Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
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Problem Solutions : Yates and Goodman,4.7.7 4.7.13 4.8.1 4.8.2 5.1.2 5.2.1 5.2.2 5.3.1 5.3.2 5.4.1 and 5.4.2

Problem 4.7.7

Since the microphone voltage V is uniformly distributed between -1 and 1 volts, V has PDF and CDF

$$f_V(v) = \begin{cases} 1/2 & -1 \le v \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad F_V(v) = \begin{cases} 0 & v < -1 \\ (v+1)/2 & -1 \le v \le 1 \\ 1 & v > 1 \end{cases}$$

The voltage is processed by a limiter whose output magnitude is given by below

$$L = \begin{cases} |V| & |V| \le 0.5 \\ 0.5 & \text{otherwise} \end{cases}$$

(a)

$$\begin{aligned} P[L = 0.5] &= P[|V| \ge 0.5] = P[V \ge 0.5] + P[V \le -0.5] \\ &= 1 - F_V(0.5) + F_V(-0.5) \\ &= 1 - 1.5/2 + 0.5/2 = 1/2 \end{aligned}$$

(b) For $0 \le l \le 0.5$,

$$F_L(l) = P[|V| \le l] = P[-l \le v \le l] = F_V(l) - F_V(-l) = 1/2(l+1) - 1/2(-l+1) = l$$

So the CDF of L is

$$F_L(l) = \begin{cases} 0 & l < 0 \\ l & 0 \le l < 0.5 \\ 1 & l \ge 0.5 \end{cases}$$

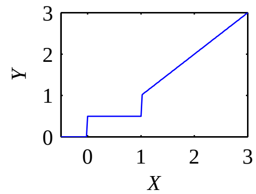
(c) By taking the derivative of $F_L(l)$, the PDF of L is

$$f_L(l) = \begin{cases} 1 + (0.5)\delta(l - 0.5) & 0 \le l \le 0.5 \\ 0 & \text{otherwise} \end{cases}$$

The expected value of L is

$$E[L] = \int_{-\infty}^{\infty} l\, f_L(l)\, dl = \int_0^{0.5} l\, dl + 0.5 \int_0^{0.5} l(0.5)\delta(l - 0.5)\, dl = 0.375$$

Problem 4.7.13
shown in the following figure:



(a) Note that $Y = 1/2$ if and only if $0 \le X \le 1$. Thus,

$$P[Y = 1/2] = P[0 \le X \le 1] = \int_0^1 f_X(x)\,dx = \int_0^1 (x/2)\,dx = 1/4$$

(b) Since $Y \ge 1/2$, we can conclude that $F_Y(y) = 0$ for $y < 1/2$. Also, $F_Y(1/2) = P[Y = 1/2] = 1/4$. Similarly, for $1/2 < y \le 1$,

$$F_Y(y) = P[0 \le X \le 1] = P[Y = 1/2] = 1/4$$

Next, for $1 < y \le 2$,

$$F_Y(y) = P[X \le y] = \int_0^y f_X(x)\,dx = y^2/4$$

Lastly, since $Y \le 2$, $F_Y(y) = 1$ for $y \ge 2$. The complete expression of the CDF is

$$F_Y(y) = \begin{cases} 0 & y < 1/2 \\ 1/4 & 1/2 \le y \le 1 \\ y^2/4 & 1 < y < 2 \\ 1 & y \ge 2 \end{cases}$$

Problem 4.8.1

$$f_X(x) = \begin{cases} 1/10 & -5 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

(a) The event B has probability

$$P[B] = P[-3 \le X \le 3] = \int_{-3}^{3} \frac{1}{10}\,dx = \frac{3}{5}$$

From Definition 4.15, the conditional PDF of X given B is

$$f_{X|B}(x) = \begin{cases} f_X(x)/P[B] & x \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1/6 & |x| \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(b) Given B, we see that X has a uniform PDF over $[a,b]$ with $a = -3$ and $b = 3$. From Theorem 4.7, the conditional expected value of X is $E[X|B] = (a+b)/2 = 0$.

(c) From Theorem 4.7, the conditional variance of X is $\text{Var}[X|B] = (b-a)^2/12 = 3$.

