

Probability and Random Processes

Course No: 14:332:321 (Fall 2000)

Final Exam

Maximum Marks : 100

Total Time : 3 hours

Instructions : *Answer all questions. The points for each question are listed below in parentheses.*

1. Suppose I look at my wristwatch exactly once every hour but at a randomly chosen time (6)
during each hour. When I look at my watch, what are the probabilities of the following events:
 - (a) The hour hand is between 6 and 7
 - (b) The minute hand is between 6 and 7
 - (c) Both the hour hand and the minute hand are between 6 and 7
 - (d) Use the result in (c) to say if the events in (a) and (b) are independent ?
2. A submarine attempts to sink an aircraft carrier by firing torpedoes at it. It will be (6)
successful only if two or more torpedoes hit the carrier. If the sub fires three torpedoes and the probability of a hit is 0.4 for each torpedo, what is the probability that the carrier will be sunk ?
3. Radars detect flying objects by measuring the power reflected from them. The power (6)
reflected from an aircraft is modelled well as a random variable P whose PDF is given as

$$f_P(p) = \begin{cases} \frac{1}{P_o} e^{-p/P_o} & p \geq 0 \\ 0 & p < 0 \end{cases}$$

where $P_o > 0$ is some constant. The aircraft is successfully identified by the radar if the received power reflected from the aircraft is larger than its average value. Show that the aircraft is successfully identified about 36.8 % of the time.

4. Assume you are a contestant appearing on the Monty Hall TV show. To refresh your (15)
memory regarding the details of the show, there are three doors that are closed and behind them are 2 goats and 1 car (only one item is behind each door). You are asked to select one of the doors without opening it. Then, your host Monty Hall opens one of the other doors to reveal a goat. You are then asked if you would like to switch your selection to the other unopened door. Since you are not sure, you toss a coin and decide on the following strategy:

Heads shows on toss \Rightarrow You switch to the other door

Tails shows on toss \Rightarrow You do not switch to the other door

- (a) What is the probability that you win the car if the probability of observing heads on the coin toss is p where $0 < p < 1$?
- (b) Using the result in (a), what is the probability of winning if $p = 1/2$?

5. The Rayleigh random variable describes the envelope of a radio signal received by a cellular telephone. Its PDF is given as

$$f_X(x) = \begin{cases} 2(x-a)e^{-(x-a)^2} & x \geq a \\ 0 & x < a \end{cases}$$

where a is a constant. Find the median of the Rayleigh random variable ?

6. Two random variables X_1 and X_2 have zero means and variances $\sigma_{X_1}^2 = 4$ and $\sigma_{X_2}^2 = 9$. Their covariance is given as $Cov(X_1, X_2) = 3$. (15)

(a) Find the covariance matrix of the vector

$$\underline{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

(b) X_1 and X_2 are linearly transformed to new variables Y_1 and Y_2 according to

$$Y_1 = X_1 - 2X_2$$

$$Y_2 = 3X_1 + 4X_2$$

Find the covariance matrix of the vector

$$\underline{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

7. Let X and Y be independent and identically distributed binomial random variables with parameters n and p , i.e., (12)

$$P_X(k) = P_Y(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & k = 0, 1, 2, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

Find the probability mass function (PMF) of $Z = X + Y$

8. Consider a collection of independent and identically distributed exponential random variables $X_1, X_2, X_3, \dots, X_{1000}$ with parameter 10. A new random variable Y is formed as (10)

$$Y = \sum_{i=1}^{1000} X_i$$

Note that the exact probability density function (PDF) of Y can be obtained by a 1000-fold convolution of the the exponential PDF. Given that this is an absurd if not tedious exercise, find a reasonable approximation for the PDF of Y

9. Consider the random process (15)

$$X(t) = A \cos(\omega_o t) + B \sin(\omega_o t),$$

where A and B are uncorrelated, zero mean random variables with the same variance σ^2 . Find the autocorrelation function $R_X(t, \tau)$. Is $X(t)$ wide-sense stationary ?

Good luck and Happy Holidays!