Detection & Estimation Theory

Course No: 16:332:549

Solutions to Homework 4

- 3.4

$$H_1: \quad Z_1 = V_1 + 1, \quad E[Z_1] = 1, Var(Z_1) = \sigma^2$$
$$Z_2 = 0.5V_2 + 0.5, \quad E[Z_2] = 0.5, Var(Z_2) = 0.25\sigma^2$$

$$H_0: \quad Z_1 = V_1 - 1, \quad E[Z_1] = -1, Var(Z_1) = \sigma^2$$
$$Z_2 = 0.5V_2 - 0.5, \quad E[Z_2] = -0.5, Var(Z_2) = 0.25\sigma^2$$

For minimum probability of error and equally likely hypothesis, the LRT is

$$\Lambda(\underline{z}) = \frac{\prod_{i=1}^{2} f(z_i|H_1)}{\prod_{i=1}^{2} f(z_i|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

Since the z_i are Gaussian and independent under each hypothesis, taking log and simplifying $\Rightarrow$ the LRT is

$$z_1 + 2z_2 \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

Note that $Z = Z_1 + 2Z_2$ is distributed under each hypothesis as

$$H_1: \quad Z \sim \mathcal{N}(2, 2\sigma^2)$$
$$H_0: \quad Z \sim \mathcal{N}(-2, 2\sigma^2)$$

Therefore, the probability of error is

$$P_e = \frac{1}{2}[P(Z > 0|H_0) + P(Z < 0|H_1)]$$

$\Rightarrow$

$$P_e = Q(\frac{\sqrt{2}}{\sigma})$$

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  $\sigma_0 = 1, \sigma_1 = 2, m_0 = -1, m_1 = 1$

$$p(z_i|H_j) = \frac{1}{\sqrt{2\pi}\sigma_j} \exp(-(z_i - m_j)^2/2\sigma_j^2) \;\; j = 0,1 \;\; i = 1, \cdots, N$$

The LRT is

$$\Lambda(\underline{z}) = \frac{\prod_{i=1}^{N} f(z_i|H_1)}{\prod_{i=1}^{N} f(z_i|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

Substituting appropriate values and taking log $\Rightarrow$

$$\frac{3}{8}l_2 + \frac{5}{4}l_1 \underset{H_0}{\overset{H_1}{\gtrless}} [ln(2^N\eta) - \frac{3}{8}N] = t$$

$\Rightarrow$ the decision regions in the $l_1 - l_2$ plane are governed by

$$l_2 + \frac{10}{3}l_1 - t \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

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  $V_k \sim \mathcal{N}(0, \sigma^2)$

$$\begin{aligned} H_1 : \;\; Z_k &= \;\; V_k, \quad k = 1, \cdots, K \\ H_0 : \;\; Z_k &= \;\; 1 + V_k, \quad k = 1, \cdots, K \end{aligned}$$

(a) $C_{00} = C_{11} = 0$, $C_{01} = 2$ and $C_{10} = 1$; $P_0 = 0.7$, $P_1 = 0.3$

The likelihood ratio is

$$\Lambda(\underline{z}) = \frac{(\frac{1}{\sqrt{2\pi}\sigma})^K \exp(-\underline{z}^T\underline{z}/2\sigma^2)}{(\frac{1}{\sqrt{2\pi}\sigma})^K \exp(-(\underline{z} - \underline{m})^T(\underline{z} - \underline{m})/2\sigma^2)},$$

where $\underline{z} = [z_1 \cdots z_K]^T$ and $\underline{m} = [1 \cdots 1]^T$

The LRT is

$$\Lambda(\underline{z}) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{7}{6},$$

Taking log and simplifying $\Rightarrow$

$$\bar{z} = \frac{1}{K}\sum_{i=1}^{K} z_i \mathop{\gtrless}_{H_1}^{H_0} \frac{1}{2} - \frac{1}{K}\sigma^2 ln(\frac{7}{6})$$

(b)

$$P_F = P(\bar{z} < \frac{1}{2} - \frac{1}{K}\sigma^2 ln(\frac{7}{6})|H_0)$$

$\Rightarrow$

$$P_F = \Phi(-\frac{K + 2\sigma^2 ln(\frac{7}{6})}{2\sigma\sqrt{K}}),$$

where $\Phi(\cdot)$ is the cdf of a normal random variable.

$$P_M = P(\bar{z} > \frac{1}{2} - \frac{1}{K}\sigma^2 ln(\frac{7}{6})|H_1)$$

$\Rightarrow$

$$P_M = Q(\frac{K - 2\sigma^2 ln(\frac{7}{6})}{2\sigma\sqrt{K}})$$

(c) ROC follows by plotting $P_D = 1 - P_M$ versus P_F for values of $K = 1, \sigma^2 = 2$

$$P_F = \Phi(-\frac{1 + 4ln(\frac{7}{6})}{2\sqrt{2}}),$$

and

$$P_D = 1 - P_M = 1 - Q(\frac{1 - 4ln(\frac{7}{6})}{2\sqrt{2}}) = \Phi(\frac{1 - 4ln(\frac{7}{6})}{2\sqrt{2}})$$

Therefore

$$P_D = \Phi(\Phi^{-1}(P_F) + \frac{1}{\sqrt{2}})$$

(d) Given $C_{00} = C_{11} = 0$, $C_{01} = 2$ and $C_{10} = 1$; $P_0 = 0.7$, $P_1 = 0.3$, we require

$$\bar{C}_K \leq 0.5\bar{C}_1$$

We can evaluate $\bar{C}_1$ as

$$\bar{C}_1 = 2P_1P_M + P_0P_F = 0.468$$

Therefore $0.5\bar{C}_1 = 0.234$.

We evaluate $\bar{C}_K$ as

$$\bar{C}_K = 0.6Q(\frac{K - 2\sigma^2 ln(\frac{7}{6})}{2\sigma\sqrt{K}}) + 0.7\Phi(-\frac{K + 2\sigma^2 ln(\frac{7}{6})}{2\sigma\sqrt{K}}),$$

and the value of K follows as $K \geq 7$.

When $K = 7$, $\bar{C}_K = 0.2217$.

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  We have

$$P_D = 1 - l(\frac{\gamma_1}{\sqrt{M+1}}, M)$$

$$P_F = 1 - l(\frac{\gamma_0}{\sqrt{M+1}}, M)$$

$N = 2, \sigma_1^2 = 4\sigma_0^2$.

Therefore, $M = 0$ and

$\gamma_0 = \frac{\gamma}{2\sigma_0^2}$

$\gamma_1 = \frac{\gamma}{8\sigma_0^2}$

$$l(u, 0) = \int_0^u \exp(-l)dl$$

Therefore

$$P_D = 1 - \int_0^{\gamma_1} \exp(-l)dl = \exp(-\frac{\gamma}{8\sigma_0^2})$$

$$P_F = 1 - \int_0^{\gamma_0} \exp(-l)dl = \exp(-\frac{\gamma}{2\sigma_0^2})$$

and

$$P_D = \exp(\frac{ln(P_F)}{4})$$

Some sample values are

$$P_D = P_F = 1$$

$$P_D = P_F = 0$$

$$P_D = 0.88, P_F = 0.606$$