Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
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Problem Solutions : Yates and Goodman,3.6.7 3.6.8 3.7.1 3.7.7 4.1.1 and 4.2.1

Problem 3.6.7

The key to solving this problem is to find the joint PMF of M and N. Note that $N \geq M$. For $n > m$, the joint event $\{M = m, N = n\}$ has probability

$$\begin{aligned}
P[M = m, N = n] &= P[\overbrace{dd\cdots d}^{\substack{m-1 \\ \text{calls}}} v \overbrace{dd\cdots d}^{\substack{n-m-1 \\ \text{calls}}} v] \\
&= (1-p)^{m-1} p (1-p)^{n-m-1} p \\
&= (1-p)^{n-2} p^2
\end{aligned}$$

A complete expression for the joint PMF of M and N is

$$P_{M,N}(m,n) = \begin{cases} (1-p)^{n-2} p^2 & m = 1, 2, \ldots, n-1;\ n = m+1, m+2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

For $n = 2, 3, \ldots$, the marginal PMF of N satisfies

$$P_N(n) = \sum_{m=1}^{n-1} (1-p)^{n-2} p^2 = (n-1)(1-p)^{n-2} p^2$$

Similarly, for $m = 1, 2, \ldots$, the marginal PMF of M satisfies

$$\begin{aligned}
P_M(m) &= \sum_{n=m+1}^{\infty} (1-p)^{n-2} p^2 \\
&= p^2 [(1-p)^{m-1} + (1-p)^m + \cdots] \\
&= (1-p)^{m-1} p
\end{aligned}$$

The complete expressions for the marginal PMF's are

$$P_M(m) = \begin{cases} (1-p)^{m-1} p & m = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$P_N(n) = \begin{cases} (n-1)(1-p)^{n-2} p^2 & n = 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Not surprisingly, if we view each voice call as a successful Bernoulli trial, M has a geometric PMF since it is the number of trials up to and including the first success. Also, N has a Pascal PMF since it is the number of trials required to see 2 successes. The conditional PMF's are now easy to find.

$$P_{N|M}(n|m) = \frac{P_{M,N}(m,n)}{P_M(m)} = \begin{cases} (1-p)^{n-m-1} p & n = m+1, m+2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

The interpretation of the conditional PMF of N given M is that given $M = m$, $N = m + N'$ where N' has a geometric PMF with mean $1/p$. The conditional PMF of M given N is

$$P_{M|N}(m|n) = \frac{P_{M,N}(m,n)}{P_N(n)} = \begin{cases} 1/(n-1) & m = 1,\ldots,n-1 \\ 0 & \text{otherwise} \end{cases}$$

Given that call $N = n$ was the second voice call, the first voice call is equally likely to occur in any of the previous $n-1$ calls.

Problem 3.6.8

(a) The number of buses, N, must be greater than zero. Also, the number of minutes that pass cannot be less than the number of buses. Thus, $P[N = n, T = t] > 0$ for integers n,t satisfying $1 \le n \le t$.

(b) First, we find the joint PMF of N and T by carefully considering the possible sample paths. In particular, $P_{N,T}(n,t) = P[ABC] = P[A]P[B]P[C]$ where the events A, B and C are

$$\begin{aligned} A &= \{n-1 \text{ buses arrive in the first } t-1 \text{ minutes}\} \\ B &= \{\text{none of the first } n-1 \text{ buses are boarded}\} \\ C &= \{\text{at time } t \text{ a bus arrives and is boarded}\} \end{aligned}$$

These events are independent since each trial to board a bus is independent of when the buses arrive. These events have probabilities

$$\begin{aligned} P[A] &= \binom{t-1}{n-1} p^{n-1}(1-p)^{t-1-(n-1)} \\ P[B] &= (1-q)^{n-1} \\ P[C] &= pq \end{aligned}$$

Consequently, the joint PMF of N and T is

$$P_{N,T}(n,t) = \begin{cases} \binom{t-1}{n-1} p^{n-1}(1-p)^{t-n}(1-q)^{n-1}pq & n \ge 1, t \ge n \\ 0 & \text{otherwise} \end{cases}$$

