Detection & Estimation Theory

Course No: 16:332:549

Solutions to Homework 4

3.4

\( H_1 : \quad Z_1 = V_1 + 1, \quad E[Z_1] = 1, Var(Z_1) = \sigma^2 \\
\quad Z_2 = 0.5V_2 + 0.5, \quad E[Z_2] = 0.5, Var(Z_2) = 0.25\sigma^2 \)

\( H_0 : \quad Z_1 = V_1 - 1, \quad E[Z_1] = -1, Var(Z_1) = \sigma^2 \\
\quad Z_2 = 0.5V_2 - 0.5, \quad E[Z_2] = -0.5, Var(Z_2) = 0.25\sigma^2 \)

For minimum probability of error and equally likely hypothesis, the LRT is

\[ \Lambda(\mathbf{z}) = \frac{\prod_{i=1}^{2} f(z_i|H_1)}{\prod_{i=1}^{2} f(z_i|H_0)} \overset{H_1}{\gtrsim} \overset{H_0}{\lesssim} 1 \]

Since the \( z_i \) are Gaussian and independent under each hypothesis, taking log and simplifying \( \Rightarrow \) the LRT is

\[ z_1 + 2z_2 \overset{H_1}{\gtrsim} \overset{H_0}{\lesssim} 0 \]

Note that \( Z = Z_1 + 2Z_2 \) is distributed under each hypothesis as

\( H_1 : \quad Z \sim \mathcal{N}(2, 2\sigma^2) \)
\( H_0 : \quad Z \sim \mathcal{N}(-2, 2\sigma^2) \)

Therefore, the probability of error is

\[ P_e = \frac{1}{2}[P(Z > 0|H_0) + P(Z < 0|H_1)] \]

\( \Rightarrow \)

\[ P_e = Q\left(\frac{\sqrt{2}}{\sigma}\right) \]
• 3.11

\[ \sigma_0 = 1, \sigma_1 = 2, m_0 = -1, m_1 = 1 \]

\[ p(z_i | H_j) = \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left(-\frac{(z_i - m_j)^2}{2\sigma_j^2}\right) \quad j = 0, 1 \quad i = 1, \ldots, N \]

The LRT is

\[ \Lambda(\tilde{z}) = \frac{\prod_{i=1}^{N} f(z_i | H_1)}{\prod_{i=1}^{N} f(z_i | H_0)} \overset{H_1}{\underset{H_0}{\gtrless}} \eta \]

Substituting appropriate values and taking log \( \Rightarrow \)

\[ \frac{3}{8} l_2 + \frac{5}{4} l_1 \overset{H_1}{\underset{H_0}{\gtrless}} \left[ \ln(2^N \eta) - \frac{3}{8} N \right] = t \]

\( \Rightarrow \) the decision regions in the \( l_1 - l_2 \) plane are governed by

\[ l_2 + \frac{10}{3} l_1 - t \overset{H_1}{\underset{H_0}{\gtrless}} 0 \]

• 3.12

\( V_k \sim \mathcal{N}(0, \sigma^2) \)

\[ H_1 : \quad Z_k = V_k, \quad k = 1, \ldots, K \]

\[ H_0 : \quad Z_k = 1 + V_k, \quad k = 1, \ldots, K \]

(a) \( C_{00} = C_{11} = 0, \ C_{01} = 2 \) and \( C_{10} = 1; \ P_0 = 0.7, \ P_1 = 0.3 \)

The likelihood ratio is

\[ \Lambda(\tilde{z}) = \frac{\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^K \exp\left(-\frac{\tilde{z}^T \tilde{z}}{2\sigma^2}\right)}{\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^K \exp\left(-\left(\tilde{z} - \tilde{m}\right)^T \left(\tilde{z} - \tilde{m}\right)/2\sigma^2\right)} \]

where \( \tilde{z} = [z_1 \cdots z_K]^T \) and \( \tilde{m} = [1 \cdots 1]^T \)

The LRT is

\[ \Lambda(\tilde{z}) \overset{H_1}{\underset{H_0}{\gtrless}} \frac{7}{6}, \]