Problem 4.8.2
Y is

$$f_Y(y) = \begin{cases} (1/5)e^{-y/5} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) The event A has probability

$$P[A] = P[Y < 2] = \int_0^2 (1/5)e^{-y/5}\,dy = -e^{-y/5}\Big|_0^2 = 1 - e^{-2/5}$$

From Definition 4.15, the conditional PDf of Y given A is

$$f_{Y|B}(y) = \begin{cases} f_Y(y)/P[A] & x \in A \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} (1/5)e^{-y/5}/(1-e^{-2/5}) & 0 \leq y < 2 \\ 0 & \text{otherwise} \end{cases}$$

(b) The conditional expected value of Y given A is

$$E[Y|A] = \int_{-\infty}^{\infty} y f_{Y|A}(y)\,dy = \frac{1/5}{1-e^{-2/5}} \int_0^2 y e^{-y/5}\,dy$$

Using the integration by parts formula $\int u\,dv = uv - \int v\,du$ with $u = y$ and $dv = e^{-y/5}\,dy$ yields

$$E[Y|A] = \frac{1/5}{1-e^{-2/5}} \left(-5ye^{-y/5}\Big|_0^2 + \int_0^2 5e^{-y/5}\,dy \right)$$
$$= \frac{1/5}{1-e^{-2/5}} \left(-10e^{-2/5} - 25e^{-y/5}\Big|_0^2 \right)$$
$$= \frac{5 - 7e^{-2/5}}{1-e^{-2/5}}$$

Problem 5.1.2

(a) Because the probability that any random variable is less than $-\infty$ is zero, we have

$$F_{X,Y}(x, -\infty) = P[X \leq x, Y \leq -\infty] \leq P[Y \leq -\infty] = 0$$

(b) The probability that any random variable is less than infinity is always one.

$$F_{X,Y}(x, \infty) = P[X \leq x, Y \leq \infty] = P[X \leq x] = F_X(x)$$

(c) Although $P[Y \leq \infty] = 1$, $P[X \leq -\infty] = 0$. Therefore the following is true.

$$F_{X,Y}(-\infty, \infty) = P[X \leq -\infty, Y \leq \infty] \leq P[X \leq -\infty] = 0$$

(d) Part (d) follows the same logic as that of part (a).

$$F_{X,Y}(-\infty, y) = P[X \leq -\infty, Y \leq y] \leq P[X \leq -\infty] = 0$$

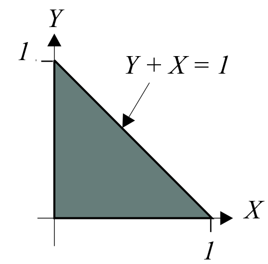
(e) Analogous to Part (b), we find that

$$F_{X,Y}(\infty, y) = P[X \leq \infty, Y \leq y] = P[Y \leq y] = F_Y(y)$$

Problem 5.2.1

(a) The joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} c & x+y \leq 1, x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



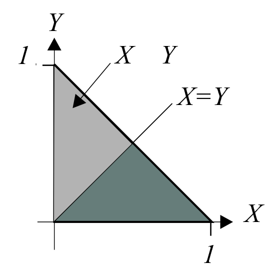
To find the constant c we integrate over the region shown. This gives

$$\int_0^1 \int_0^{1-x} c\, dy\, dx = cx - \frac{cx}{2}\Big|_0^1 = \frac{c}{2} = 1$$

Therefore $c = 2$.

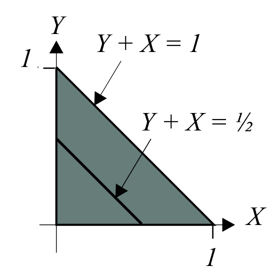
(b) To find the $P[X \leq Y]$ we look to integrate over the area indicated by the graph

$$\begin{aligned} P[X \leq Y] &= \int_0^{1/2} \int_x^{1-x} dy\, dx \\ &= \int_0^{1/2} (2-4x)\, dx \\ &= 1/2 \end{aligned}$$



(c) The probability $P[X+Y \leq 1/2]$ can be seen in the figure at right. Here we can set up the following integrals

$$\begin{aligned} P[X+Y \leq 1/2] &= \int_0^{1/2} \int_0^{1/2-x} 2\, dy\, dx \\ &= \int_0^{1/2} (1-2x)\, dx \\ &= 1/2 - 1/4 = 1/4 \end{aligned}$$



Problem 5.2.2

Given the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cxy^2 & 0 \leq x, y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

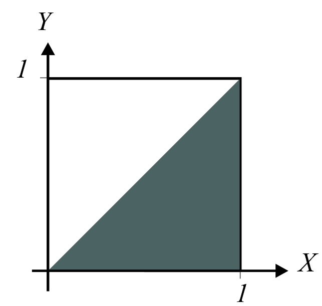
(a) To find the constant c integrate $f_{X,Y}(x,y)$ over the all possible values of X and Y to get

$$1 = \int_0^1 \int_0^1 cxy^2 \, dx \, dy = c/6$$

Therefore $c = 6$.

