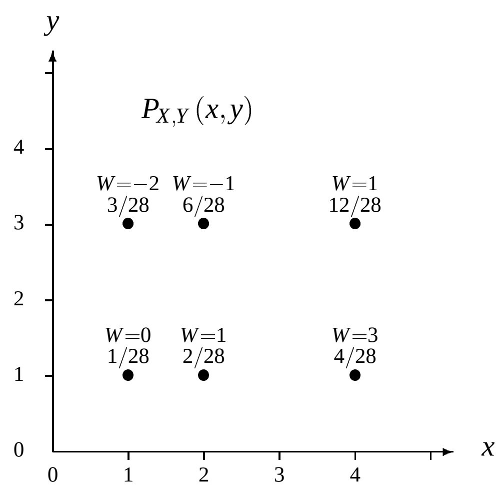
Probability and Stochastic Processes: A Friendly Introduction for Electrical and Computer Engineers

Roy D. Yates and David J. Goodman

Problem Solutions : Yates and Goodman,3.3.1 3.4.1 3.4.4 3.5.2 3.6.1 and 3.6.3

Problem 3.3.1

In this problem, it is helpful to label points X,Y with nonzero probability along with the corresponding values of $W = X - Y$. From the statement of Problem 3.3.1, we have



(a) To find the PMF of W, we simply add the probabilities associated with each possible value of W:

$$P_W(-2) = P_{X,Y}(1,3) = 3/28 \qquad P_W(-1) = P_{X,Y}(2,3) = 6/28$$
$$P_W(0) = P_{X,Y}(1,1) = 1/28 \qquad P_W(1) = P_{X,Y}(2,1) + P_{X,Y}(4,3) = 14/28$$
$$P_W(3) = P_{X,Y}(4,1) = 4/28$$

For all other values of w, $P_W(w) = 0$.

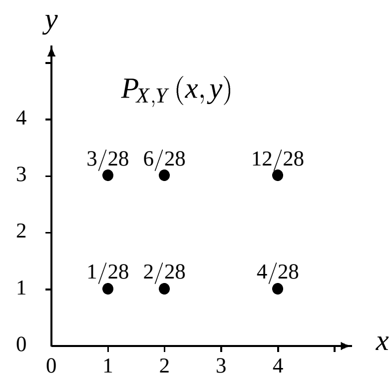
(b) The expected value of W is

$$E[W] = \sum_w w P_W(w) = -2(3/28) + -1(6/28) + 0(1/28) + 1(14/28) + 3(4/28) = 1/2$$

(c)

$$P[W > 0] = P_W(1) + P_W(3) = 18/28$$

Problem 3.4.1
In Problem 3.2.1, we found that the joint PMF of X and Y is



The expected values and variances were found to be

$$E[X] = 3 \qquad \text{Var}[X] = 10/7$$
$$E[Y] = 5/2 \qquad \text{Var}[Y] = 3/4$$

We will need these results in the solution to this problem.

(a) Random variable $W = Y/X$ has expected value

$$\begin{aligned} E[Y/X] &= \sum_{x=1,2,4} \sum_{y=1,3} \frac{y}{x} P_{X,Y}(x,y) \\ &= \frac{1}{1}\frac{1}{28} + \frac{3}{1}\frac{3}{28} + \frac{1}{2}\frac{2}{28} + \frac{3}{2}\frac{6}{28} + \frac{1}{4}\frac{4}{28} + \frac{3}{4}\frac{12}{28} \\ &= 30/28 = 15/14 \end{aligned}$$

(b) The correlation of X and Y is

$$\begin{aligned} r_{X,Y} = E[XY] &= \sum_{x=1,2,4} \sum_{y=1,3} xy P_{X,Y}(x,y) \\ &= \frac{1 \cdot 1 \cdot 1}{28} + \frac{1 \cdot 2 \cdot 3}{28} + \frac{2 \cdot 1 \cdot 2}{28} + \frac{2 \cdot 3 \cdot 6}{28} + \frac{4 \cdot 1 \cdot 4}{28} + \frac{4 \cdot 3 \cdot 12}{28} \\ &= 207/28 \end{aligned}$$

