Communications Engineering

Course No: 16:332:421 - (Fall 2007)

Solution to Homework 1

1. (a) Integral of pdf has to be 1 so $C = \lambda$.
   (b) $E(X) = 1/\lambda$ (integrate by parts)
   (c) $P(A) = \int_a^\infty f_X(x)dx = e^{-a\lambda}$. $f_{X|A} = \lambda e^{-\lambda(x-A)}u(x-a)$. Yes it does depend on a (the step function). However, the form does not depend on a... it's still an exponential distribution with the same parameters only shifted to the right by a.

2. (a) $E(Y) = 2E(X) - 6 = 0$, $\sigma_Y^2 = 4\sigma_X^2 = 16$
   (b) Mean and variance completely define distribution of Y since it's Gaussian so, $f_Y(y) = N(0, 16)$.
   (c) Hell no! Just show that $E(XY) \neq E(X)E(Y)$.

3. (a) Derived distribution problem with X uniform on $(0, 4)$. The probability that $Y \leq y$ is the probability that $X^2 \leq y$ which is

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \int_0^{\sqrt{y}} \frac{1}{4} dx & 0 \leq y \leq 16 \\ 1 & y > 16 \end{cases}$$

   which reduces to

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{\sqrt{y}}{4} & 0 \leq y \leq 16 \\ 1 & y > 16 \end{cases}$$

   (b) Take the derivative:

$$f_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{8\sqrt{y}} & 0 \leq y \leq 16 \\ 0 & y > 16 \end{cases}$$

4. Let the event $\mathcal{A} = \{$ k heads in a specific order$\}$

   Let the event $\mathcal{C}_i = \{$The event of selecting the i^{th} coin$\}$

   Then $P(\mathcal{C}_i) = 1/m$. Further, $P(\mathcal{A} \mid \mathcal{C}_i) = p_i^k(1 - p_i)^{n-k}$.

   Then $q_r = P(\mathcal{C}_r \mid \mathcal{A})$ follows by Bayes rule and the total probability theorem.

5. (a) The characteristic function of $X_{i,n}$ is

$M_{X_{i,n}}(u) = E[\exp(juX_{i,n})] = (1-\lambda/n) + \exp(ju)\lambda/n = 1 + \frac{\lambda}{n}[\exp(ju) - 1]$

Therefore,

$M_{Y_n}(u) = \{1 + \frac{\lambda}{n}[\exp(ju) - 1]\}^n$

(b) $\lim_{n\to\infty} M_{Y_n}(u) = \exp(\lambda(\exp(ju) - 1))$,

which implies $\lim_{n\to\infty} Y_n$ is a Poisson random variable with mean and variance λ. (This follows from the fact that the mapping from a characteristic function to a distribution is 1:1)

6. (a) To show $R_X(\tau)$ is an even function

   By definition,

$$R_X(\tau) = E[X(t)X(t-\tau)] = E[X(t-\tau)X(t)] = R_X(-\tau)$$

   (b) To show $\mid R_X(\tau) \mid \leq \ R_X(0)$

   Note that

$$E[(X(t) \pm X(t-\tau))^2] \geq 0.$$

   Expanding the above result results in

$$E[(X(t)^2] + E[X(t-\tau)^2] \pm 2E[X(t)X(t-\tau)] \geq 0$$

   $\Rightarrow$

$$\pm R_X(\tau) \leq R_X(0) \Rightarrow \mid R_X(\tau) \mid \leq \ R_X(0)$$

7. $E[Y_t] = \mu D$

By definition $\phi_Y(t,s) = E[(X_{t+D} - X_t)(X_{s+D} - X_s)]$. Therefore,

$\phi_Y(t,s) = \sigma^2[\min(t+D, s+D) - \min(t+D, s) - \min(t, s+D) + \min(t,s)] + \mu^2[(t+D)(s+D) - (t+D)s - t(s+D) + ts]$

There are 2 cases:

Case 1: $\mid t-s \mid \leq D$

$\phi_Y(t,s) = \sigma^2[D - \mid t-s \mid] + \mu^2 D^2$

Case 2: $\mid t-s \mid > D$

$\phi_Y(t,s) = \mu^2 D^2$

From Case 1 and Case 2, it is clear that Y_t is wide-sense-stationary. Further, since Y_t is Gaussian, it is strictly stationary!