Communications Engineering

Course No: 16:332:421 - (Fall 2007)

Solution to Homework 1-a

1. By definition, the mean of X is $E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$

   Since $f_X(x)$ is a PDF, it must integrate to 1 $\Rightarrow$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$$

$\Rightarrow$

$$\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \sqrt{2\pi\sigma^2}$$

   Taking derivative with respect to μ on both sides of the above equation and rearranging terms results in

$$\int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \mu$$

2. Consider a random vector

$$\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

   whose entries X_1, X_2, X_3 are Gaussian random variables. The covariance matrix C of $\underline{X}$ is given as

$$C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

   and the correlation matrix R of $\underline{X}$ is given as

$$R = \begin{bmatrix} 4 & 2\sqrt{2} & \sqrt{6} \\ 2\sqrt{2} & 8 & 2\sqrt{3} \\ \sqrt{6} & 2\sqrt{3} & 6 \end{bmatrix}$$

   (a) $E[X_1] = \sqrt{E[X_1^2] - Var[X_1]} = \sqrt{4-2} = \sqrt{2}$. The PDF of X_1 is Gaussian given as

$$f_{X_1}(x_1) = \frac{1}{\sqrt{4\pi}} e^{-\frac{(x_1-\sqrt{2})^2}{4}} \quad -\infty \leq x_1 \leq \infty$$

   (b) It can be seen that X_2 is a Guassian random variable with parameters $E[X_2] = \sqrt{E[X_2^2] - Var[X_2]} = \sqrt{8-4} = 2 \equiv \mu_2$ and $Var[X_2] = 4 \equiv \sigma_2^2$. Therefore

$$P[0 < X_2 \leq 2] = \phi(\frac{2-\mu_2}{\sigma_2}) - \phi(\frac{0-\mu_2}{\sigma_2}) = \phi(0) - \phi(-1) = \phi(0) - (1-\phi(1)) = 0.3413,$$

   where $\phi(\cdot)$ is the standard normal CDF.

(c) The correlation coefficient $\rho_{X_2,X_3} = 0$ since $Cov[X_2, X_3] = 0$ from the entries of matrix C.

(d) From the entries of matrix C, $Cov[X_1, X_3] = 0 \Rightarrow X_1$ and X_3 are uncorrelated. Since they are Gaussian, they are also independent. Therefore, the PDF

$$f_{X_1|X_3(x_1|x_3)} = f_{X_1}(x_1) = \frac{1}{\sqrt{4\pi}} e^{-\frac{(x_1-\sqrt{2})^2}{4}} \quad -\infty \leq x_1 \leq \infty$$

3. Consider the random process

$$X(t) = A\cos(\omega t + \theta)$$

where A is an exponential random variable with parameter 1 and θ is a continuous uniform random variable over the interval $[0, 2\pi]$. Assume that A and θ are independent of each other and ω is a constant carrier frequency.

(a) $E[X(t)] = E[A\cos(\omega t+\theta)]$. Since A and θ are independent $E[X(t)] = E[A]E[\cos(\omega t+\theta)]$.
Since A is an exponential random variable with parameter $1 E[A] = 1$.
$E[\cos(\omega t + \theta)] = \int_0^{2\pi} \cos(\omega t + \theta)\frac{1}{2\pi}d\theta = 0$
Therefore, $E[X(t)] = 0$.

(b) To find the autocorrelation function $R_X(t, \tau)$, we need to evaluate $R_X(t, \tau) = E[X(t)X(t+\tau)]$

$$R_X(t, \tau) = E[X(t)X(t + \tau)] = E[A^2 \cos(\omega t + \theta) \cos(\omega t + \omega\tau + \theta]$$

Using independence of A and θ

$$R_X(t, \tau) = E[A^2]E[\cos(\omega t + \theta) \cos(\omega t + \omega\tau + \theta]$$

Using $\cos(A)\cos(B) = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$

$$E[\cos(\omega t + \theta) \cos(\omega t + \omega\tau + \theta] = \frac{1}{2}E[\cos(\omega\tau) + \cos(2\omega t + \omega\tau + 2\theta)] = \frac{1}{2}\cos(\omega\tau)$$

Since A is exponential random variable, $E[A^2] = Var[A] + (E[A])^2 = 2$
Therefore

$$R_X(t, \tau) = \cos(\omega_o\tau)$$

(c) Since $E[X(t)] = 0$ and $R_X(t, \tau)$ is a function of τ only, $X(t)$ wide-sense stationary!