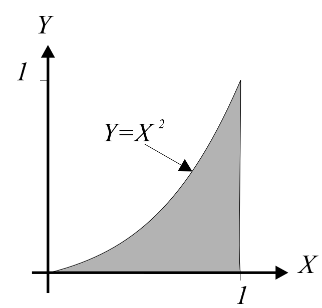
(b) The probability $P[X \geq Y]$ is the integral of the joint PDF $f_{X,Y}(x,y)$ over the indicated shaded region.

$$P[X \geq Y] = \int_0^1 \int_0^x 6xy^2 \, dy \, dx = \int_0^1 2x^4 \, dx = 2/5$$



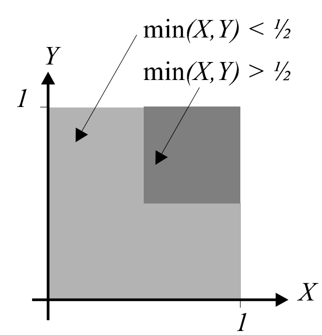
Similarly, to find $P[Y \leq X^2]$ we can integrate over the region shown in the figure.

$$P[Y \leq X^2] = \int_0^1 \int_0^{x^2} 6xy^2 \, dy \, dx = 1/4$$



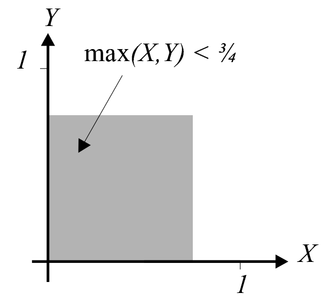
(c) Here we can choose to either integrate $f_{X,Y}(x,y)$ over the lighter shaded region, which would require the evaluation of two integrals, or we can perform one integral over the darker region by recognizing

$$P[\min(X,Y) \leq 1/2] = 1 - P[\min(X,Y) > 1/2] = 1 - \int_{1/2}^1 \int_{1/2}^1 6xy^2 \, dx \, dy = 1 - \int_{1/2}^1 \frac{9y^2}{4} \, dy = \frac{11}{32}$$



(d) The $P[\max(X,Y) \leq 3/4]$ can be found be integrating over the shaded region shown below.

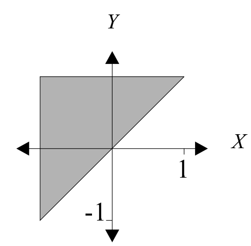
$$
\begin{aligned}
P[\max(X,Y) \leq 3/4] &= P[X \leq 3/4, Y \leq 3/4] \\
&= \int_0^{\frac{3}{4}} \int_0^{\frac{3}{4}} 6xy^2 \, dx \, dy \\
&= \left(x^2 \Big|_0^{3/4} \right) \left(y^3 \Big|_0^{3/4} \right) \\
&= (3/4)^5 = 0.237
\end{aligned}
$$



Problem 5.3.1

(a) The joint PDF (and the corresponding region of nonzero probability) are

$$
f_{X,Y}(x,y) = \begin{cases} 1/2 & -1 \leq x \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}
$$



(b)

$$
P[X > 0] = \int_0^1 \int_x^1 \frac{1}{2} \, dy \, dx = \int_0^1 \frac{1-x}{2} \, dx = 1/4
$$

This result can be deduced by geometry. The shaded triangle of the X,Y plane corresponding to the event $X > 0$ is $1/4$ of the total shaded area.

