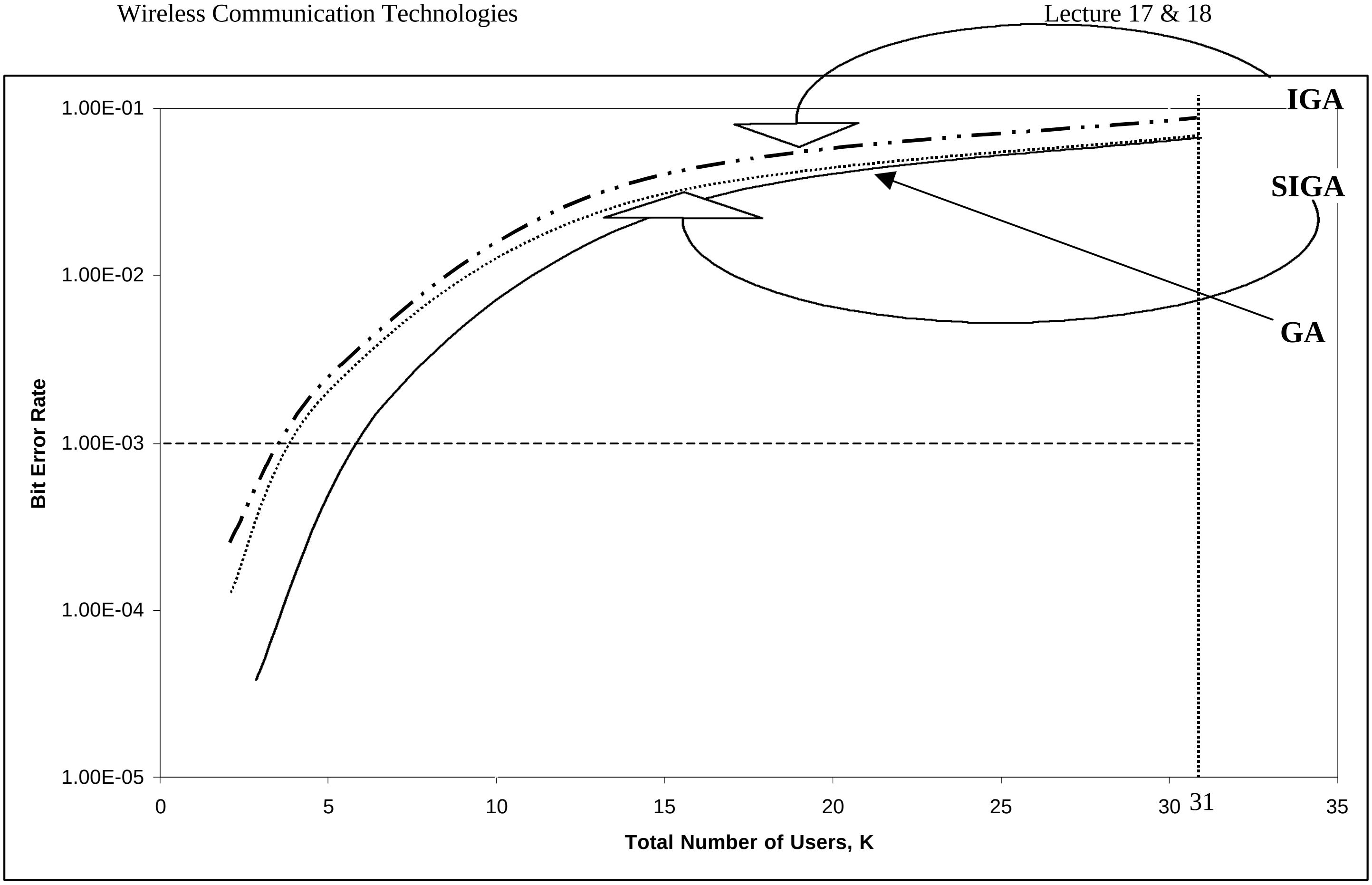


Figure 4: Comparison GA, IGA and SIGA when interfering users have random but identically distributed power levels

The interferer power levels obey a log-normal distribution with a standard deviation of 5dB. The power level of the desired user is constant and equal to the mean value of the power level of each individual interfering user.

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[4] Kemal Karakayali, Lectures notes for "16:332:559", Spring 2001