Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
Roy D. Yates and David J. Goodman

Problem Solutions : Yates and Goodman,2.6.3 2.7.7 2.8.11 2.9.6 3.1.1 3.2.1 and 3.2.2

Problem 2.6.3

Problem 2.4.3, the PMF of X is

$$P_X(x) = \begin{cases} 0.4 & x = -3 \\ 0.4 & x = 5 \\ 0.2 & x = 7 \\ 0 & \text{otherwise} \end{cases}$$

(a) The PMF of $W = -X$ satisfies

$$P_W(w) = P[-X = w] = P_X(-w)$$

This implies

$$P_W(-7) = P_X(7) = 0.2 \qquad P_W(-5) = P_X(5) = 0.4 \qquad P_W(3) = P_X(-3) = 0.4$$

The complete PMF for W is

$$P_W(w) = \begin{cases} 0.2 & w = -7 \\ 0.4 & w = -5 \\ 0.4 & w = 3 \\ 0 & \text{otherwise} \end{cases}$$

(b) From the PMF, the CDF of W is

$$F_W(w) = \begin{cases} 0 & w < -7 \\ 0.2 & -7 \le w < -5 \\ 0.6 & -5 \le w < 3 \\ 1 & w \ge 3 \end{cases}$$

(c) From the PMF, W has expected value

$$E[W] = \sum_w w P_W(w) = -7(0.2) + -5(0.4) + 3(0.4) = -2.2$$

Problem 2.7.7

Let W denote the event that a circuit works. The circuit works and generates revenue of k dollars if all of its 10 constituent devices work. For each implementation, standard or ultra-reliable, let R denote the profit on a device. We can express the expected profit as

$$E[R] = P[W]E[R|W] + P[W^c]E[R|W^c]$$

Let's first consider the case when only standard devices are used. In this case, a circuit works with probability $P[W] = (1-q)^{10}$. The profit made on a working device is $k-10$ dollars while a non-working circuit has a profit of -10 dollars. That is, $E[R|W] = k-10$ and $E[R|W^c] = -10$. Of course, a negative profit is actually a loss. Using R_s to denote the profit using standard circuits, the expected profit is

$$\begin{aligned} E[R_s] &= (1-q)^{10}(k-10) + (1-(1-q)^{10})(-10) \\ &= (0.9)^{10}k - 10 \end{aligned}$$

And for the ultra-reliable case, the circuit works with probability $P[W] = (1-q/2)^{10}$. The profit per working circuit is $E[R|W] = k-30$ dollars while the profit for a nonworking circuit is $E[R|W^c] = -30$ dollars. The expected profit is

$$\begin{aligned} E[R_u] &= (1-q/2)^{10}(k-30) + (1-(1-q/2)^{10})(-30) \\ &= (0.95)^{10}k - 30 \end{aligned}$$

Now we wish to determine which implementation will generate the most profit. Realizing that both profit functions are linear functions of k, we can plot them versus k and find for which values of k each plan is preferable. The two lines intersect at a value of $k = 80.21$ dollars. So for values of $k <$ \$80.21 using all standard devices results in greater revenue, and for values of $k >$ \$80.21 more revenue will be generated by implementing all ultra-reliable devices. So we can see that when the price commanded for each working circuit is sufficiently high it is worthwhile to spend the extra money to ensure that more working circuits can be produced.

Problem 2.8.11

The PMF of K is the Poisson PMF

$$P_K(k) = \begin{cases} \lambda^k e^{-\lambda}/k! & k = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases}$$

The mean of K is

$$E[K] = \sum_{k=0}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!} = \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1} e^{-\lambda}}{(k-1)!} = \lambda$$

To find $E\left[K^2\right]$, we use the hint and first find

$$E[K(K-1)] = \sum_{k=0}^{\infty} k(k-1) \frac{\lambda^k e^{-\lambda}}{k!} = \sum_{k=2}^{\infty} \frac{\lambda^k e^{-\lambda}}{(k-2)!}$$

By factoring out λ^2 and substituting $j = k-2$, we obtain

$$E[K(K-1)] = \lambda^2 \underbrace{\sum_{j=0}^{\infty} \frac{\lambda^j e^{-\lambda}}{j!}}_{1} = \lambda^2$$

