Detection & Estimation Theory

Course No: 16:332:549 - (Spring 2006)

Solutions to Midterm Exam

1. Consider the following binary hypothesis testing problem with prior probabilities P_0 and P_1.

$$\begin{aligned} H_0: \quad x = & \quad N \\ H_1: \quad x = & \quad S + N, \end{aligned}$$

where S and N are independent random variables having the densities

$$p_S(s) = ae^{-as}, \; s \geq 0$$

$$p_N(n) = be^{-bn}, \; n \geq 0$$

The PDF of x under hypothesis H_1 is obtained by convolving the densities p_S and p_N. Therefore,

$$p_{X|H_1}(x) = \int_0^x p_S(s) P_N(x-s) ds = \frac{ab}{b-a}[e^{-ax} - e^{-bx}]$$

The likelihood ratio test is now given by

$$\Lambda(x) = \frac{p_{X|H_1}(x)}{p_{X|H_0}(x)} = \frac{a}{b-a}[e^{(b-a)x} - 1] \gtrless_{H_0}^{H_1} \eta = \frac{P_0}{P_1}$$

Taking the log and rearranging terms results in

$$x \gtrless_{H_0}^{H_1} \gamma = \frac{1}{b-a} log(1 + \frac{b-a}{a}\eta)$$

2. A basket contains 10 IC chips out of which n are known to be defective. It is required to test the following two hypotheses

$$\begin{aligned} H_0 &: \quad n = 3 \\ H_1 &: \quad n = 7, \end{aligned}$$

by drawing two chips at random and noting down its quality. We need to design a Neyman-Pearson test for testing the hypothesis at a false-alarm probability of 1/15.

Since the two chips drawn at random can each be either a "good" chip or a "bad" chip, we have the following possibilities

Pr(both chips are bad) $= Pr(b,b) = \frac{n(n-1)}{10.9}$

Pr(one bad, one good) $= Pr(b,g) = Pr(g,b) = \frac{n(10-n)}{10.9}$

Event, E	$P(E\|H_0)$	$P(E\|H_1)$	$\Lambda = P(E\|H_1)/P(E\|H_0)$
(b,b)	1/15	7/15	7
(b,g)	7/30	7/30	1
(g,b)	7/30	7/30	1
(g,g)	7/15	1/15	1/7

$Pr(\text{both chips are good}) = Pr(g,g) = \frac{(10-n)(9-n)}{10.9}$

The following table shows the probabilities under both hypotheses and the likelihood ratio Λ

The N-P test is of the following form : $\Lambda \overset{H_1}{\underset{H_0}{\gtrless}} t$.

If $t = 7$ is chosen, then we get $P_F = P(\Lambda \geq t|H_0) = 1/15$.

Therefore, if (b,b) is observed $\Rightarrow$ decide H_1, else decide H_0.

3. The likelihood ratio defined as

$$\Lambda(Z) = \frac{f_{z|H_1}(Z|H_1)}{f_{z|H_0}(Z|H_0)}$$

is a random variable on each hypothesis with a different probability density function. To show that $E[\Lambda^k|H_1] = E[\Lambda^{k+1}|H_0]$ for $k \geq 0$, we proceed as follows:

Let $R(\Lambda(z)) \equiv$ The range of $\Lambda(Z)$

Let $R(z) \equiv$ The range of Z

Then by definition

$$E[\Lambda^k|H_1] = \int_{R(\Lambda(z))} \Lambda^k d\Lambda(Z|H_1) = \int_{R(z)} \Big(\frac{f_{z|H_1}(Z|H_1)}{f_{z|H_0}(Z|H_0)}\Big)^k f_{z|H_1}(Z|H_1)dz$$

$\Rightarrow$

$$E[\Lambda^k|H_1] = \int_{R(z)} \Big(\frac{f_{z|H_1}(Z|H_1)}{f_{z|H_0}(Z|H_0)}\Big)^{k+1} f_{z|H_0}(Z|H_1)dz = E[\Lambda^{k+1}|H_0]$$

Substituting $k = 1$ in the above $\Rightarrow E[\Lambda|H_0] = 1$

Substituting $k = 2$ and using the above result $\Rightarrow E[\Lambda|H_1] - E[\Lambda|H_0] = Var(\Lambda|H_0)$. Alternately, we can infer that $E[\Lambda|H_1] = E[\Lambda|H_0] + Var(\Lambda|H_0) > 1$, since the variance is necessarily greater than zero.