(c) The covariance of X and Y is

$$\sigma_{X,Y} = E[XY] - E[X]E[Y] = \frac{-3}{28}$$

(d) The correlation coefficient is

$$\rho_{X,Y} = \frac{\text{Cov}[X,Y]}{\sqrt{\text{Var}[X]\,\text{Var}[Y]}} = \frac{-3}{2\sqrt{210}}$$

Problem 3.4.4

From the joint PMF, $P_X(x)Y$, found in Example 3.4, we can find the marginal PMF for X or Y by summing over the columns or rows of the joint PMF.

$$P_Y(y) = \begin{cases} 25/48 & y=1 \\ 13/48 & y=2 \\ 7/48 & y=3 \\ 3/48 & y=4 \\ 0 & \text{otherwise} \end{cases} \qquad P_X(x) = \begin{cases} 1/4 & x=1,2,3,4 \\ 0 & \text{otherwise} \end{cases}$$

(a) The expected values are

$$E[Y] = \sum_{y=1}^{4} y P_Y(y) = 1\frac{25}{48} + 2\frac{13}{48} + 3\frac{7}{48} + 4\frac{3}{48} = 7/4$$
$$E[X] = \sum_{x=1}^{4} x P_X(x) = \frac{1}{4}(1+2+3+4) = 5/2$$

(b) To find the variances, we first find the second moments.

$$E\left[Y^2\right] = \sum_{y=1}^{4} y^2 P_Y(y) = 1^2\frac{25}{48} + 2^2\frac{13}{48} + 3^2\frac{7}{48} + 4^2\frac{3}{48} = 47/12$$
$$E\left[X^2\right] = \sum_{x=1}^{4} x^2 P_X(x) = \frac{1}{4}\left(1^2+2^2+3^2+4^2\right) = 15/2$$

Now the variances are

$$\text{Var}[Y] = E\left[Y^2\right] - (E[Y])^2 = 47/12 - (7/4)^2 = 41/48$$
$$\text{Var}[X] = E\left[X^2\right] - (E[X])^2 = 15/2 - (5/2)^2 = 5/4$$

(c) To find the correlation, we evaluate the product XY over all values of X and Y. Specifically,

$$\begin{aligned} r_{X,Y} = E[XY] &= \sum_{x=1}^{4}\sum_{y=1}^{x} xy P_{X,Y}(x,y) \\ &= \frac{1}{4} + \frac{2}{8} + \frac{3}{12} + \frac{4}{16} + \frac{4}{8} + \frac{6}{12} + \frac{8}{16} + \frac{9}{12} + \frac{12}{16} + \frac{16}{16} \\ &= 5 \end{aligned}$$

(d) The covariance of X and Y is

$$\text{Cov}[X,Y] = E[XY] - E[X]E[Y] = 5 - (7/4)(10/4) = 10/16$$

(e) The correlation coefficient is

$$\rho_{X,Y} = \frac{\text{Cov}[W,V]}{\sqrt{\text{Var}[W]\,\text{Var}[V]}} = \frac{10/16}{\sqrt{(41/48)(5/4)}} \approx 0.605$$

Problem 3.5.2

The event B occurs iff $X \leq 5$ and $Y \leq 5$ and has probability

$$P[B] = P[X \leq 5, Y \leq 5] = \sum_{x=1}^{5} \sum_{y=1}^{5} 0.01 = 0.25$$

From Theorem 3.11,

$$P_{X,Y|B}(x,y) = \begin{cases} \frac{P_{X,Y}(x,y)}{P[B]} & (x,y) \in A \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 0.04 & x = 1,\ldots,5; y = 1,\ldots,5 \\ 0 & \text{otherwise} \end{cases}$$

