Probability and Random Processes

Course No: 14:332:321 (Fall 2001)

Solutions to Exam 2

Maximum Marks : **60** **Total Time** : **1hour** & **10minutes**

Instructions : *The points for each question are listed below in parentheses.*

1. $Y = 3X + 4$, where X is an exponential random variable with parameter 3. Therefore $E[X] = 1/3$ and $Var[X] = 1/9$. **(10)**

   (a) Therefore $E[Y] = 3E[X] + 4 = 5$ and $Var[Y] = 9Var[X] = 1$ **[6]**

   (b) If Y is an exponential random variable, then the following must be true: $(E[Y])^2 = Var[Y]$. Clearly, in this case $5^2 \neq 1$, thus Y is not an exponential random variable. **[4]**

2. A total of 4 buses carrying 148 students from the same school arrives at a football game. The buses carry respectively 40, 33, 25 and 50 students. **(10)**

   (a) One of the students is randomly selected during a halftime prize giveaway event. Let X denote the number of students that were on the bus carrying this randomly selected student. The PMF of X is **[4]**

$$P_X(x) = \begin{cases} x/148 & x = 40, 33, 25, 50 \\ 0 & \text{otherwise} \end{cases}$$

   $E[X] = \sum_x x P_X(x) = [40^2 + 33^2 + 25^2 + 50^2]/148 \approx 39.28$

   (b) One of the 4 bus drivers is also randomly selected to receive a prize during the same halftime event. Let Y denote the number of students on her bus. **[4]**

$$P_Y(y) = \begin{cases} 1/4 & y = 40, 33, 25, 50 \\ 0 & \text{otherwise} \end{cases}$$

   $E[Y] = \sum_y y P_Y(y) = 148/4 = 37$

   (c) X and Y are independent, therefore they are also uncorrelated. **[2]**

3. Three distinct numbers are randomly distributed to players A, B and C. Whenever two players compare their numbers, the one with the higher number wins. Initially, players A and B compare their numbers; the winner then compares with player C. Let Y denote the number of times player A is a winner. **(10)**

   Clearly A can win 0, 1 or 2 times.

   Consider when $Y = 0$. This happens only when A loses to B right away. Therefore $P[Y = 0] = 1/2$.

Consider when $Y = 1$. This happens only when A wins against B and then loses to C. Therefore

$$P[Y = 1] = P[\text{C has largest number and A is larger than B}]$$

$$= P[\text{C has largest number}]P[\text{A is larger than B} \mid \text{C has largest number}]$$

$$= 1/3 \times 1/2 = 1/6$$

Consider when $Y = 2$. In this case, A has to have the largest number, i.e., $P[Y = 2] = 1/3$

The probability mass function of Y is therefore:

$$P_Y(y) = \begin{cases} 1/2 & y = 0 \\ 1/6 & y = 1 \\ 1/3 & y = 2 \\ 0 & \text{otherwise} \end{cases}$$

4. A salesman has scheduled two appointments to sell vacuum cleaners. His first appointment is with a "sloppy" single-guy and will lead to a sale with probability 0.3. His second appointment is with a "mysophobic"[1] woman and will lead independently to a sale with probability 0.8. Any sale made is equally likely to be either for the deluxe model which costs \$ 1000 or the standard model which costs \$ 500. Let X, be the total dollar value of all sales. **(15)**

   Let us define the following events and corresponding probabilities:

   - $S_g \equiv$ *Sale with guy* $\quad P[S_g] = 0.3$
   - $N_g \equiv$ *No Sale with guy* $\quad P[N_g] = 0.7$
   - $S_w \equiv$ *Sale with woman* $\quad P[S_w] = 0.8$
   - $N_w \equiv$ *No Sale with woman* $\quad P[N_w] = 0.2$
   - $Dx \equiv$ *Deluxe Sale* $\quad P[Dx] = 0.5$
   - $St \equiv$ *Standard Sale* $\quad P[St] = 0.5$

   $X = 0$ when both the guy and the woman do not buy a vacuum cleaner. Therefore,

$$P[X = 0] = P[N_g \textit{ and } N_w] = P[N_g]P[N_w] = 0.7 \times 0.2 = 0.14$$

   $X = 500$ when there is only one standard vacuum cleaner sale (either to the guy or the woman) from both visits . Therefore,

$$P[X = 500] = P[S_gSt \textit{ and } N_w] + P[N_g \textit{ and } S_wSt] = 0.3 \times 0.5 \times 0.2 + 0.7 \times 0.8 \times 0.2 = 0.31$$

   $X = 1000$ when there are two standard sales (one to the guy and one to the woman) or one deluxe sale (either to the guy or the woman). Therfore,

$$P[X = 1000] = P[S_gSt \textit{ and } S_wSt] + P[S_gDx \textit{ and } N_w] + P[N_g \textit{ and } S_wDx]$$

$$= 0.3 \times 0.5 \times 0.8 \times 0.5 + 0.3 \times 0.5 \times 0.2 + 0.7 \times 0.8 \times 0.5 = 0.37$$

[1] Mysophobia: Fear of Dirt

$X = 1500$ when there is one standard and one deluxe sale to the guy and woman, respectively or vice-versa. Therfore,

$$P[X = 1500] = P[S_g Dx \; and \; S_w St] + P[S_g St \; and \; S_w Dx] = 0.3 \times 0.5 \times 0.8 \times 0.5 + 0.3 \times 0.5 \times 0.8 \times 0.5 = 0.12$$

$X = 2000$ when one deluxe sold to the guy and one to the woman. Therefore,

$$P[X = 2000] = P[S_g Dx \; and \; S_w Dx] = 0.3 \times 0.5 \times 0.8 \times 0.5 = 0.06$$

$$P_X(x) = \begin{cases} 0.14 & x = 0 \\ 0.31 & x = 500 \\ 0.37 & x = 1000 \\ 0.12 & x = 1500 \\ 0.06 & x = 2000 \\ 0 & \text{otherwise} \end{cases}$$

5. If X and Y are discrete random variables such that their joint PMF is $P_{X,Y}(x, y)$ and their marginal PMFs are $P_X(x)$ and $P_Y(y)$, respectively. To prove that if X and Y are independent, then they are also uncorrelated we proceed as follows. **(5)**

   Since X and Y are independent, $P_{X,Y}(x, y) = P_X(x)P_Y(y)$. Therefore, we can write

   $$E[XY] = \sum_x \sum_y xyP_{X,Y}(x, y) = \sum_x \sum_y xyP_X(x)P_Y(y) = (\sum_x xP_X(x))(\sum_y yP_Y(y)) = E[X]E[Y]$$

   which in turn implies

   $$Cov[X, Y] = E[XY] - E[X]E[Y] = 0,$$

   thus proving that X and Y are uncorrelated.

6. X is a random variable with PDF given as **(10)**

   $$f_X(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-2)^2}{8}} \quad -\infty \leq x \leq \infty$$

   Clearly X is Gaussian with mean $\mu_X = 2$ and variance $\sigma_X^2 = 4$. Since a linear transformation of the Gaussian random variable results in a Gaussian random variable, $Y = 2X + 10$ is also Gaussian with $\mu_Y = 2\mu_X + 10 = 14$ and $\sigma_Y^2 = 4\sigma_X^2 = 16$. The PDF of Y is

   $$f_Y(y) = \frac{1}{4\sqrt{2\pi}} e^{-\frac{(y-14)^2}{32}} \quad -\infty \leq y \leq \infty$$