Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
Roy D. Yates and David J. Goodman

Problem Solutions : Yates and Goodman,6.2.1 6.2.3 6.3.2 6.4.1 6.4.2 6.7.1 6.7.2 6.7.5 6.8.1 6.8.3 and 6.8.5

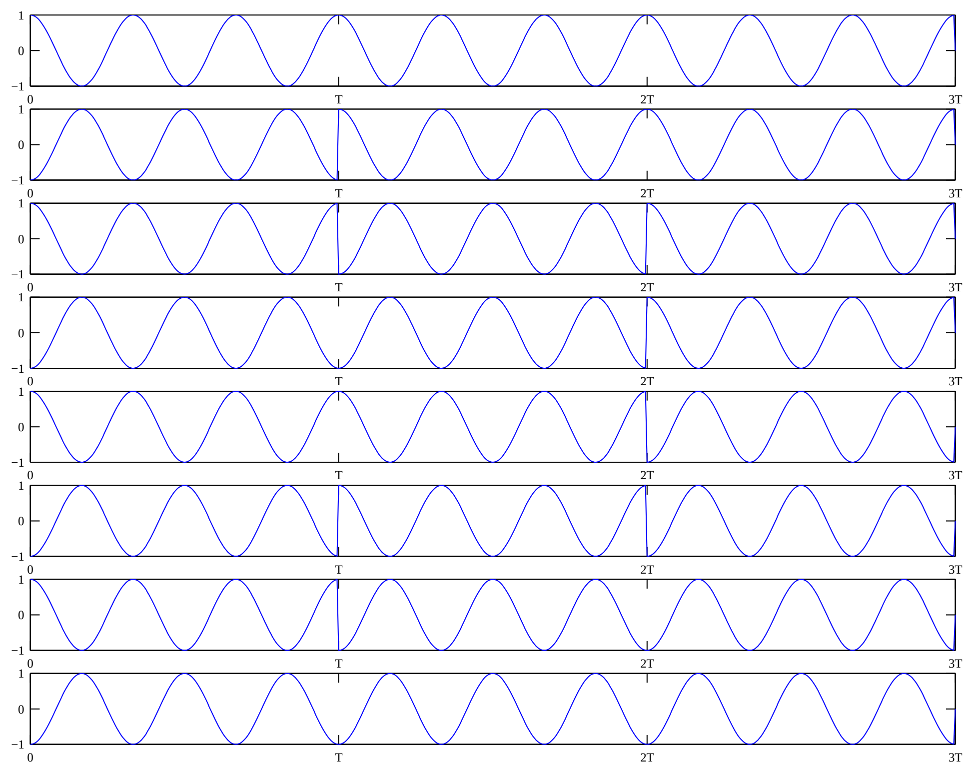
Problem 6.2.1

- In Example 6.3, the daily noontime temperature at Newark Airport is a discrete time, continuous value random process. However, if the temperature is recorded only in units of one degree, then the process was would be discrete value.
- In Example 6.4, the the number of active telephone calls is discrete time and discrete value.
- The dice rolling experiment of Example 6.5 yields a discrete time, discrete value random process.
- The QPSK system of Example 6.6 is a continuous time and continuous value random process.

Problem 6.2.3
The eight possible waveforms correspond to the the bit sequences

$$\{(0,0,0),(1,0,0),(1,1,0),\ldots,(1,1,1)\}$$

The corresponding eight waveforms are:



Problem 6.3.2

(a) Each resistor has frequency W in Hertz with uniform PDF

$$f_R(r) = \begin{cases} 0.025 & 9980 \le r \le 1020 \\ 0 & \text{otherwise} \end{cases}$$

The probability that a test yields a one part in 10^4 oscillator is

$$p = P[9999 \le W \le 10001] = \int_{9999}^{10001} (0.025)\, dr = 0.05$$

(b) To find the PMF of T_1, we view each oscillator test as an independent trial. A success occurs on a trial with probability p if we find a one part in 10^4 oscillator. The first one part in 10^4 oscillator is found at time $T_1 = t$ if we observe failures on trials $1, \ldots, t-1$ followed by a success on trial t. Hence, just as in Example 2.11, T_1 has the geometric PMF

$$P_{T_1}(t) = \begin{cases} (1-p)^{t-1} p & t = 1, 2, \ldots \\ 9 & \text{otherwise} \end{cases}$$

A geometric random variable with success probability p has mean $1/p$. This is derived in Theorem 2.5. The expected time to find the first good oscillator is $E[T_1] = 1/p = 20$ minutes.

(c) Since $p = 0.05$, the probability the first one part in 10^4 oscillator is found in exactly 20 minutes is $P_{T_1}(20) = (0.95)^{19}(0.05) = 0.0189$.

