Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
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Problem Solutions : Yates and Goodman,1.5.1 1.5.2 1.5.3 1.6.3 1.6.4 and 1.6.7

Problem 1.5.1

probability.

(a) Note that the probability a call is brief is

$$P[B] = P[H_0B] + P[H_1B] + P[H_2B] = 0.6.$$

The probability a brief call will have no handoffs is

$$P[H_0|B] = \frac{P[H_0B]}{P[B]} = \frac{0.4}{0.6} = \frac{2}{3}$$

(b) The probability of one handoff is $P[H_1] = P[H_1B] + P[H_1L] = 0.2$. The probability that a call with one handoff will be long is

$$P[L|H_1] = \frac{P[H_1L]}{P[H_1]} = \frac{0.1}{0.2} = \frac{1}{2}$$

(c) The probability a call is long is $P[L] = 1 - P[B] = 0.4$. The probability that a long call will have one or more handoffs is

$$P[H_1 \cup H_2|L] = \frac{P[H_1L \cup H_2L]}{P[L]} = \frac{P[H_1L] + P[H_2L]}{P[L]} = \frac{0.1+0.2}{0.4} = \frac{3}{4}$$

Problem 1.5.2

Let s_i denote the outcome that the roll is i. So, for $1 \le i \le 6$, $R_i = \{s_i\}$. Similarly, $G_j = \left\{s_{j+1}, \dots, s_6\right\}$.

(a) Since $G_1 = \{s_2, s_3, s_4, s_5, s_6\}$ and all outcomes have probability $1/6$, $P[G_1] = 5/6$. The event $R_3G_1 = \{s_3\}$ and $P[R_3G_1] = 1/6$ so that

$$P[R_3|G_1] = \frac{P[R_3G_1]}{P[G_1]} = \frac{1}{5}$$

(b) The conditional probability that 6 is rolled given that the roll is greater than 3 is

$$P[R_6|G_3] = \frac{P[R_6G_3]}{P[G_3]} = \frac{P[s_6]}{P[s_4, s_5, s_6]} = \frac{1/6}{3/6}$$

(c) The event E that the roll is even is $E = \{s_2, s_4, s_6\}$ and has probability $3/6$. The joint probability of G_3 and E is

$$P[G_3E] = P[s_4, s_6] = 1/3$$

The conditional probabilities of G_3 given E is

$$P[G_3|E] = \frac{P[G_3E]}{P[E]} = \frac{1/3}{1/2} = \frac{2}{3}$$

(d) The conditional probability that the roll is even given that it's greater than 3 is

$$P[E|G_3] = \frac{P[EG_3]}{P[G_3]} = \frac{1/3}{1/2} = \frac{2}{3}$$

Problem 1.5.3

Since the 2 of clubs is an even numbered card, $C_2 \subset E$ so that $P[C_2E] = P[C_2] = 1/3$. Since $P[E] = 2/3$,

$$P[C_2|E] = \frac{P[C_2E]}{P[E]} = \frac{1/3}{2/3} = 1/2$$

The probability that an even numbered card is picked given that the 2 is picked is

$$P[E|C_2] = \frac{P[C_2E]}{P[C_2]} = \frac{1/3}{1/3} = 1$$

Problem 1.6.3

(a) Since A and B are disjoint, $P[A \cap B] = 0$.

(b) Since $P[A \cap B] = 0$,

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] = 3/8$$

(c) A Venn diagram should convince you that $A \subset B^c$ so that $A \cap B^c = A$. This implies

$$P[A \cap B^c] = P[A] = 1/4$$

(d) $P[A \cup B^c] = P[B^c] = 1 - 1/8 = 7/8$

(e) Events A and B are dependent since $P[AB] \neq P[A]P[B]$.

(f) Since C and D are independent,

$$P[C \cap D] = P[C]P[D] = 15/64$$

(g) The previous part implies

$$P[C \cup D] = P[C] + P[D] - P[C \cap D] = 5/8 + 3/8 - 15/64 = 49/64$$

(h) Since C and D are independent, $P[C|D] = P[C] = 5/8$.

(i) The next few items are a little trickier. From Venn diagrams, we see

$$P[C \cap D^c] = P[C] - P[C \cap D] = 5/8 - 15/64 = 25/64$$

(j) It follows that

$$\begin{aligned} P[C \cup D^c] &= P[C] + P[D^c] - P[C \cap D^c] \\ &= 5/8 + (1 - 3/8) - 25/64 = 55/64 \end{aligned}$$

(k) Using DeMorgan's law, we have

$$P[C^c \cap D^c] = P[(C \cup D)^c] = 1 - P[C \cup D] = 15/64$$

(l) Since $P[C^cD^c] = P[C^c]P[D^c]$, C^c and D^c are independent.

Problem 1.6.4

(a) Since $A \cap B = \emptyset$, $P[A \cap B] = 0$.

(b) To find $P[B]$, we can write

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$
$$5/8 = 3/8 + P[B] - 0$$

Thus, $P[B] = 1/4$.

(c) Since A is a subset of B^c, $P[A \cap B^c] = P[A] = 3/8$.

(d) Since A is a subset of B^c, $P[A \cup B^c] = P[B^c] = 3/4$.

(e) The events A and B are dependent because

$$P[AB] = 0 \neq 3/32 = P[A]P[B].$$

(f) Since C and D are independent $P[CD] = P[C]P[D]$. So

$$P[D] = \frac{P[CD]}{P[C]} = \frac{1/3}{1/2} = 2/3$$

(g) This permits us to write

$$P[C \cup D] = P[C] + P[D] - P[C \cap D] = 1/2 + 2/3 - 1/3 = 5/6$$

(h) Since C and D are independent events, $P[C|D] = P[C] = 1/2$.

(i) $P[C \cap D^c] = P[C] - P[C \cap D] = 1/2 - 1/3 = 1/6$

(j) $P[C \cup D^c] = P[C] + P[D] - P[C \cap D^c] = 1/2 + (1 - 2/3) - 1/6 = 2/3$

(k) By De Morgan's Law, $C^c \cap D^c = (C \cup D)^c$. This implies

$$P[C^c \cap D^c] = P[(C \cup D)^c] = 1 - P[C \cup D] = 1/6$$

(l) By Definition 1.6, events C and D^c are independent because

$$P[C \cap D^c] = 1/6 = (1/2)(1/3) = P[C]P[D^c]$$

Problem 1.6.7

(a) For any events A and B, we can write the law of total probability in the form of

$$P[A] = P[AB] + P[AB^c]$$

Since A and B are independent, $P[AB] = P[A]P[B]$. This implies

$$P[AB^c] = P[A] - P[A]P[B] = P[A](1 - P[B]) = P[A]P[B^c]$$

Thus A and B^c are independent.

(b) Proving that A^c and B are independent is not really necessary. Since A and B are arbitrary labels, it is really the same claim as in part (a). That is, simply reversing the labels of A and B proves the claim. Alternatively, one can construct exactly the same proof as in part (a) with the labels A and B reversed.

(c) To prove that A^c and B^c are independent, we apply the result of part (a) to the sets A and B^c. Since we know from part (a) that A and B^c are independent, part (b) says that A^c and B^c are independent.