# Probability and Random Processes 

Course No: 14:332:321 (Fall 2000)
Final Exam Solutions
Maximum Marks : 100
Total Time : 3 hours
Instructions : Answer all questions. The points for each question are listed below in parentheses.

1. Suppose I look at my wristwatch exactly once every hour but at a randomly chosen time during each hour. When I look at my watch, the probabilities of the following events are :
(a) $P[$ The hour hand is between 6 and 7$]=1 / 12$
(b) $P[$ The minute hand is between 6 and 7$]=1 / 12$
(c) $P[$ Both the hour hand and the minute hand are between 6 and 7$]=$ $P[$ look at my watch between $6: 30$ and $6: 35]=1 / 144$
(d) Yes, independent.
2. A submarine attempts to sink an aircraft carrier by firing torpedoes at it. It will be successful only if two or more torpedoes hit the carrier. If the sub fires three torpedoes and the probability of a hit is 0.4 for each torpedo, the probability $p$ that the carrier will be sunk is

$$
\begin{gather*}
p=P[\text { exactly } 2 \text { hits }]+P[\text { exactly } 3 \text { hits }] \\
p=\binom{3}{2}(0.4)^{2}(1-0.4)^{1}+\binom{3}{3}(0.4)^{3}(1-0.4)^{0}=0.352 \tag{6}
\end{gather*}
$$

3. Radars detect flying objects by measuring the power reflected from them. The power reflected from an aircraft is modelled well as a random variable $P$ whose PDF is given as

$$
f_{P}(p)= \begin{cases}\frac{1}{P_{o}} e^{-p / P_{o}} & p \geq 0 \\ 0 & p<0\end{cases}
$$

where $P_{o}>0$ is some constant. The aircraft is successfully identified by the radar if the received power reflected from the aircraft is larger than its average value. The probability of this happening is

$$
\begin{equation*}
P\left[P>P_{o}\right]=1-P\left[P \leq P_{o}\right]=1-\int_{o}^{P_{o}} \frac{1}{P_{o}} e^{-p / P_{o}}=1 / e \approx 36.8 \% \tag{15}
\end{equation*}
$$

4. Assume you are a contestant appearing on the Monty Hall TV show. To refresh your memory regarding the details of the show, there are three doors that are closed and behind them are 2 goats and 1 car (only one item is behind each door). You are asked to select one of the doors without opening it. Then, your host Monty Hall opens one of the other doors to reveal a goat. You are then asked if you would like to switch your selection to the other unopened door. Since you are not sure, you toss a coin and decide on the following strategy:

Heads shows on toss $\Rightarrow$ You switch to the other door
Tails shows on toss $\Rightarrow$ You do not switch to the other door
The tree diagram for the procedure is given as

(a) The probability that you win the car if the probability of observing heads on the coin toss is $p$ is $P[W]=\frac{1}{3}(1-p)+\frac{1}{3} p+\frac{1}{3} p=\frac{1}{3}+\frac{1}{3} p$
(b) Using the result in (a), the probability of winning if $p=1 / 2$ is $P[W]=1 / 2$
5. The Rayleigh random variable describes the envelope of a radio signal received by a cellular (15) telephone. Its PDF is given as

$$
f_{X}(x)= \begin{cases}2(x-a) e^{-(x-a)^{2}} & x \geq a \\ 0 & x<a\end{cases}
$$

where $a$ is a constant. The distribution function of the Rayleigh random variable is given as

$$
\begin{aligned}
F_{x}(x) & =\int_{a}^{x} 2(y-a) e^{-(y-a)^{2}} d y, x \geq a \\
& =\int_{0}^{x-a} 2 t e^{-t^{2}} d t, \quad x \geq a \\
& = \begin{cases}1-e^{-(x-a)^{2}} & x \geq a \\
0 & x<a\end{cases}
\end{aligned}
$$

The median of the Rayleigh random variable is found by solving for $x_{\text {med }}$ where $F_{X}\left(x_{m e d}\right)=0.5$ which results in $x_{\text {med }}=a+[\ln (2)]^{1 / 2}$
6. Two random variables $X_{1}$ and $X_{2}$ have zero means and variances $\sigma_{X_{1}}^{2}=4$ and $\sigma_{X_{2}}^{2}=9$.

Their covariance is given as $\operatorname{Cov}\left(X_{1}, X_{2}\right)=3$.
(a) Since $X_{1}$ and $X_{2}$ are zero mean, their correlation is same as their covariance. Therefore, [3] the covariance matrix of the vector $\underline{X}=\binom{X_{1}}{X_{2}}$ is same as its correlation matrix

$$
R_{X}=\left[\begin{array}{ll}
\sigma_{X_{1}}^{2} & \operatorname{Cov}\left(X_{1}, X_{2}\right)  \tag{12}\\
\operatorname{Cov}\left(X_{1}, X_{2}\right) & \sigma_{X_{2}}^{2}
\end{array}\right]=\left[\begin{array}{ll}
4 & 3 \\
3 & 9
\end{array}\right]
$$

(b) $X_{1}$ and $X_{2}$ are linearly transformed to new variables $Y_{1}$ and $Y_{2}$ according to

$$
\begin{gathered}
Y_{1}=X_{1}-2 X_{2} \\
Y_{2}=3 X_{1}+4 X_{2}
\end{gathered}
$$

