Detection & Estimation Theory

Course No: 16:332:549

Solutions to Homework 5

- 3.19

$$p(z|H_1) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{z^2}{2\sigma^2})$$

$$p(z|H_0) = \frac{1}{2}, \ \ -1 < z < 1; \ \ 0 \ \text{otherwise}$$

(a) σ is nonrandom and lies in $[0, 1]$.

The LRT a given value of σ is

$$\Lambda(z) = \frac{\frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{z^2}{2\sigma^2})}{\frac{1}{2} I_{(-1,1)}(z)} \ \overset{H_1}{\underset{H_0}{\gtrless}} \ \eta$$

Due to the range of H_0, it follows that for $-\infty < z < -1$ and $1 < z < \infty$ always decide H_1.

For $-1 < z < 1$, we need to evaluate the decision regions as follows :

Simplifying the LRT results in

$$-\frac{z^2}{2\sigma^2} \ \overset{H_1}{\underset{H_0}{\gtrless}} \ ln(\sqrt{\frac{\pi}{2}}\sigma\eta)$$

$\Rightarrow$

$$z^2 \ \overset{H_0}{\underset{H_1}{\gtrless}} \ -2\sigma^2 ln(\sqrt{\frac{\pi}{2}}\sigma\eta); \ \text{when the RHS is } > 0$$

Therefore, decide H_1 if :

$$-\sqrt{2\sigma^2 ln(\sqrt{\frac{\pi}{2}}\sigma\eta)} < z < \sqrt{2\sigma^2 ln(\sqrt{\frac{\pi}{2}}\sigma\eta)}$$

(b) σ is random and $U(0, 1)$.

The LRT now is

$$\Lambda(z) = \frac{\int_0^1 \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{z^2}{2\sigma^2}) d\sigma}{\frac{1}{2} I_{(-1,1)}(z)} \ \overset{H_1}{\underset{H_0}{\gtrless}} \ \eta$$

Again decide H_1 if $-\infty < z < -1$ and $1 < z < \infty$.

For $-1 < z < 1$, we have

$$J(z^2) = \frac{1}{\sqrt{2\pi}} \int_0^1 \frac{1}{\sigma} \exp(-\frac{z^2}{2\sigma^2}) d\sigma \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

Note that $J(z^2)$ is a monotonic decreasing function in z^2. Therefore, an equivalent test is

$$z^2 \underset{H_1}{\overset{H_0}{\gtrless}} t_1$$

$\Rightarrow$

$$|z| \underset{H_1}{\overset{H_0}{\gtrless}} \sqrt{t_1} = t,$$

or decide H_1 if $-t < z < t$

What is t ?

Note that

$$P_F = P(D_1|H_0) = \int_{-t}^{t} \frac{1}{2} dz = t$$

Therefore the threshold t is chosen according to the value of P_F desired, i.e., $t = P_F$

- 3.23

$$p(x|\theta) = \frac{1}{\pi(1 + (x - \theta)^2)}$$

(a)

$$\begin{aligned} H_1 : \ \ \theta = & \ \ \theta_1 \\ H_0 : \ \ \theta = & \ \ \theta_0 \end{aligned}$$

The LRT for minimum probability of error and equally likely hypothesis is

$$\Lambda(x) = \frac{\pi(1 + (x - \theta_0)^2)}{\pi(1 + (x - \theta_1)^2)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

$\Rightarrow$ the test is

$$x \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2}(\theta_1 + \theta_0) \ \text{ if } \theta_1 > \theta_0$$

or

$$x \underset{H_1}{\overset{H_0}{\gtrless}} \frac{1}{2}(\theta_1 + \theta_0) \ \text{ if } \theta_1 < \theta_0$$

(b) For the Neyman-Pearson test, the LRT is

$$\frac{\pi(1+(x-\theta_0)^2)}{\pi(1+(x-\theta_1)^2)} \gtrless_{H_0}^{H_1} t$$

$\Rightarrow$

$$(1-t)x^2 - 2x(\theta_0 - t\theta_1) \gtrless_{H_0}^{H_1} t - 1 + t\theta_1^2 - \theta_0^2$$

If $t < 1$, the test is

$$(x - \frac{\theta_0 - t\theta_1}{1-t})^2 \gtrless_{H_0}^{H_1} p = \frac{t(\theta_1 - \theta_0)^2}{(1-t)^2} - 1$$

