Communications Engineering

Course No: 16:332:421 - (Fall 2007)

Solutions to Homework 5

1. Problem 6.2

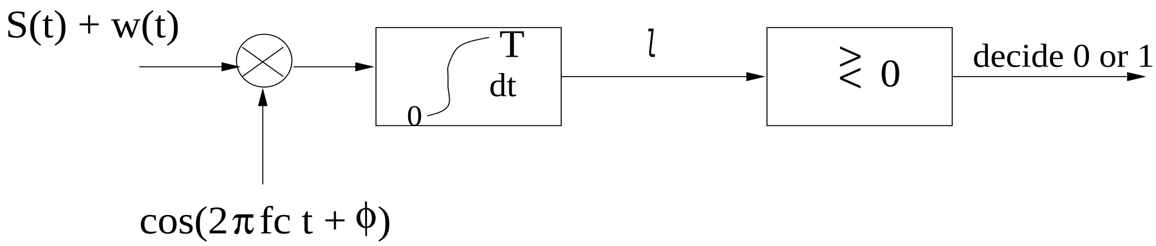

Figure 1: Receiver

The signal $s(t)$ is given as $s(t) = \pm A\cos(2\pi f_c t)$, depending on whether a 0 or 1 is transmitted.

The receiver output l is given as

$$l = \pm\frac{AT}{2}\ \cos(\phi) + \int_0^T w(t)\cos(2\pi f_c t\ +\ \phi)dt$$

Note that l is again a Gaussian random variable with mean $\pm\frac{AT}{2}\ \cos(\phi)$ and variance $N_0/2$. The average probability of error for the system is given as

$$P_e = \frac{1}{2}\text{erfc}(\sqrt{\frac{E_b}{N_0}}),$$

where E_b is the energy at the output of the correlator and the noise variance is $N_0/2$. In this case $E_b = \frac{A^2T}{2}\cos(\phi) \Rightarrow$

$$P_e = \frac{1}{2}\text{erfc}(\sqrt{\frac{A^2T}{2N_0}}\cos(\phi))$$

Note that the performance is degraded due to the unknown phase!

2. Problem 6.4

In a coherent PSK system, $s(t)$ is given as

$$s(t) = A_c\sin(2\pi f_c t) \pm A_c\sqrt{1-k^2}\cos(2\pi f_c t),$$

where the first signal is to aid in carrier synchronization and the second signal is for data transmission.

(a) The signal space diagram (see Figure 2) now has to be 2-dimensional since there are two basis functions corresponding to sin(.) and cos(.) terms. This is different from conventional BPSK which requires only one basis function!

The points in the Figure are $\mathbf{a} = [-A_c\sqrt{\frac{T_b}{2}(1-k^2)}\ \ 0]$, $\mathbf{b} = [A_c\sqrt{\frac{T_b}{2}(1-k^2)}\ \ 0]$ and $\mathbf{c} = [0\ \ A_c k\sqrt{\frac{T_b}{2}}]$

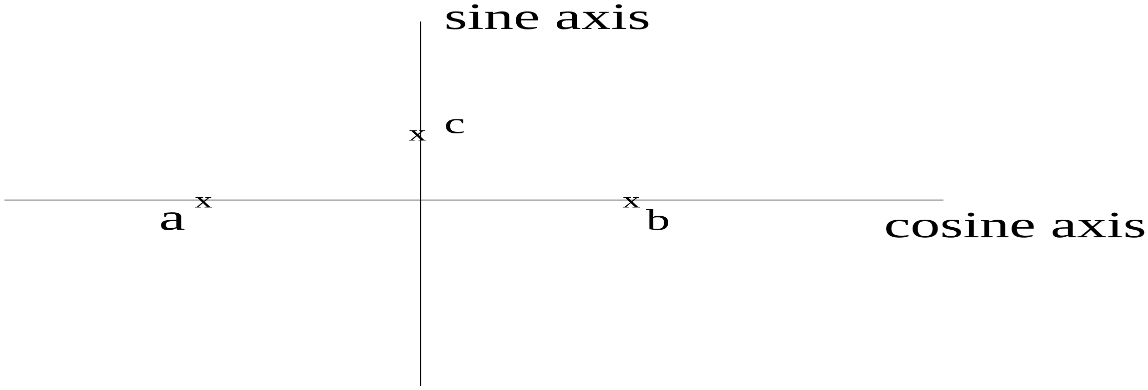

Figure 2: Signal Space

(b) The probability of error is evaluated by observing the output of the correlator l in Figure 3 and evaluating the signal energy and noise variance.