In the limit as the number of observations $n \to \infty$, the sample mean $g(\underline{Z})$ will converge to $E[\Lambda|H_0]$ under hypothesis H_0 or $E[\Lambda|H_1]$ under hypothesis H_1. Therefore the test

$$\text{Decide} \quad H_0 \quad \text{if} \quad g(\underline{Z}) \quad = \quad 1$$

and

$$\text{Decide } H_1 \text{ if } g(\underline{Z}) \; > \; 1$$

should be error-free.

4. Let $X_1, X_2, \cdots, X_m$ represent i.i.d. random samples from an exponential distribution with parameter θ_1, and let $Y_1, Y_2, \cdots, Y_n$ represent i.i.d. random samples from an exponential distribution with parameter θ_2. Specifically,

$$X_i \sim p(x_i|\theta_1) = \frac{1}{\theta_1} \exp(-\frac{x_i}{\theta_1}), \quad i = 1, \cdots, m$$

$$Y_i \sim p(y_i|\theta_2) = \frac{1}{\theta_2} \exp(-\frac{y_i}{\theta_2}), \quad i = 1, \cdots, n$$

Assume that $\{X_i\}$ are independent of $\{Y_i\}$ as well.

The observations for the above problem are $X_1, X_2, \cdots, X_m, Y_1, Y_2, \cdots, Y_n$.

We need to find the generalized likelihood ratio test (GLRT) for testing

$$\begin{aligned} H_0 &: \; \theta_1 = \theta_2 \\ H_1 &: \; \theta_1 \neq \theta_2 \end{aligned}$$

The Likelihood ratio for GLRT is obtained by maximizing both the numerator and denominator separately. Therefore

$$\Lambda = \frac{\max_{\theta_1,\theta_2} \prod_{i=1}^m p(x_i|\theta_1) \prod_{j=1}^n p(y_j|\theta_2)}{\max_{\theta_1=\theta_2=\theta} \prod_{i=1}^m p(x_i|\theta_1) \prod_{j=1}^n p(y_j|\theta_2)} = \frac{\max_{\theta_1} \prod_{i=1}^m p(x_i|\theta_1) \max_{\theta_2} \prod_{j=1}^n p(y_j|\theta_2)}{\max_{\theta} \prod_{i=1}^m p(x_i|\theta) \prod_{j=1}^n p(y_j|\theta)}$$

Note that the maximum is attained for each case as

$$\hat{\theta}_1 = \frac{1}{m} \sum_{i=1}^m X_i$$

$$\hat{\theta}_2 = \frac{1}{n} \sum_{j=1}^n Y_j$$

$$\hat{\theta} = \frac{1}{n+m} (\sum_{i=1}^m X_i + \sum_{j=1}^n Y_j)$$

Let us define $t_1 = \sum_{i=1}^m X_i$ and $t_2 = \sum_{j=1}^n Y_j$.

Substituting the above estimates in the likelihood ratio yields

$$\Lambda = \frac{(\frac{m}{t_1})^m (\frac{n}{t_2})^n \exp(-m-n)}{(\frac{m+n}{t_1+t_2})^{m+n} \exp(-m-n)}$$

The GLRT is now given as

$$\Lambda = \frac{(\frac{m}{t_1})^m (\frac{n}{t_2})^n}{(\frac{m+n}{t_1+t_2})^{m+n}} \gtrless_{H_0}^{H_1} \eta$$

$\Rightarrow$

$$\frac{(t_1+t_2)^m (t_1+t_2)^n}{t_1^m t_2^n} \gtrless_{H_0}^{H_1} \eta_1$$

Observe that $T = \frac{t_1}{t_1+t_2}$ and $1-T = \frac{t_2}{t_1+t_2}$.
Therefore, the GLRT in terms of the statistic T is given as

$$\frac{1}{T^m} \frac{1}{(1-T)^n} \gtrless_{H_0}^{H_1} \eta_1$$

$\Rightarrow$

$$T^{-m}(1-T)^{-n} \gtrless_{H_0}^{H_1} \eta_1$$