$\Rightarrow$

$$|x - \frac{\theta_0 - t\theta_1}{1-t}| \gtrless_{H_0}^{H_1} \sqrt{p}$$

Therefore decide H_0 if

$$\frac{\theta_0 - t\theta_1}{1-t} - \sqrt{p} < x < \frac{\theta_0 - t\theta_1}{1-t} + \sqrt{p}$$

The probability of false-alarm should be such that

$$P_F = \int_{R_1} \frac{1}{\pi(1+(x-\theta_0)^2)} dx = \alpha$$

where R_1 is the decision region corresponding to H_1, i.e.,

$$R_1 \equiv (-\infty, \frac{\theta_0 - t\theta_1}{1-t} - \sqrt{p}) \bigcup (\frac{\theta_0 - t\theta_1}{1-t} + \sqrt{p}, \infty)$$

- 3.24

$$\begin{aligned} H_1 : \quad \theta = & \quad \theta_1 \\ H_0 : \quad \theta = & \quad \theta_0 \end{aligned}$$

$(x_1, \cdots, x_n)$, $(x_{n+1}, \cdots, x_{2n})$ are i.i.d. with

$$p(x|H_j) = \theta_j \exp(-\theta_j x), \; j = 0, 1; \; \theta_0 = 1$$

(a) Optimal Centralized Rule :

$$\Lambda(\underline{x}) = \frac{\prod_{i=1}^{2n} f(x_i|H_1)}{\prod_{i=1}^{2n} f(x_i|H_0)} = \frac{\prod_{i=1}^{2n} \theta_1 \exp(-\theta_1 x_i)}{\prod_{i=1}^{2n} \exp(-x_i)} \gtrless_{H_0}^{H_1} \eta$$

If $1-\theta_1 > 0, \Rightarrow$

$$\sum_{i=1}^{2n} x_i \ \overset{H_1}{\underset{H_0}{\gtrless}} \ t$$

where $y = \sum_{i=1}^{2n} x_i \ \sim \ \Gamma(2n, \theta_1) = \frac{\theta_1^{2n} y^{2n-1} \exp(-\theta_1 y)}{\Gamma(2n)}$

Therefore we solve for t as

$$P_F = \alpha = \int_t^\infty \frac{y^{2n-1}\exp(-y)}{\Gamma(2n)} dy$$

If $1-\theta_1 < 0, \Rightarrow$

$$\sum_{i=1}^{2n} x_i \ \overset{H_0}{\underset{H_1}{\gtrless}} \ t,$$

with

$$P_F = \alpha = \int_0^t \frac{y^{2n-1}\exp(-y)}{\Gamma(2n)} dy$$

(b)

$$\Lambda(\underline{x}) = \frac{\prod_{i=1}^n f(x_i|H_1)}{\prod_{i=1}^n f(x_i|H_0)} = \frac{\prod_{i=1}^n \theta_1 \exp(-\theta_1 x_i)}{\prod_{i=1}^n \exp(-x_i)} \ \overset{H_1}{\underset{H_0}{\gtrless}} \ \eta$$

If $1-\theta_1 > 0, \Rightarrow$

$$\sum_{i=1}^{n} x_i \ \overset{H_1}{\underset{H_0}{\gtrless}} \ t_k \quad k = 1, 2,$$

with

$$P_{F_k} = \alpha_k = \int_{t_k}^\infty \frac{y^{n-1}\exp(-y)}{\Gamma(n)} dy$$

If $1-\theta_1 < 0, \Rightarrow$

$$\sum_{i=1}^{n} x_i \ \overset{H_0}{\underset{H_1}{\gtrless}} \ t_k \quad k = 1, 2,$$

with

$$P_{F_k} = \alpha_k = \int_0^{t_k} \frac{y^{n-1}\exp(-y)}{\Gamma(n)} dy$$

(c) AND Rule : $u_o = u_1 u_2$

(i)

$$P_{F_o} = \alpha_o = P(u_o = 1|H_0) = P(u_1 = 1, u_2 = 1|H_0)$$

Since decisions are independent at each site $\Rightarrow$

$$P_{F_o} = \alpha_o = P(u_1 = 1|H_0)P(u_2 = 1|H_0) = \alpha_1\alpha_2$$

(ii)

$$P_{D_o} = P(u_o = 1|H_1) = P(u_1 = 1|H_1)P(u_2 = 1|H_1)$$

If $1 - \theta_1 > 0$, $\Rightarrow$

$$P_{D_o} = [\int_{t_1}^{\infty} \frac{\theta_1^n y^{n-1} \exp(-\theta_1 y)}{\Gamma(n)} dy][\int_{t_2}^{\infty} \frac{\theta_1^n y^{n-1} \exp(-\theta_1 y)}{\Gamma(n)} dy]$$