Taking log and simplifying \( \Rightarrow \)
\[
\bar{z} = \frac{1}{K} \sum_{i=1}^{K} z_i \begin{cases}
> H_0 & \frac{1}{2} - \frac{1}{K} \sigma^2 \ln(\frac{7}{6}) \\
< H_1 & \frac{1}{2} - \frac{1}{K} \sigma^2 \ln(\frac{7}{6})
\end{cases}
\]

(b) \[
P_F = P(\bar{z} < \frac{1}{2} - \frac{1}{K} \sigma^2 \ln(\frac{7}{6}) | H_0)
\]
\[
\Rightarrow P_F = \Phi(-\frac{K + 2\sigma^2 \ln(\frac{7}{6})}{2\sigma\sqrt{K}}),
\]
where \(\Phi(\cdot)\) is the cdf of a normal random variable.

\[
P_M = P(\bar{z} > \frac{1}{2} - \frac{1}{K} \sigma^2 \ln(\frac{7}{6}) | H_1)
\]
\[
\Rightarrow P_M = Q\left(\frac{K - 2\sigma^2 \ln(\frac{7}{6})}{2\sigma\sqrt{K}}\right)
\]

(c) ROC follows by plotting \(P_D = 1 - P_M\) versus \(P_F\) for values of \(K = 1, \sigma^2 = 2\)
\[
P_F = \Phi\left(-\frac{1 + 4\ln(\frac{7}{6})}{2\sqrt{2}}\right),
\]
and
\[
P_D = 1 - P_M = 1 - Q\left(\frac{1 - 4\ln(\frac{7}{6})}{2\sqrt{2}}\right) = \Phi\left(\frac{1 - 4\ln(\frac{7}{6})}{2\sqrt{2}}\right)
\]
Therefore
\[
P_D = \Phi(\Phi^{-1}(P_F) + \frac{1}{\sqrt{2}})
\]

(d) Given \(C_{00} = C_{11} = 0, C_{01} = 2\) and \(C_{10} = 1; P_0 = 0.7, P_1 = 0.3\), we require
\[
\bar{C}_K \leq 0.5 \bar{C}_1
\]
We can evaluate \(\bar{C}_1\) as
\[
\bar{C}_1 = 2P_1P_M + P_0P_F = 0.468
\]
Therefore \(0.5\bar{C}_1 = 0.234\).
We evaluate \(\bar{C}_K\) as
\[
\bar{C}_K = 0.6Q\left(\frac{K - 2\sigma^2 \ln(\frac{7}{6})}{2\sigma\sqrt{K}}\right) + 0.7\Phi\left(-\frac{K + 2\sigma^2 \ln(\frac{7}{6})}{2\sigma\sqrt{K}}\right),
\]
and the value of \(K\) follows as \(K \geq 7\).
When \(K = 7\), \(\bar{C}_K = 0.2217\).
We have

\[ P_D = 1 - l\left(\frac{\gamma_1}{\sqrt{M + 1}}, M\right) \]

\[ P_F = 1 - l\left(\frac{\gamma_0}{\sqrt{M + 1}}, M\right) \]

\[ N = 2, \sigma_1^2 = 4\sigma_0^2. \]

Therefore, \( M = 0 \) and

\[ \gamma_0 = \frac{\gamma}{2\sigma_0} \]
\[ \gamma_1 = \frac{\gamma}{8\sigma_0} \]

\[ l(u, 0) = \int_0^u \exp(-l)dl \]

Therefore

\[ P_D = 1 - \int_0^{\gamma_1} \exp(-l)dl = \exp\left(-\frac{\gamma}{8\sigma_0}\right) \]

\[ P_F = 1 - \int_0^{\gamma_0} \exp(-l)dl = \exp\left(-\frac{\gamma}{2\sigma_0}\right) \]

and

\[ P_D = \exp\left(\frac{\ln(P_F)}{4}\right) \]

Some sample values are

\[ P_D = P_F = 1 \]
\[ P_D = P_F = 0 \]
\[ P_D = 0.88, P_F = 0.606 \]