Note that $Y_{1}$ and $Y_{2}$ are also zero mean $\Rightarrow$ their covariance is same as their correlation. The covariance matrix of the vector $\underline{Y}=\binom{Y_{1}}{Y_{2}}$ is the same as the correlation matrix

$$
R_{Y}=\left[\begin{array}{ll}
\sigma_{Y_{1}}^{2} & \operatorname{Cov}\left(Y_{1}, Y_{2}\right) \\
\operatorname{Cov}\left(Y_{1}, Y_{2}\right) & \sigma_{Y_{2}}^{2}
\end{array}\right]=\left[\begin{array}{ll}
28 & -66 \\
-66 & 252
\end{array}\right]
$$

The entries above are obtained as follows:

$$
\begin{gather*}
\sigma_{Y_{1}}^{2}=\sigma_{X_{1}}^{2}+4 \sigma_{X_{2}}^{2}+2 \operatorname{Cov}\left(X_{1},-2 X_{2}\right)=\sigma_{X_{1}}^{2}+4 \sigma_{X_{2}}^{2}-4 \operatorname{Cov}\left(X_{1}, X_{2}\right)=28 \\
\sigma_{Y_{2}}^{2}=9 \sigma_{X_{1}}^{2}+16 \sigma_{X_{2}}^{2}+2 \operatorname{Cov}\left(3 X_{1}, 4 X_{2}\right)=9 \sigma_{X_{1}}^{2}+16 \sigma_{X_{2}}^{2}+24 \operatorname{Cov}\left(X_{1}, X_{2}\right)=252 \\
\operatorname{Cov}\left(Y_{1}, Y_{2}\right)=E\left[Y_{1} Y_{2}\right]=E\left[\left(X_{1}-2 X_{2}\right)\left(3 X_{1}+4 X_{2}\right)\right]=3 \sigma_{X_{1}}^{2}-8 \sigma_{X_{2}}^{2}-2 \operatorname{Cov}\left(X_{1}, X_{2}\right)=-66 \tag{12}
\end{gather*}
$$

7. Let $X$ and $Y$ be independent and identically distributed binomial random variables with parameters $n$ and $p$, i.e.,

$$
P_{X}(k)=P_{Y}(k)= \begin{cases}\binom{n}{k} p^{k}(1-p)^{n-k} & k=0,1,2, \cdots, n \\ 0 & \text { otherwise }\end{cases}
$$

The moment generating function of $Z$ is $\phi_{Z}(s)=\phi_{X}(s) \phi_{Y}(s)=\left(\phi_{X}(s)\right)^{2}$

$$
\begin{aligned}
\phi_{X}(s)=\sum_{k=0}^{n} e^{s k}\binom{n}{k} p^{k}(1-p)^{n-k} & =\sum_{k=0}^{n}\binom{n}{k}\left(p e^{s}\right)^{k}(1-p)^{n-k}=\left(p e^{s}+1-p\right)^{n} \\
\phi_{Z}(s) & =\left(p e^{s}+1-p\right)^{2 n}
\end{aligned}
$$

$\Rightarrow Z$ is binomial with parameters $2 n$ and $p$, i.e., the probability mass function (PMF) of $Z=X+Y$ is

$$
P_{Z}(k)= \begin{cases}\binom{2 n}{k} p^{k}(1-p)^{n-k} & k=0,1,2,, \cdots, 2 n \\ 0 & \text { otherwise }\end{cases}
$$

8. Consider a collection of independent and identically distributed exponential random variables $X_{1}, X_{2}, X_{3}, \cdots, X_{1000}$ with parameter 10 . A new random variable $Y$ is formed as

$$
Y=\sum_{i=1}^{1000} X_{i}
$$

Note that the exact probability density function (PDF) of $Y$ can be obtained by a 1000-fold convolution of the the exponential PDF. A Central Limit Theorem approximation for the PDF of $Y$ is

$$
f_{Y}(y)=\frac{1}{\sqrt{2 \pi \sigma^{2}}} e^{-\frac{(y-\mu)^{2}}{2 \sigma^{2}}}-\infty \leq y \leq \infty
$$

where $E[Y]=\mu=100$ and $\operatorname{Var}[Y]=\sigma^{2}=10$
9. Consider the random process

$$
X(t)=A \cos \left(\omega_{o} t\right)+B \sin \left(\omega_{o} t\right)
$$

where $A$ and $B$ are uncorrelated, zero mean random variables with the same variance $\sigma^{2}$.
To find the autocorrelation function $R_{X}(t, \tau)$, we need to evaluate $R_{X}(t, \tau)=E[X(t) X(t+\tau)]$
Using the facts that $E[A B]=0$ and $E\left[A^{2}\right]=E\left[B^{2}\right]=\sigma^{2}$, we can see that

$$
R_{X}(t, \tau)=E[X(t) X(t+\tau)]=\sigma^{2}\left[\cos \left(\omega_{o} t\right) \cos \left(\omega_{o} t+\omega_{o} \tau\right)+\sin \left(\omega_{o} t\right) \sin \left(\omega_{o} t+\omega_{o} \tau\right)\right]
$$

Using $\cos (A-B)=\cos (A) \cos (B)+\sin (A) \sin (B)$,

$$
R_{X}(t, \tau)=\sigma^{2} \cos \left(\omega_{o} \tau\right)
$$

Since $E[X(t)]=0$ and $R_{X}(t, \tau)$ is a function of $\tau$ only, $X(t)$ wide-sense stationary!