If $1 - \theta_1 < 0$, $\Rightarrow$

$$P_{D_o} = [\int_{0}^{t_1} \frac{\theta_1^n y^{n-1} \exp(-\theta_1 y)}{\Gamma(n)} dy][\int_{0}^{t_2} \frac{\theta_1^n y^{n-1} \exp(-\theta_1 y)}{\Gamma(n)} dy]$$

Note that t_k is a function of $\alpha_k,\ k = 1, 2$. Further, $\alpha_2 = \alpha_o/\alpha_1$.
Therefore, P_{D_o} is a function of $\alpha_o, \alpha_1, \theta_1$.
(iii) $n = 1$
Let $1 - \theta_1 > 0$, then

$$\alpha_k = \int_{t_k}^{\infty} \frac{y^0 \exp(-y)}{\Gamma(1)} dy = \exp(-t_k)$$

$\Rightarrow t_k = -ln(\alpha_k)$ for $k = 1, 2$.

$$P_{D_o} = [\int_{t_1}^{\infty} \frac{\theta_1 \exp(-\theta_1 y)}{\Gamma(1)} dy][\int_{t_2}^{\infty} \frac{\theta_1 \exp(-\theta_1 y)}{\Gamma(1)} dy] = \alpha_o^{\theta_1}$$

(d) OR Rule : $u_o = 1 - (1 - u_1)(1 - u_2)$
(i)

$\alpha_o = P(u_o = 1|H_0) = P(u_1 = 1, u_2 = 0|H_0) + P(u_1 = 0, u_2 = 1|H_0) + P(u_1 = 1, u_2 = 1|H_0)$

$\Rightarrow$

$$\alpha_o = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$$

(ii)

$P_{D_o} = P(u_o = 1|H_1) = P(u_1 = 1, u_2 = 0|H_1) + P(u_1 = 0, u_2 = 1|H_1) + P(u_1 = 1, u_2 = 1|H_1)$

$\Rightarrow$

$$P_{D_o} = P_{D_1} + P_{D_2} - P_{D_1}P_{D_2}$$

Consider $1 - \theta_1 > 0$, $\Rightarrow t_k = -ln(\alpha_k)$ for $k = 1, 2$.

Therefore,

$$P_{D_o} = \alpha_1^{\theta_1} + \alpha_2^{\theta_1} - \alpha_1^{\theta_1}\alpha_2^{\theta_1}$$

Consider $1-\theta_1 < 0$, $\Rightarrow t_k = -ln(1-\alpha_k)$ for $k = 1, 2$.

$P_{D_o} = (1-\exp(\theta_1 ln(1-\alpha_1))) + (1-\exp(\theta_1 ln(1-\alpha_2))) - (1-\exp(\theta_1 ln(1-\alpha_1)))(1-\exp(\theta_1 ln(1-\alpha_2)))$

$\Rightarrow$

$$P_{D_o} = 1 - ((1-\alpha_1)(1-\alpha_2))^{\theta_1}$$

(iii)

$$\alpha_2 = \frac{\alpha_o - \alpha_1}{1-\alpha_1}$$

Let $1-\theta_1 > 0$, $\Rightarrow$

$$P_{D_o} = \alpha_1^{\theta_1} + \alpha_2^{\theta_1} - \alpha_1^{\theta_1}\alpha_2^{\theta_1}$$

Maximizing P_{D_o} is equivalent to minimizing $P_{M_o} = 1 - P_{D_o}$. Now

$$P_{M_o} = (1-\alpha_1^{\theta_1})(1-\alpha_2^{\theta_1})$$

Taking derivatives and setting to zero yields the desired α_1, i.e., the solution is of the form

$$(1-\alpha_1^{\theta_1})\alpha_1^{\theta_1} - \alpha_2^{\theta_1-1}\frac{(1-\alpha_1^{\theta_1})}{(1-\alpha_1)} + \alpha_2^{\theta_1-1}(\alpha_o-\alpha_1)\frac{(1-\alpha_1^{\theta_1})}{(1-\alpha_1)^2} = 0$$

For example, if $\theta_1 = 0.5$, $\alpha_o = 0.1$, then substituting for $\alpha_2 = \frac{\alpha_o-\alpha_1}{1-\alpha_1}$ and solving the above equation yields $\alpha_1 = 0.05126$.

(e) For illustration, let us compare the probabilities of the centralized and decentralized schemes for $n = 1$ and $\theta_1 = 0.5$, $\alpha_o = 0.1$.