(d) The time T_5 required to find the 5th one part in 10^4 oscillator is the number of trials needed for 5 successes. T_5 is a Pascal random variable. If this is not clear, see Example 2.15 where the Pascal PMF is derived. When we are looking for 5 successes, the Pascal PMF is

$$P_{T_5}(t) = \begin{cases} \binom{t-1}{4} p^5 (1-p)^{t-5} & t = 5, 6, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Looking up the Pascal PMF in Appendix A, we find that $E[T_5] = 5/p = 100$ minutes. The following argument is a second derivation of the mean of T_5. Once we find the first one part in 10^4 oscillator, the number of additional trials needed to find the next one part in 10^4 oscillator once again has a geometric PMF with mean $1/p$ since each independent trial is a success with probability p. Similarly, the time required to find 5 one part in 10^4 oscillators is the sum of five independent geometric random variables. That is,

$$T_5 = K_1 + K_2 + K_3 + K_4 + K_5$$

where each K_i is identically distributed to T_1. Since the expectation of the sum equals the sum of the expectations,

$$E[T_5] = E[K_1 + K_2 + K_3 + K_4 + K_5] = 5E[K_i] = 5/p = 100 \text{ minutes}$$

Problem 6.4.1
independent Gaussian random variables. Hence, each Y_k must have the same PDF. That is, the Y_k are identically distributed. Next, we observe that the sequence of Y_k is independent. To see this, we observe that each Y_k is composed of two samples of X_k that are unused by any other Y_j for $j \neq k$.

Problem 6.4.2
independent Gaussian random variables. Hence, each W_n must have the same PDF. That is, the W_n are identically distributed. However, since W_{n-1} and W_n both use X_{n-1} in their averaging, W_{n-1} and W_n are dependent. We can verify this observation by calculating the covariance of W_{n-1} and W_n. First, we observe that for all n,

$$E[W_n] = (E[X_n] + E[X_{n-1}])/2 = 30$$

Next, we observe that W_{n-1} and W_n have covariance

$$\begin{aligned}\operatorname{Cov}[W_{n-1}, W_n] &= E[W_{n-1}W_n] - E[W_n]E[W_{n-1}] \\ &= \frac{1}{4}E[(X_{n-1} + X_{n-2})(X_n + X_{n-1})] - 900\end{aligned}$$

We observe that for $n \neq m$, $E[X_nX_m] = E[X_n]E[X_m] = 900$ while

$$E\left[X_n^2\right] = \operatorname{Var}[X_n] + (E[X_n])^2 = 916$$

Thus,

$$\operatorname{Cov}[W_{n-1}, W_n] = \frac{900 + 916 + 900 + 900}{4} - 900 = 4$$

Since $\operatorname{Cov}[W_{n-1}, W_n] \neq 0$, W_n and W_{n-1} must be dependent.

Problem 6.7.1
The discrete time autocovariance function is

$$C_X[m,k] = E[(X_m - \mu_X)(X_{m+k} - \mu_X)]$$

for $k = 0$, $C_X[m,0] = \operatorname{Var}[X_m] = \sigma_X^2$. For $k \neq 0$, X_m and X_{m+k} are independent so that

$$C_X[m,k] = E[(X_m - \mu_X)]E[(X_{m+k} - \mu_X)] = 0$$

Thus the autocovariance of X_n is

$$C_X[m,k] = \begin{cases} \sigma_X^2 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Problem 6.7.2
Recall that $X(t) = t - W$ where $E[W] = 1$ and $E\left[W^2\right] = 2$.

(a) The mean is $\mu_X(t) = E[t - W] = t - E[W] = t - 1$.

(b) The autocovariance is

$$\begin{aligned}
C_X(t,\tau) &= E[X(t)X(t+\tau)] - \mu_X(t)\mu_X(t+\tau) \\
&= E[(t-W)(t+\tau-W)] - (t-1)(t+\tau-1) \\
&= t(t+\tau) - E[(t+t+\tau)W] + E\left[W^2\right] - t(t+\tau) + t + t + \tau - 1 \\
&= -(2t+\tau)E[W] + 2 + 2t + \tau - 1 \\
&= 1
\end{aligned}$$

Problem 6.7.5

The output sequence has mean

$$E[Y_m] = E[X_{m+1} + X_m + X_{m-1}] = E[X_{m+1}] + E[X_m] + E[X_{m-1}] = 0$$

Thus, the autocovariance function is

$$\begin{aligned}
C_Y[m,k] &= E[Y_m Y_{m+k}] \\
&= E[(X_{m+1} + X_m + X_{m-1})(X_{m+k+1} + X_{m+k} + X_{m+k-1})] \\
&= C_X[m+1,k] + C_X[m+1,k-1] + C_X[m+1,k-2] + C_X[m,k+1] + C_X[m,k] \\
&\quad + C_X[m,k-1] + C_X[m-1,k+2] + C_X[m-1,k+1] + C_X[m-1,k]
\end{aligned}$$

The value of the autocovariance depends on how close k is to zero. For a given m, we will consider each possible value of k. The answers we obtain will be fairly simple because

$$C_X[m,k] = C_X[k] = \begin{cases} \sigma^2 & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