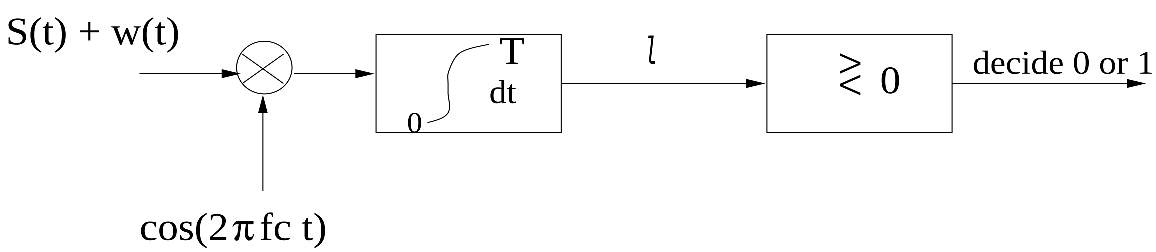

Figure 3: Receiver

It follows that

$$l = \pm\frac{A_c T}{2} + \int_0^T w(t)\cos(2\pi f_c t)dt$$

Similar to problem 8.6, we can now evaluate the probability of error as

$$P_e = \frac{1}{2}\text{erfc}(\sqrt{\frac{E_b}{N_0}(1-k^2)}),$$

where $E_b = \frac{A_c^2 T_b}{2}$

(c) If 10 % of the transmitted energy is allocated to the carrier component, then $k^2 = 0.1$. Therefore, for $P_e = 10^{-4}$, the required $\frac{E_b}{N_0}$ is given by solving

$$10^{-4} = \frac{1}{2}\text{erfc}(\sqrt{\frac{E_b}{N_0}(0.9)}),$$

which results in $\frac{E_b}{N_0} = 7.74$ which in dB is given as 8.9 dB

(d) For a conventional PSK system,

$$P_e = \frac{1}{2}\text{erfc}(\sqrt{\frac{E_b}{N_0}}),$$

In this case we get $\frac{E_b}{N_0} = 6.92$ which in dB is given as 8.4 dB
Therefore, the conventional PSK system requires 0.5 dB less in $\frac{E_b}{N_0}$ than the modified system described herein.

3. Problem 6.20

For binary FSK, the two signal vectors are

$$\mathbf{s}_1 = [\sqrt{E_b}\ \ 0]^\top$$

and

$$\mathbf{s}_2 = [0\ \ \sqrt{E_b}]^\top,$$

where E_b is the signal energy/bit. The inner product of these two signal vectors with the observation vector $\mathbf{x} = [x_1\ \ x_2]^\top$ results in

$$\mathbf{x}^\top\mathbf{s}_1 = \sqrt{E_b}x_1$$

and

$$\mathbf{x}^\top\mathbf{s}_2 = \sqrt{E_b}x_2$$

Therefore the condition (for deciding in favor of symbol 1)

$$\mathbf{x}^\top\mathbf{s}_1 > \mathbf{x}^\top\mathbf{s}_2$$

is equivalent to

$$\sqrt{E_b}x_1 > \sqrt{E_b}x_2,$$

which is equivalent to $x_1 > x_2$ which is the desired condition for making a decision in favor of symbol 1.

4. Problem 6.21 (a) and (c)

The bit duration is $T_b = \frac{1}{2.5\times10^6} = 0.4\mu sec$, and the signal amplitude is $A_c = 1\mu\, volt$.

The signal energy is $E_b = \frac{A_c^2 T_b}{2} = 2 \times 10^{-19}\ Joules$

(a) Coherent binary FSK

The probability of error is

$$P_e = \frac{1}{2}\text{erfc}(\sqrt{\frac{E_b}{2N_0}})$$

$\Rightarrow P_e = 0.85 \times 10^{-3}$

(b) Noncoherent binary FSK

The probability of error is

$$P_e = \frac{1}{2}\exp(-\frac{E_b}{2N_0})$$

$\Rightarrow P_e = 3.37 \times 10^{-3}$

5. Problem 6.22

(a) By definition, the correlation coefficient between two signals $s_1(t)$ and $s_2(t)$ is given as

$$\rho = \frac{\int_0^{T_b} s_1(t)s_2(t)dt}{[\int_0^{T_b} s_1^2(t)dt]^{1/2}\ [\int_0^{T_b} s_2^2(t)dt]^{1/2}} \quad (1)$$

Substituting

$$s_1(t) = A_c\cos(2\pi(f_c + \frac{\Delta f}{2})t),$$

and

$$s_2(t) = A_c\cos(2\pi(f_c - \frac{\Delta f}{2})t),$$

in equation (1), and using the cosine formula results in

$$\rho = \frac{1}{T_b}\int_0^{T_b}[\cos(2\pi\Delta f t) + \cos(4\pi f_c t]dt$$

$\Rightarrow$

$$\rho = \frac{1}{2\pi T_b}[\frac{\sin(2\pi\Delta f T_b)}{\Delta f} + \frac{\sin(4\pi f_c T_b)}{2f_c}$$

Since $f_c \gg \Delta f$, we can ignore the second term above $\Rightarrow$

$$\rho = \text{sinc}(2\Delta f T_b)$$

(b) For orthogonality, $\rho = 0 \Rightarrow$ the minimum value of $\Delta f = 1/2T_b$

(c) The average probability of error is

$$P_e = \frac{1}{2}\text{erfc}(\sqrt{\frac{E_b(1-\rho)}{2N_0}})$$

$\Rightarrow$ the minimum value of P_e occurs for the most negative value of ρ, which is $\rho = -0.216$ which occurs at $\Delta f = 0.7/T_b \Rightarrow$

$$P_{e,min} = \frac{1}{2}\text{erfc}(\sqrt{\frac{0.608E_b}{N_0}})$$

(d) For a conventional BPSK system,

$$P_e = \frac{1}{2}\text{erfc}(\sqrt{\frac{E_b}{N_0}})$$

Comparing with $P_{e,min}$ above shows that the increase in $\frac{E_b}{N_0}$ required for same probability of error is $1/0.608 = 1.645$ (or 2.16 dB).