Probability and Random Processes

Course No: 14:332:321 (Fall 2000)

Solutions to Exam 2

Maximum Marks : **30** **Total Time** : **1hour** & **10minutes**

Instructions : *Answer all questions. The points for each question are listed below in parentheses.*

1. Fill in the blanks **(5)**

   (a) If random variable X is Bernoulli with parameter 0.5, $E[X] = 0.5$

   (b) If random variable X is Bernoulli with parameter 0.5, $Var[X] = 0.25$

   (c) If random variable X is geometric with parameter 0.5, $E[X] = 2$

   (d) If random variable X is geometric with parameter 0.5, $Var[X] = 2$

   (e) If random variable X is Poisson with parameter 0.5, $Var[X] = 0.5$

2. X is a uniform random variable taking values over the interval $[1, 5]$. Consider $K = \lceil X \rceil$. **(8)**

   (a) The probability mass function (PMF) of K is that of a dicrete uniform random variable given as

$$P_K(k) = \begin{cases} 1/4 & k = 2, 3, 4, 5 \\ 0 & otherwise \end{cases}$$

   (b) $E[K] = \sum_{k=2}^{5} k P_K(k) = \frac{1}{4} \sum_{k=2}^{5} k = 7/2$

   (c) $Var[K] = E[K^2] - (E[K])^2 = \sum_{k=2}^{5} k^2 P_K(k) - (E[K])^2 = \frac{1}{4} \sum_{k=2}^{5} k^2 - (7/2)^2 = 5/4$

3. Flip a fair coin until heads occurs twice. Let X_1 equal the number of flips up to and including the first head. Let X_2 equal the number of additional flips up to and including the second head. **(6)**

   (a)

$$P_{X_1}(x_1) = \begin{cases} 0.5^{x_1 - 1} 0.5 & x_1 = 1, 2, 3, \cdots \\ 0 & otherwise \end{cases}$$

   (b)

$$P_{X_2}(x_2) = \begin{cases} 0.5^{x_2 - 1} 0.5 & x_2 = 1, 2, 3, \cdots \\ 0 & otherwise \end{cases}$$

   (c) X_1 and X_2 are independent since the flips upto the first head do not affect the additional flips in any way.

   (d)

$$P_{X_1, X_2}(x_1, x_2) = \begin{cases} 0.5^{x_1 + x_2} & x_1 = 1, 2, 3, \cdots;\ x_2 = 1, 2, 3, \cdots \\ 0 & otherwise \end{cases}$$

4. If X and Y are random variables such that $Y = aX + b$, prove that **(6)**

$$\rho_{X,Y} = \begin{cases} -1 & a < 0 \\ 0 & a = 0 \\ 1 & a > 0 \end{cases}$$

There are three cases, $a > 0$, $a < 0$ and $a = 0$.

Consider $a = 0$. In this case $Y = b$ is a constant and not a random variable. Therefore, X cannot be correlated with a constant and thus $\rho_{X,Y} = 0$.

When $a \neq 0$, we can see that

$$\rho_{X,Y} = \frac{Cov[X,Y]}{\sqrt{Var[X]Var[Y]}} = \frac{E[XY] - E[X]E[Y]}{\sqrt{Var[X]Var[Y]}} = \frac{E[X(aX+b)] - E[X]E[aX+b]}{\sqrt{Var[X]Var[aX+b]}}$$

$$= \frac{aE[X^2] - a(E[X])^2}{\sqrt{Var[X]a^2Var[X]}} = \frac{aVar[X]}{\mid a \mid Var[X]} = \frac{a}{\mid a \mid} = \begin{cases} -1 & a < 0 \\ 1 & a > 0 \end{cases}$$

5. X is a random variable with PDF given as **(5)**

$$f_X(x) = \frac{1}{\sqrt{18\pi}} e^{-\frac{(x-3)^2}{18}} \quad -\infty \leq x \leq \infty$$

Note that X is a Gaussian random variable with mean $\mu = 3$ and variance $\sigma^2 = 9$. Therefore, the second moment of X is $E[X^2] = \sigma^2 + \mu^2 = 9 + 9 = 18$