Communications Engineering

Course No: 16:332:421 - (Fall 2007)

Solution to Homework 4

1. 5.3 This problem was solved in class.

2. 5.4

   (a)

   We first observe that the signals $\{s_i(t)\}$ $i = 1, 2, 3$ are linearly independent.

   The energy of signal $s_1(t)$ is given as

$$E_1 = \int_0^T s_1^2(t)dt = 4,$$

   where $T = 3$. Therefore, the first basis function is

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

   Based on the definition of the coefficients as

$$s_{ij} = \int_0^T s_i(t)\phi_j(t)dt, \qquad (1)$$

   we can find that $s_{21} = -4$.

   Based on definition of the function $g_i(t)$ as

$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij}\phi_j(t), \qquad (2)$$

   we can evaluate $g_2(t)$ as

$$g_2(t) = \begin{cases} -4, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

   The second basis function is now given as

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)dt}} = \begin{cases} -1, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

   Using equation (1), we can now compute

$$s_{31} = 3, \; s_{32} = -3$$

Using the above coefficients in (2), we get

$$g_3(t) = \begin{cases} 3, & 2 \le t \le 3 \\ 0, & \text{otherwise} \end{cases}$$

Hence, the third basis function is given as

$$\phi_3(t) = \frac{g_3(t)}{\sqrt{\int_0^T g_3^2(t)dt}} = \begin{cases} 1, & 2 \le t \le 3 \\ 0, & \text{otherwise} \end{cases}$$

(b)

We can write the signals in terms of the basis functions as

$$s_1(t) = 2\phi_1(t)$$

$$s_2(t) = -4\phi_1(t) + 4\phi_2(t)$$

$$s_3(t) = 3\phi_1(t) - 3\phi_2(t) + 3\phi_3(t)$$

3. 5.8

(a) Let the signals $s_i(t)$ and $s_k(t)$ be expanded over the interval $[0, T]$ in terms of a set of orthonormal basis functions $\{\phi_j(t)\}_{j=1}^N$, i.e.,

$$s_i(t) = \sum_{j=1}^N s_{ij}\phi_j(t),$$

where

$$s_{ij} = \int_0^T s_i(t)\phi_j(t)dt,$$

and

$$\int_0^T \phi_i(t)\phi_j(t)dt == \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}$$

It now follows that

$$\int_0^T s_i(t)s_k(t)dt = \int_0^T \sum_{j=1}^N s_{ij}\phi_j(t) \sum_{l=1}^N s_{kl}\phi_l(t)dt = \sum_{j=1}^N s_{ij}s_{kj} \int_0^T \phi_j^2(t)dt = \sum_{j=1}^N s_{ij}s_{kj},$$

which is the inner product of the vectors, i.e., $\mathbf{s_i}^\top \mathbf{s_k}$.

(b) Substituting $s_i(t) = \sum_{j=1}^N s_{ij}\phi_j(t)$ and $s_k(t) = \sum_{j=1}^N s_{kj}\phi_j(t)$ in the expression $\int_0^T [s_i(t) - s_k(t)]^2 dt$, and using the definition of the orthonormality of the basis functions results in

$$\int_0^T [s_i(t) - s_k(t)]^2 dt = \int_0^T [\sum_{j=1}^N s_{ij}\phi_j(t) - \sum_{l=1}^N s_{kj}\phi_l(t)]^2 dt = [\sum_{j=1}^N (s_{ij}^2 + s_{kj}^2 - 2s_{ij}s_{kj})]^2 = [\sum_{j=1}^N (s_{ij} - s_{kj})^2]^2,$$

which is the squared distance between the vectors $\mathbf{s_i}$ and $\mathbf{s_k}$.