The above sum equals 1 because it is the sum of a Poisson PMF over all possible values. Since $E[K] = \lambda$, the variance of K is

$$\begin{aligned} \text{Var}[K] &= E\left[K^2\right] - (E[K])^2 \\ &= E[K(K-1)] + E[K] - (E[K])^2 \\ &= \lambda^2 + \lambda - \lambda^2 = \lambda \end{aligned}$$

Problem 2.9.6

(a) Consider each circuit test as a binomial trial such that a failed circuit is called a success. The number of trias until the first success (i.e. a failed circuit) has the geometric PMF

$$P_N(n) = \begin{cases} (1-p)^{n-1}p & n = 1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$$

(b) The probability there are at least 20 tests is

$$P[B] = P[N \geq 20] = \sum_{n=20}^{\infty} P_N(n) = (1-p)^{19}$$

Note that $(1-p)^{19}$ is just the probability that the first 19 circuits pass the test, which is what we would expect since there must be at least 20 tests if the first 19 circuits pass. The conditional PMF of N given B is

$$P_{N|B}(n) = \begin{cases} \frac{P_N(n)}{P[B]} & n \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} (1-p)^{n-20}p & n = 20, 21, \ldots \\ 0 & \text{otherwise} \end{cases}$$

(c) Given the event B the conditional expectation of N is

$$E[N|B] = \sum_n n P_{N|B}(n) = \sum_{n=20}^{\infty} n(1-p)^{n-20}p$$

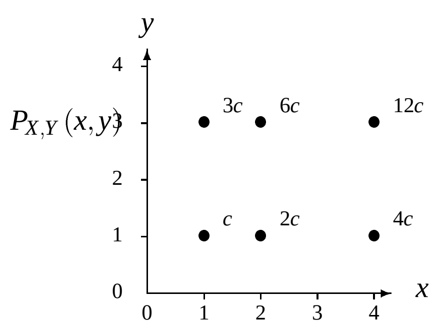
Making the substitution $j = n - 19$ yields

$$E[N|B] = \sum_{j=1}^{\infty} (j+19)(1-p)^{j-1}p = 1/p + 19$$

We see that in the above sum, we effectively have the expected value of $J + 19$ where J is geometric random variable with parameter p. This is not surprising since the $N \geq 20$ iff we observed 19 successful tests. After 19 successful tests, the number of additional tests needed to find the first failure is still a geometric random variable with mean $1/p$.

Problem 3.1.1

In this problem, it is helpful to label points with nonzero probability on the X, Y plane:



(a) We must choose c so the PMF sums to one:

$$\sum_{x=1,2,4}\sum_{y=1,3} P_{X,Y}(x,y) = c\sum_{x=1,2,4} x \sum_{y=1,3} y = c[1(1+3)+2(1+3)+4(1+3)] = 28c$$

Thus $c = 1/28$.

(b) The event $\{Y < X\}$ has probability

$$P[Y<X] = \sum_{x=1,2,4}\sum_{y<x} P_{X,Y}(x,y) = \frac{1(0)+2(1)+4(1+3)}{28} = \frac{18}{28}$$

(c) The event $\{Y > X\}$ has probability

$$P[Y>X] = \sum_{x=1,2,4}\sum_{y>x} P_{X,Y}(x,y) = \frac{1(3)+2(3)+4(0)}{28} = \frac{9}{28}$$

(d) There are two ways to solve this part. The direct way is to calculate

$$P[Y=X] = \sum_{x=1,2,4}\sum_{y=x} P_{X,Y}(x,y) = \frac{1(1)+2(0)}{28} = \frac{1}{28}$$