(c) For $x > 1$ or $x < -1$, $f_X(x) = 0$. For $-1 \leq x \leq 1$,

$$
f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy = \int_x^1 \frac{1}{2} \, dy = (1-x)/2
$$

The complete expression for the marginal PDF is

$$
f_X(x) = \begin{cases} (1-x)/2 & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}
$$

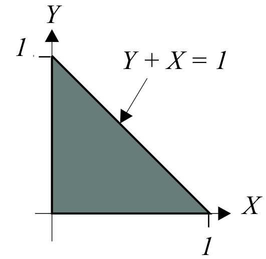
(d) From the marginal PDF $f_X(x)$, the expected value of X is

$$
E[X] = \int_{-\infty}^{\infty} x f_X(x) \, dx = \frac{1}{2} \int_{-1}^{1} x(1-x) \, dx = \left. \frac{x^2}{4} - \frac{x^3}{6} \right|_{-1}^{1} = -\frac{1}{3}
$$

Problem 5.3.2

$$f_{X,Y}(x,y) = \begin{cases} 2 & x+y \leq 1, x,y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Using the figure to the left we can find the marginal PDFs by integrating over the appropriate regions.



$$f_X(x) = \int_0^{1-x} 2\,dy = \begin{cases} 2(1-x) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

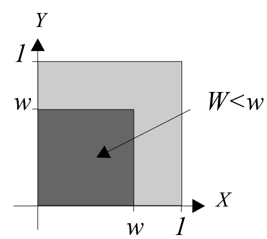
Likewise for $f_Y(y)$:

$$f_Y(y) = \int_0^{1-y} 2\,dx = \begin{cases} 2(1-y) & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 5.4.1

(a) The minimum value of W is $W = 0$, which occurs when $X = 0$ and $Y = 0$. The maximum value of W is $W = 1$, which occurs when $X = 1$ or $Y = 1$. The range of W is $S_W = \{w|0 \leq w \leq 1\}$.

(b) For $0 \leq w \leq 1$, the CDF of W is



$$\begin{aligned} F_W(w) &= P[\max(X,Y) \leq w] \\ &= P[X \leq w, Y \leq w] \\ &= \int_0^w \int_0^w f_{X,Y}(x,y)\,dy\,dx \end{aligned}$$

Substituting $f_{X,Y}(x,y) = x+y$ yields

$$F_W(w) = \int_0^w \int_0^w (x+y)\,dy\,dx = \int_0^w \left(xy + \frac{y^2}{2}\bigg|_{y=0}^{y=w} \right) dx = \int_0^w (wx + w^2/2)\,dx = w^3$$

The complete expression for the CDF is

$$F_W(w) = \begin{cases} 0 & w < 0 \\ w^3 & 0 \leq w \leq 1 \\ 1 & \text{otherwise} \end{cases}$$

(c) The PDF of W is found by differentiating the CDF.

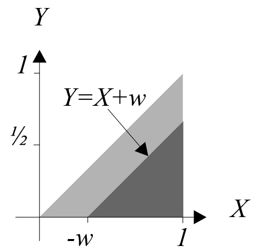
$$f_Y(y) = \frac{dF_W(w)}{dw} = \begin{cases} 3w^2 & 0 \leq w \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 5.4.2

(a) Since the joint PDF $f_{X,Y}(x,y)$ is nonzero only for $0 \le y \le x \le 1$, we observe that $W = Y - X \le 0$ since $Y \le X$. In addition, the most negative value of W occurs when $Y = 0$ and $X = 1$ and $W = -1$. Hence the range of W is $S_W = \{w | -1 \le w \le 0\}$.

(b) For $w < -1$, $F_W(w) = 0$. For $w > 0$, $F_W(w) = 1$. For $-1 \le w \le 0$, the CDF of W is

$$
\begin{aligned}
F_W(w) &= P[Y - X \le w] \\
&= \int_{-w}^{1} \int_{0}^{x+w} 6y \, dy \, dx \\
&= \int_{-w}^{1} 3(x+w)^2 \, dx = \left. (x+w)^3 \right|_{-w}^{1} = (1+w)^3
\end{aligned}
$$



Therefore, the complete CDF of W is

$$
F_W(w) = \begin{cases} 0 & w < -1 \\ (1+w)^3 & -1 \le w \le 0 \\ 1 & w > 0 \end{cases}
$$

(c) By taking the derivative of $f_W(w)$ with respect to w, we obtain the PDF

$$
f_W(w) = \begin{cases} 3(w+1)^2 & -1 \le w \le 0 \\ 0 & \text{otherwise} \end{cases}
$$