Centralized Scheme :

$$P_{F_c} = \int_{t_c}^{\infty} y\exp(-y)dy = \alpha_o = 0.1$$

$\Rightarrow t_c = 3.89$.

$$P_{D_c} = \int_{t_c}^{\infty} \theta_1^2 y\exp(-\theta_1 y)dy$$

$\Rightarrow$ for $t_c = 3.89$, $P_{D_c} = 0.42$.

Decentralized Scheme :

For the AND Rule

$$P_{D_{and}} = \alpha_o^{\theta_1} = 0.316$$

For the OR Rule

$$P_{D_{or}} = \alpha_1^{\theta_1} + \alpha_2^{\theta_1} - \alpha_1^{\theta_1}\alpha_2^{\theta_1} = 0.4$$

Therefore

$$P_{D_c} > P_{D_{or}} > P_{D_{and}}$$

- 4.1

  The receiver based on the minimum probability of error criterion and equiprobable hypothesis results in the following test :

$$\int_0^1 z(t)(y_1(t) - y_0(t))dt \gtrless_{H_0}^{H_1} \frac{1}{2}\int_0^1 (y_1^2(t) - y_0^2(t))dt$$

Now $\int_0^1 (y_1^2(t)dt = \epsilon$ and $\int_0^1 (y_0^2(t) = 2\int_0^{1/2}(2\sqrt{3\epsilon})^2 t^2 dt = \epsilon$. Therefore the test simplifies to

$$l = \int_0^1 z(t)(y_1(t) - y_0(t))dt \gtrless_{H_0}^{H_1} 0$$

The probability of error is given as

$$P_e = \frac{1}{2}P(l < 0|H_1) + \frac{1}{2}P(l > 0|H_0)$$

Under each hypothesis, l is Gaussian with

$$E(l|H_1) = \int_0^1 y_1(t)(y_1(t) - y_0(t))dt = \epsilon - \rho,$$

and

$$E(l|H_0) = \int_0^1 y_0(t)(y_1(t) - y_0(t))dt = -\epsilon + \rho,$$

where $\rho = \int_0^1 y_0(t)y_1(t)dt = \frac{\sqrt{3}\epsilon}{2}$

Further,

$$Var(l|H_i) = N_0(\epsilon - \rho) \quad i = 0, 1$$

$\Rightarrow$

$$P_e = Q(\sqrt{\frac{\epsilon - \rho}{N_0}}) = Q(\sqrt{\frac{\epsilon}{N_0}\frac{2-\sqrt{3}}{2}})$$

- 4.2

  This is an example of amplitude shift keying.

  Let $g(t) = \frac{\sin(\omega_o t)}{\sqrt{T/2}}$ and $A_1 < A_2$

  (a) For equal prior probabilities, the minimum probability of error receiver is same as the minimum Euclidean distance receiver! The decision regions are as shown above.

  (b)To compute $P(D_0|H_0)$, consider

  $$z(t) = \nu(t)$$

  and

  $$l = \int_0^T z(t)g(t)dt$$

  $\Rightarrow E(l|H_0) = 0$, $Var(l|H_0) = \frac{N_0}{2}$ and

  $$P(D_0|H_0) = P(l < \frac{A_1\sqrt{T}}{2\sqrt{2}}) = \Phi(\frac{A_1\sqrt{T}}{2\sqrt{N_0}})$$

  Similarly,

  $$P(D_1|H_1) = P(\frac{A\sqrt{T}}{2\sqrt{2}} < l < \frac{A_1 + A_2}{2}\sqrt{T/2}|H_1)$$

  Note that $E(l|H_1) = A_1\sqrt{T/2}$ and $Var(l|H_i) = \frac{N_0}{2}$ for $i = 0, 1, 2$

  Therefore,

  $$P(D_1|H_1) = \Phi(\frac{\frac{A_2 - A_1}{2}\sqrt{T/2}}{\sqrt{N_0/2}}) - \Phi(\frac{\frac{-A_1}{2}\sqrt{T/2}}{\sqrt{N_0/2}})$$

  Similarly,

  $$P(D_2|H_2) = P(l > \frac{A_2 + A_1}{2}\sqrt{T/2})$$

  $\Rightarrow$

  $$P(D_2|H_2) = Q(\frac{\frac{A_2 + A_1}{2}\sqrt{T/2} - A_2\sqrt{T/2}}{\sqrt{N_0/2}})$$

  The probability of a correct decision is

  $$P(c) = \frac{1}{3}[P(D_0|H_0) + P(D_1|H_1) + P(D_2|H_2)]$$