(c) It is possible to find the marginal PMF's by summing the joint PMF. However, it is much easier to obtain the marginal PMFs by consideration of the experiment. Specifically, when a bus arrives, it is boarded with probability q. Moreover, the experiment ends when a bus is boarded. By viewing whether each arriving bus is boarded as an independent trial, N is the number of trials until the first success. Thus, N has the geometric PMF

$$P_N(n) = \begin{cases} (1-q)^{n-1}q & n = 1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$$

To find the PMF of T, suppose we regard each minute as an independent trial in which a success occurs if a bus arrives and that bus is boarded. In this case, the success probability is pq and T is the number of minutes up to and including the first success. The PMF of T is also geometric.

$$P_T(t) = \begin{cases} (1-pq)^{t-1}pq & t = 1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$$

(d) Once we have the marginal PMFs, the conditional PMFs are easy to find.

$$P_{N|T}(n|t) = \frac{P_{N,T}(n,t)}{P_T(t)} = \begin{cases} \binom{t-1}{n-1}\left(\frac{p(1-q)}{1-pq}\right)^{n-1}\left(\frac{1-p}{1-pq}\right)^{t-1-(n-1)} & n = 1,2,\ldots,t \\ 0 & \text{otherwise} \end{cases}$$

That is, given you depart at time $T = t$, the number of buses that arrive during minutes $1,\ldots,t-1$ has a binomial PMF since in each minute a bus arrives with probability p. Similarly, the conditional PMF of T given N is

$$P_{T|N}(t|n) = \frac{P_{N,T}(n,t)}{P_N(n)} = \begin{cases} \binom{t-1}{n-1}p^n(1-p)^{t-n} & t = n, n+1, \ldots \\ 0 & \text{otherwise} \end{cases}$$

This result can be explained. Given that you board bus $N = n$, the time T when you leave is the time for n buses to arrive. If we view each bus arrival as a success of an independent trial, the time for n buses to arrive has the above Pascal PMF.

Problem 3.7.1

Flip a fair coin 100 times and let X be the number of heads in the first 75 flips and Y be the number of heads in the last 25 flips. We know that X and Y are independent and can find their PMFs easily.

$$P_X(x) = \begin{cases} \binom{75}{x}(1/2)^{75} & x = 0,1,\ldots,75 \\ 0 & \text{otherwise} \end{cases} \qquad P_Y(y) = \begin{cases} \binom{25}{y}(1/2)^25 & y = 0,1,\ldots,25 \\ 0 & \text{otherwise} \end{cases}$$

The joint PMF of X and N can be expressed as the product of the marginal PMFs because we know that X and Y are independent.

$$P_{X,Y}(x,y) = \begin{cases} \binom{75}{x}\binom{25}{y}(1/2)^{100} & x = 0,1,\ldots,75 \;\; y = 0,1,\ldots,25 \\ 0 & \text{otherwise} \end{cases}$$

Problem 3.7.7

The key to this problem is understanding that “short order” and “long order” are synonyms for $N = 1$ and $N = 2$. Similarly, “vanilla”, “chocolate”, and “strawberry” correspond to the events $D = 20$, $D = 100$ and $D = 300$.

(a) The following table is given in the problem statement.

	vanilla	choc.	strawberry
short order	0.2	0.2	0.2
long order	0.1	0.2	0.1

This table can be translated directly into the joint PMF of N and D.