This implies

$$\begin{aligned}
C_Y[m,k] &= 3C_X[k] + 2C_X[k-1] + 2C_X[k+1] + C_X[k-2] + C_X[k+2] \\
&= \begin{cases} 3\sigma^2 & k = 0 \\ 2\sigma^2 & |k| = 1 \\ \sigma^2 & |k| = 2 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Problem 6.8.1

$Y(t_1), \ldots, Y(t_k)$, we observe that $Y(t_j) = X(t_j + a)$. This implies

$$f_{Y(t_1),\ldots,Y(t_k)}(y_1,\ldots,y_k) = f_{X(t_1+a),\ldots,X(t_k+a)}(y_1,\ldots,y_k)$$

Thus,

$$f_{Y(t_1+\tau),\ldots,Y(t_k+\tau)}(y_1,\ldots,y_k) = f_{X(t_1+\tau+a),\ldots,X(t_k+\tau+a)}(y_1,\ldots,y_k)$$

Since $X(t)$ is a stationary process,

$$f_{X(t_1+\tau+a),\ldots,X(t_k+\tau+a)}(y_1,\ldots,y_k) = f_{X(t_1+a),\ldots,X(t_k+a)}(y_1,\ldots,y_k)$$

This implies

$$f_{Y(t_1+\tau),\ldots,Y(t_k+\tau)}(y_1,\ldots,y_k) = f_{X(t_1+a),\ldots,X(t_k+a)}(y_1,\ldots,y_k) = f_{Y(t_1),\ldots,Y(t_k)}(y_1,\ldots,y_k)$$

We can conclude that $Y(t)$ is a stationary process.

Problem 6.8.3

$n_1, \ldots, n_m$ and an offset k, we note that $Y_{n_i+k} = X((n_i+k)\Delta)$. This implies

$$f_{Y_{n_1+k},\ldots,Y_{n_m+k}}(y_1,\ldots,y_m) = f_{X((n_1+k)\Delta),\ldots,X((n_m+k)\Delta)}(y_1,\ldots,y_m)$$

Since $X(t)$ is a stationary process,

$$f_{X((n_1+k)\Delta),\ldots,X((n_m+k)\Delta)}(y_1,\ldots,y_m) = f_{X(n_1\Delta),\ldots,X(n_m\Delta)}(y_1,\ldots,y_m)$$

Since $X(n_i\Delta) = Y_{n_i}$, we see that

$$f_{Y_{n_1+k},\ldots,Y_{n_m+k}}(y_1,\ldots,y_m) = f_{Y_{n_1},\ldots,Y_{n_m}}(y_1,\ldots,y_m)$$

Hence Y_n is a stationary random sequence.

Problem 6.8.5

Since $g(\cdot)$ is an unspecified function, we will work with the joint CDF of $Y(t_1+\tau),\ldots,Y(t_n+\tau)$. To show $Y(t)$ is a stationary process, we will show that for all τ,

$$F_{Y(t_1+\tau),\ldots,Y(t_n+\tau)}(y_1,\ldots,y_n) = F_{Y(t_1),\ldots,Y(t_n)}(y_1,\ldots,y_n)$$

By taking partial derivatives with respect to $y_1,\ldots,y_n$, it should be apparent that this implies that the joint PDF $f_{Y(t_1+\tau),\ldots,Y(t_n+\tau)}(y_1,\ldots,y_n)$ will not depend on τ. To proceed, we write

$$\begin{aligned} F_{Y(t_1+\tau),\ldots,Y(t_n+\tau)}(y_1,\ldots,y_n) &= P[Y(t_1+\tau) \le y_1,\ldots,Y(t_n+\tau) \le y_n] \\ &= P\Bigg[\underbrace{g(X(t_1+\tau)) \le y_1,\ldots,g(X(t_n+\tau)) \le y_n}_{A_\tau}\Bigg] \end{aligned}$$

In principle, we can calculate $P[A_\tau]$ by integrating $f_{X(t_1+\tau),\ldots,X(t_n+\tau)}(x_1,\ldots,x_n)$ over the region corresponding to event A_τ. Since $X(t)$ is a stationary process,

$$f_{X(t_1+\tau),\ldots,X(t_n+\tau)}(x_1,\ldots,x_n) = f_{X(t_1),\ldots,X(t_n)}(x_1,\ldots,x_n)$$

This implies $P[A_\tau]$ does not depend on τ. In particular,

$$\begin{aligned} F_{Y(t_1+\tau),\ldots,Y(t_n+\tau)}(y_1,\ldots,y_n) &= P[A_\tau] \\ &= P[g(X(t_1)) \le y_1,\ldots,g(X(t_n)) \le y_n] \\ &= F_{Y(t_1),\ldots,Y(t_n)}(y_1,\ldots,y_n) \end{aligned}$$