Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
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Problem Solutions : Yates and Goodman,4.3.1 4.3.4 4.3.5 4.4.1 4.4.4 4.4.6 4.5.1 4.5.7 4.6.1 and 4.6.8

Problem 4.3.1

$$f_X(x) = \begin{cases} 1/4 & -1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

We recognize that X is a uniform random variable from [-1,3].

(a) $E[X] = 1$ and $\mathrm{Var}[X] = \frac{(3+1)^2}{12} = 4/3$.

(b) The new random variable Y is defined as $Y = h(X) = X^2$. Therefore

$$h(E[X]) = h(1) = 1$$

and

$$E[h(X)] = E\left[X^2\right] = \mathrm{Var}[X] + E[X]^2 = 4/3 + 1 = 7/3$$

Finally

$$E[Y] = E[h(X)] = E\left[X^2\right] = 7/3$$
$$\mathrm{Var}[Y] = E\left[X^4\right] - E\left[X^2\right]^2 = \int_{-1}^{3} \frac{x^4}{4}\,dx - \frac{49}{9} = \frac{61}{5} - \frac{49}{9}$$

Problem 4.3.4

(a) We can find the expected value of X by direct integration of the given PDF.

$$f_Y(y) = \begin{cases} y/2 & 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

The expectation is

$$E[Y] = \int_0^2 \frac{y^2}{2}\,dy = 4/3$$

(b)

$$\begin{aligned} E\left[Y^2\right] &= \int_0^2 \frac{y^3}{2}\,dy = 2 \\ \mathrm{Var}[Y] &= E\left[Y^2\right] - E[Y]^2 = 2 - (4/3)^2 = 2/9 \end{aligned}$$

Problem 4.3.5
The CDF of Y is

$$F_Y(y) = \begin{cases} 0 & y < -1 \\ (y+1)/2 & -1 \leq y < 1 \\ 1 & y \geq 1 \end{cases}$$

(a) We can find the expected value of Y by first find the PDF by differentiating the above CDF.

$$f_Y(y) = \begin{cases} 1/2 & -1 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

And

$$E[Y] = \int_{-1}^{1} y/2\, dy = 0$$

(b)

$$\begin{aligned} E\left[Y^2\right] &= \int_{-1}^{1} \frac{y^2}{2}\, dy = 1/3 \\ \operatorname{Var}[Y] &= E\left[Y^2\right] - E[Y]^2 = 1/3 - 0 = 1/3 \end{aligned}$$

Problem 4.4.1
observe that an exponential PDF Y with parameter $\lambda > 0$ has PDF

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

In addition, the mean and variance of Y are

$$E[Y] = \frac{1}{\lambda} \qquad \operatorname{Var}[Y] = \frac{1}{\lambda^2}$$

(a) Since $\operatorname{Var}[Y] = 25$, we must have $\lambda = 1/5$.

(b) The expected value of Y is $E[Y] = 1/\lambda = 5$.

(c)

$$P[Y > 5] = \int_{5}^{\infty} f_Y(y)\, dy = -e^{-y/5}\Big|_{5}^{\infty} = e^{-1}$$

Problem 4.4.4

(a) The PDF of a continuous uniform random variable distributed from $[-5, 5)$ is

$$f_X(x) = \begin{cases} 1/10 & -5 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

(b) For $x < -5$, $F_X(x) = 0$. For $x \geq 5$, $F_X(x) = 1$. For $-5 \leq x \leq 5$, the CDF is

$$F_X(x) = \int_{-5}^{x} f_X(\tau)\, d\tau = \frac{x+5}{10}$$

The complete expression for the CDF of X is

$$F_X(x) = \begin{cases} 0 & x < -5 \\ (x+5)/10 & 5 \leq x \leq 5 \\ 1 & x > 5 \end{cases}$$

(c) the expected value of X is

$$\int_{-5}^{5} \frac{x}{10}\, dx = \left.\frac{x^2}{20}\right|_{-5}^{5} = 0$$

Another way to obtain this answer is to use Theorem 4.7 which says the expected value of X is

$$E[X] = \frac{5 + -5}{2} = 0$$

(d) The fifth moment of X is

$$\int_{-5}^{5} \frac{x^5}{10}\, dx = \left.\frac{x^6}{60}\right|_{-5}^{5} = 0$$

The expected value of e^X is

$$\int_{-5}^{5} \frac{e^x}{10}\, dx = \left.\frac{e^x}{10}\right|_{-5}^{5} = \frac{e^5 - e^{-5}}{10} = 14.84$$

Problem 4.4.6
Given that

$$f_X(x) = \begin{cases} (1/2)e^{-x/2} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(a)

$$P[1 \leq X \leq 2] = \int_{1}^{2} (1/2)e^{-x/2}\, dx = e^{-1/2} - e^{-1} = 0.2387$$

(b) The CDF of X may be be expressed as

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \int_0^x (1/2)e^{-x/2}\, d\tau & x \geq 0 \end{cases} = \begin{cases} 0 & x < 0 \\ 1 - e^{-x/2} & x \geq 0 \end{cases}$$

(c) X is an exponential random variable with parameter $a = 1/2$. By Theorem 4.9, the expected value of X is $E[X] = 1/a = 2$.

(d) By Theorem 4.9, the variance of X is $\text{Var}[X] = 1/a^2 = 4$.