Problem 3.6.1

The main part of this problem is just interpreting the problem statement. No calculations are necessary. Since a trip is equally likely to last 2, 3 or 4 days,

$$P_D(d) = \begin{cases} 1/3 & d = 2,3,4 \\ 0 & \text{otherwise} \end{cases}$$

Given a trip lasts d days, the weight change is equally likely to be any value between $-d$ and d pounds. Thus,

$$P_{W|D}(w|d) = \begin{cases} 1/(2d+1) & w = -d, -d+1, \ldots, d \\ 0 & \text{otherwise} \end{cases}$$

The joint PMF is simply

$$P_{D,W}(d,w) = P_{W|D}(w|d)\, P_D(d) = \begin{cases} 1/(6d+3) & d = 2,3,4; w = -d,\ldots,d \\ 0 & \text{otherwise} \end{cases}$$

Problem 3.6.3

(a) First we observe that A takes on the values $S_A = \{-1, 1\}$ while B takes on values from $S_B = \{0, 1\}$. To construct a table describing $P_{A,B}(a,b)$ we build a table for all possible values of pairs (A,B). The general form of the entries is

$P_{A,B}(a,b)$	$b=0$	$b=1$
$a=-1$	$P_{B\|A}(0\|-1)\, P_A(-1)$	$P_{B\|A}(1\|-1)\, P_A(-1)$
$a=1$	$P_{B\|A}(0\|1)\, P_A(1)$	$P_{B\|A}(1\|1)\, P_A(1)$

Now we fill in the entries using the conditional PMFs $P_{B|A}(b|a)$ and the marginal PMF $P_A(a)$. This yields

$P_{A,B}(a,b)$	$b=0$	$b=1$
$a=-1$	$(1/3)(1/3)$	$(2/3)(1/3)$
$a=1$	$(1/2)(2/3)$	$(1/2)(2/3)$

which simplifies to

$P_{A,B}(a,b)$	$b=0$	$b=1$
$a=-1$	$1/9$	$2/9$
$a=1$	$1/3$	$1/3$

(b) If $A = 1$, then the conditional expectation of B is

$$E[B|A=1] = \sum_{b=0}^{1} bP_{B|A}(b|1) = P_{B|A}(1|1) = 1/2$$

(c) Before finding the conditional PMF $P_{A|B}(a|1)$, we first sum the columns of the joint PMF table to find

$$P_B(b) = \begin{cases} 4/9 & b = 0 \\ 5/9 & b = 1 \end{cases}$$

The conditional PMF of A given $B = 1$ is

$$P_{A|B}(a|1) = \frac{P_{A,B}(a,1)}{P_B(1)} = \begin{cases} 2/5 & a = -1 \\ 3/5 & a = 1 \end{cases}$$

(d) Now that we have the conditional PMF $P_{A|B}(a|1)$, calculating conditional expectations is easy.

$$E[A|B=1] = \sum_{a=-1,1} aP_{A|B}(a|1) = -1(2/5) + (3/5) = 1/5$$
$$E\left[A^2|B=1\right] = \sum_{a=-1,1} a^2 P_{A|B}(a|1) = 2/5 + 3/5 = 1$$

The conditional variance is then

$$\mathrm{Var}[A|B=1] = E\left[A^2|B=1\right] - (E[A|B=1])^2 = 1 - (1/5)^2 = 24/25$$

(e) To calculate the covariance, we need

$$E[A] = \sum_{a=-1,1} aP_A(a) = -1(1/3) + 1(2/3) = 1/3$$
$$E[B] = \sum_{b=0}^{1} bP_B(b) = 0(4/9) + 1(5/9) = 5/9$$
$$E[AB] = \sum_{a=-1,1} \sum_{b=0}^{1} abP_{A,B}(a,b)$$
$$= -1(0)(1/9) + -1(1)(2/9) + 1(0)(1/3) + 1(1)(1/3) = 1/9$$

The covariance is just

$$\mathrm{Cov}[A,B] = E[AB] - E[A]E[B] = 1/9 - (1/3)(5/9) = -2/27$$