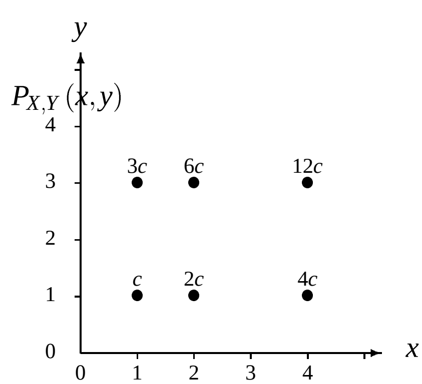
The indirect way is to use the previous results and the observation that

$$P[Y=X] = 1 - P[Y<X] - P[Y>X] = (1-18/28-9/28) = 1/28$$

(e)

$$P[Y=3] = \sum_{x=1,2,4} P_{X,Y}(x,3) = \frac{(1)(3)+(2)(3)+(4)(3)}{28} = \frac{21}{28} = \frac{3}{4}$$

Problem 3.2.1
On the X,Y plane, the joint PMF $P_{X,Y}(x,y)$ is



By choosing $c = 1/28$, the PMF sums to one.

(a) The marginal PMFs of X and Y are

$$P_X(x) = \sum_{y=1,3} P_{X,Y}(x,y) = \begin{cases} 4/28 & x=1 \\ 8/28 & x=2 \\ 16/28 & x=4 \\ 0 & \text{otherwise} \end{cases}$$

$$P_Y(y) = \sum_{x=1,2,4} P_{X,Y}(x,y) = \begin{cases} 7/28 & y=1 \\ 21/28 & y=3 \\ 0 & \text{otherwise} \end{cases}$$

(b) The expected values of X and Y are

$$E[X] = \sum_{x=1,2,4} xP_X(x) = (4/28) + 2(8/28) + 4(16/28) = 3$$

$$E[Y] = \sum_{y=1,3} yP_Y(y) = 7/28 + 3(21/28) = 5/2$$

(c) The second moments are

$$E\left[X^2\right] = \sum_{x=1,2,4} xP_X(x) = 1^2(4/28) + 2^2(8/28) + 4^2(16/28) = 73/7$$

$$E\left[Y^2\right] = \sum_{y=1,3} yP_Y(y) = 1^2(7/28) + 3^2(21/28) = 7$$

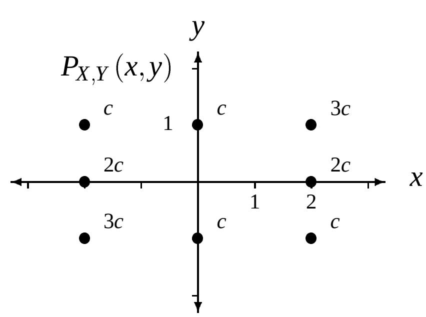
The variances are

$$\text{Var}[X] = E\left[X^2\right] - (E[X])^2 = 10/7 \qquad \text{Var}[Y] = E\left[Y^2\right] - (E[Y])^2 = 3/4$$

The standard deviations are $\sigma_X = \sqrt{10/7}$ and $\sigma_Y = \sqrt{3/4}$.

Problem 3.2.2

On the X,Y plane, the joint PMF is



The PMF sums to one when $c = 1/14$.

(a) The marginal PMFs of X and Y are

$$P_X(x) = \sum_{y=-1,0,1} P_{X,Y}(x,y) = \begin{cases} 6/14 & x = -2, 2 \\ 2/14 & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$P_Y(y) = \sum_{x=-2,0,2} P_{X,Y}(x,y) = \begin{cases} 5/14 & y = -1, 1 \\ 4/14 & y = 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) The expected values of X and Y are

$$E[X] = \sum_{x=-2,0,2} x P_X(x) = -2(6/14) + 2(6/14) = 0$$

$$E[Y] = \sum_{y=-1,0,1} y P_Y(y) = -1(5/14) + 1(5/14) = 0$$

(c) Since X and Y both have zero mean, the variances are

$$\text{Var}[X] = E\left[X^2\right] = \sum_{x=-2,0,2} x^2 P_X(x) = (-2)^2(6/14) + 2^2(6/14) = 24/7$$

$$\text{Var}[Y] = E\left[Y^2\right] = \sum_{y=-1,0,1} y^2 P_Y(y) = (-1)^2(5/14) + 1^2(5/14) = 5/7$$

The standard deviations are $\sigma_X = \sqrt{24/7}$ and $\sigma_Y = \sqrt{5/7}$.