Problem 4.5.1
Given that the peak temperature, T, is a Gaussian random variable with mean 85 and standard deviation 10 we can use the fact that $F_T(t) = \Phi((t-\mu_T)/\sigma_T)$ and Table 4.1 on page 142 to evaluate the following

$$
\begin{aligned}
P[T > 100] &= 1 - P[T \leq 100] = 1 - F_T(100) = 1 - \Phi\left(\frac{100-85}{10}\right) \\
&= 1 - \Phi(1.5) = 1 - 0.933 = 0.066 \\
P[T < 60] &= \Phi\left(\frac{60-85}{10}\right) = \Phi(-2.5) \\
&= 1 - \Phi(2.5) = 1 - .993 = 0.007 \\
P[70 \leq T \leq 100] &= F_T(100) - F_T(70) \\
&= \Phi(1.5) - \Phi(-1.5) = 2\Phi(1.5) - 1 = .866
\end{aligned}
$$

Problem 4.5.7
$N[\mu, \sigma^2]$ distribution, the integral we wish to evaluate is

$$
I = \int_{-\infty}^{\infty} f_W(w)\, dw = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-(w-\mu)^2/2\sigma^2}\, dw
$$

(a) Using the substitution $x = (w-\mu)/\sigma$, we have $dx = dw/\sigma$ and

$$
I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx
$$

(b) When we write I^2 as the product of integrals, we use y to denote the other variable of integration so that

$$
\begin{aligned}
I^2 &= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx\right)\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\, dy\right) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dx\, dy
\end{aligned}
$$

(c) By changing to polar coordinates, $x^2 + y^2 = r^2$ and $dx\, dy = r\, dr\, d\theta$ so that

$$
\begin{aligned}
I^2 &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-r^2/2} r\, dr\, d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} -e^{-r^2/2}\Big|_0^{\infty}\, d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} d\theta = 1
\end{aligned}
$$

Problem 4.6.1

(a) Using the given CDF

$$\begin{aligned} P[X < -1] = F_X\left(-1^-\right) &= 0 \\ P[X \leq -1] = F_X(-1) &= -1/3 + 1/3 = 0 \end{aligned}$$

Where $F_X\left(-1^-\right)$ denotes the limiting value of the CDF found by approaching -1 from the left. Likewise, $F_X\left(-1^+\right)$ is interpreted to be the value of the CDF found by approaching -1 from the right. We notice that these two probabilities are the same and therefore the probability that X is exactly -1 is zero.

(b)

$$\begin{aligned} P[X < 0] &= F_X\left(0^-\right) = 1/3 \\ P[X \leq 0] &= F_X(0) = 2/3 \end{aligned}$$

Here we see that there is a discrete jump at $X = 0$. Approached from the left the CDF yields a value of 1/3 but approached from the right the value is 2/3. This means that there is a non-zero probability that $X = 0$, in fact that probability is the difference of the two values.

$$P[X = 0] = P[X \leq 0] - P[X < 0] = 2/3 - 1/3 = 1/3$$

(c)

$$\begin{aligned} P[0 < X \leq 1] &= F_X(1) - F_X\left(0^+\right) = 1 - 2/3 = 1/3 \\ P[0 \leq X \leq 1] &= F_X(1) - F_X\left(0^-\right) = 1 - 1/3 = 2/3 \end{aligned}$$

The difference in the last two probabilities above is that the first was concerned with the probability that X was strictly greater then 0, and the second with the probability that X was greater than or equal to zero. Since the the second probability is a larger set (it includes the probability that $X = 0$) it should always be greater than or equal to the first probability. The two differ by the probability that $X = 0$, and this difference is non-zero only when the random variable exhibits a discrete jump in the CDF.

Problem 4.6.8

good, that is, no foul occurs. The CDF of D obeys

$$F_D(y) = P[D \leq y|G]P[G] + P[D \leq y|G^c]P[G^c]$$

Given the event G,

$$P[D \leq y|G] = P[X \leq y - 60] = 1 - e^{-(y-60)/10} \quad (y \geq 60)$$

Of course, for $y < 60$, $P[D \leq y|G] = 0$. From the problem statement, if the throw is a foul, then $D = 0$. This implies

$$P[D \leq y|G^c] = u(y)$$

where $u(\cdot)$ denotes the unit step function. Since $P[G] = 0.7$, we can write

$$\begin{aligned} F_D(y) &= P[G]P[D \leq y|G] + P[G^c]P[D \leq y|G^c] \\ &= \begin{cases} 0.3u(y) & y < 60 \\ 0.3 + 0.7(1 - e^{-(y-60)/10}) & y \geq 60 \end{cases} \end{aligned}$$

Another way to write this CDF is

$$F_D(y) = 0.3u(y) + 0.7u(y-60)(1 - e^{-(y-60)/10})$$

However, when we take the derivative, either expression for the CDF will yield the PDF. However, taking the derivative of the first expression perhaps may be simpler:

$$f_D(y) = \begin{cases} 0.3\delta(y) & y < 60 \\ 0.07e^{-(y-60)/10} & y \geq 60 \end{cases}$$

Taking the derivative of the second expression for the CDF is a little tricky because of the product of the exponential and the step function. However, applying the usual rule for the differentation of a product does give the correct answer:

$$\begin{aligned} f_D(y) &= 0.3\delta(y) + 0.7\delta(y-60)(1 - e^{-(y-60)/10}) + 0.07u(y-60)e^{-(y-60)/10} \\ &= 0.3\delta(y) + 0.07u(y-60)e^{-(y-60)/10} \end{aligned}$$

The middle term $\delta(y-60)(1 - e^{-(y-60)/10})$ dropped out because at $y = 60$, $e^{-(y-60)/10} = 1$.