$P_{N,D}(n,d)$	$d = 20$	$d = 100$	$d = 300$
$n = 1$	0.2	0.2	0.2
$n = 2$	0.1	0.2	0.1

(b) To find the conditional PMF $P_{D|N}(d|2)$, we first need to find the probability of the conditioning event

$$P_N(2) = P_{N,D}(2,20) + P_{N,D}(2,100) + P_{N,D}(2,300) = 0.4$$

The conditional PMF of N D given $N = 2$ is

$$P_{D|N}(d|2) = \frac{P_{N,D}(2,d)}{P_N(2)} = \begin{cases} 1/4 & d = 20 \\ 1/2 & d = 100 \\ 1/4 & d = 300 \\ 0 & \text{otherwise} \end{cases}$$

(c) The conditional expectation of D given $N = 2$ is

$$E[D|N=2] = \sum_d dP_{D|N}(d|2) = 20(1/4) + 100(1/2) + 300(1/4) = 130$$

(d) To check independence, we calculate the marginal PMFs of N and D. An easy way to do this is to find the row and column sums from the table for $P_{N,D}(n,d)$. This yields

$P_{N,D}(n,d)$	$d = 20$	$d = 100$	$d = 300$	$P_N(n)$
$n = 1$	0.2	0.2	0.2	0.6
$n = 2$	0.1	0.2	0.1	0.4
$P_D(d)$	0.3	0.4	0.3	

Some study of the table will show that $P_{N,D}(n,d) \neq P_N(n) P_D(d)$. Hence, N and D are dependent.

(e) In terms of N and D, the cost (in cents) of a fax is $C = ND$. The expected value of C is

$$\begin{aligned} E[C] &= \sum_{n,d} ndP_{N,D}(n,d) \\ &= 1(20)(0.2) + 1(100)(0.2) + 1(300)(0.2) \\ &\quad + 2(20)(0.3) + 2(100)(0.4) + 2(300)(0.3) = 356 \end{aligned}$$

Problem 4.1.1

$$F_X(x) = \begin{cases} 0 & x < -1 \\ (x+1)/2 & -1 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

Each question can be answered by expressing the requested probability in terms of $F_X(x)$.

(a)

$$P[X > 1/2] = 1 - P[X \leq 1/2] = 1 - F_X(1/2) = 1 - 3/4 = 1/4$$

(b) This is a little trickier than it should be. Being careful, we can write

$$P[-1/2 \leq X < 3/4] = P[-1/2 < X \leq 3/4] + P[X = -1/2] - P[X = 3/4]$$

Since the CDF of X is a continuous function, the probability that X takes on any specific value is zero. This implies $P[X = 3/4] = 0$ and $P[X = -1/2] = 0$. (If this is not clear at this point, it will become clear in Section 4.6.) Thus,

$$P[-1/2 \leq X < 3/4] = P[-1/2 < X \leq 3/4] = F_X(3/4) - F_X(-1/2) = 5/8$$

(c)

$$P[|X| \leq 1/2] = P[-1/2 \leq X \leq 1/2] = P[X \leq 1/2] - P[X < -1/2]$$

Note that $P[X \leq 1/2] = F_X(1/2) = 3/4$. Since the probability that $P[X = -1/2] = 0$, $P[X < -1/2] = P[X \leq 1/2]$. Hence $P[X < -1/2] = F_X(-1/2) = 1/4$. This implies

$$P[|X| \leq 1/2] = P[X \leq 1/2] - P[X < -1/2] = 3/4 - 1/4 = 1/2$$

(d) Since $F_X(1) = 1$, we must have $a \leq 1$. For $a \leq 1$, we need to satisfy

$$P[X \leq a] = F_X(a) = \frac{a+1}{2} = 0.8$$

Thus $a = 0.6$.

Problem 4.2.1

$$f_X(x) = \begin{cases} cx & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) From the above PDF we can determine the value of c by integrating the PDF and setting it equal to 1.

$$\int_0^2 cx\, dx = 2c = 1$$

Therefore $c = 1/2$.

(b) $P[0 \leq X \leq 1] = \int_0^1 \frac{x}{2}\, dx = 1/4$

(c) $P[-1/2 \leq X \leq 1/2] = \int_0^{1/2} \frac{x}{2}\, dx = 1/16$

(d) The CDF of X is found by integrating the PDF from 0 to x.

$$F_X(x) = \int_0^x f_X(x')\, dx' = \begin{cases} 0 & x < 0 \\ x^2/4 & 0 \leq x \leq 2 \\ 1 & x > 2 \end{cases}$$