Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
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Problem Solutions : Yates and Goodman,7.1.1 7.1.2 7.2.2 7.3.1 7.4.2 and 7.5.2

Problem 7.1.1

random variable that indicates the result of flip 33. The PMF of X_{33} is

$$P_{X_{33}}(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & \text{otherwise} \end{cases}$$

Note that each X_i has expected value $E[X] = p$ and variance $\text{Var}[X] = p(1-p)$. The random variable $Y = X_1 + \cdots + X_{100}$ is the number of heads in 100 coin flips. Hence, Y has the binomial PMF

$$P_Y(y) = \begin{cases} \binom{100}{y} p^y (1-p)^{100-y} & y = 0,1,\ldots,100 \\ 0 & \text{otherwise} \end{cases}$$

Since the X_i are independent, by Theorems 7.1 and 7.3, the mean and variance of Y are

$$E[Y] = 100E[X] = 100p \qquad \text{Var}[Y] = 100\,\text{Var}[X] = 100p(1-p)$$

Problem 7.1.2

(a) Since $Y = X_1 + (-X_2)$, Theorem 7.1 says that the expected value of the difference is

$$E[Y] = E[X_1] + E[-X_2] = E[X] - E[X] = 0$$

(b) By Theorem 7.2, the variance of the difference is

$$\text{Var}[Y] = \text{Var}[X_1] + \text{Var}[-X_2] = 2\,\text{Var}[X]$$

Problem 7.2.2

$$f_{X,Y}(x,y) = \begin{cases} 1 & 0 \le x,y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Proceeding as in Problem 7.2.1, we must first find $F_W(w)$ by integrating over the square defined by $0 \le x,y \le 1$. Again we are forced to find $F_W(w)$ in parts as we did in Problem 7.2.1 resulting in the following integrals for their appropriate regions. For $0 \le w \le 1$,

$$F_W(w) = \int_0^w \int_0^{w-x} dx\,dy = w^2/2$$

For $1 \le w \le 2$,

$$F_W(w) = \int_0^{w-1} \int_0^1 dx\,dy + \int_{w-1}^1 \int_0^{w-y} dx\,dy = 2w - 1 - w^2/2$$

The complete expression for the CDF of W is

$$F_W(w) = \begin{cases} 0 & w < 0 \\ w^2/2 & 0 \le w \le 1 \\ 2w - 1 - w^2/2 & 1 \le w \le 2 \\ 1 & \text{otherwise} \end{cases}$$

With the CDF, we can find $f_W(w)$ by differentiating with respect to w.

$$f_W(w) = \begin{cases} w & 0 \le w \le 1 \\ 2 - w & 1 \le w \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Problem 7.3.1

For a constant $a > 0$, a zero mean Laplace random variable X has PDF

$$f_X(x) = \frac{a}{2} e^{-a|x|} \qquad -\infty < x < \infty$$

The moment generating function of X is

$$\begin{aligned} \phi_X(s) = E\left[e^{sX}\right] &= \frac{a}{2}\int_{-\infty}^{0} e^{sx} e^{ax}\,dx + \frac{a}{2}\int_{0}^{\infty} e^{sx} e^{-ax}\,dx \\ &= \frac{a}{2}\left.\frac{e^{(s+a)x}}{s+a}\right|_{-\infty}^{0} + \frac{a}{2}\left.\frac{e^{(s-a)x}}{s-a}\right|_{0}^{\infty} \\ &= \frac{a}{2}\left(\frac{1}{s+a} - \frac{1}{s-a}\right) \\ &= \frac{a^2}{a^2 - s^2} \end{aligned}$$

Problem 7.4.2

points X_i that you earn for game i has PMF

$$P_{X_i}(x) = \begin{cases} 1/3 & x = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) The MGF of X_i is

$$\phi_{X_i}(s) = E\left[e^{sX_i}\right] = 1/3 + e^s/3 + e^{2s}/3$$

Since $Y = X_1 + \cdots + X_n$, Theorem 7.10 implies

$$\phi_Y(s) = [\phi_{X_i}(s)]^n = [1 + e^s + e^{2s}]^n / 3^n$$

(b) First we observe that first and second moments of X_i are

$$\begin{aligned} E[X_i] &= \sum_x x P_{X_i}(x) = 1/3 + 2/3 = 1 \\ E\left[X_i^2\right] &= \sum_x x^2 P_{X_i}(x) = 1^2/3 + 2^2/3 = 5/3 \end{aligned}$$

Hence, $\text{Var}[X_i] = E\left[X_i^2\right] - (E[X_i])^2 = 2/3$. By Theorems 7.1 and 7.3, the mean and variance of Y are

$$E[Y] = nE[X] = n$$
$$\text{Var}[Y] = n\,\text{Var}[X] = 2n/3$$

Problem 7.5.2

Using the moment generating function of X, $\phi_X(s) = e^{\sigma^2 s^2/2}$. We can find the nth moment of X, $E[X^n]$ by taking the nth derivative of $\phi_X(s)$ and setting $s = 0$.

$$\begin{aligned} E[X] &= \sigma^2 s e^{\sigma^2 s^2/2}\Big|_{s=0} = 0 \\ E\left[X^2\right] &= \sigma^2 e^{\sigma^2 s^2/2} + \sigma^4 s^2 e^{\sigma^2 s^2/2}\Big|_{s=0} = \sigma^2 \end{aligned}$$

Continuing in this manner we find that

$$E\left[X^3\right] = \left(3\sigma^4 s + \sigma^6 s^3\right) e^{\sigma^2 s^2/2}\Big|_{s=0} = 0$$
$$E\left[X^4\right] = \left(3\sigma^4 + 6\sigma^6 s^2 + \sigma^8 s^4\right) e^{\sigma^2 s^2/2}\Big|_{s=0} = 3\sigma^4$$