

Decode-and-Forward with Offset Encoding for Multiaccess Relay Channels

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Abstract—An offset encoding technique is presented that improves sliding-window decoding with decode-and-forward for K -user multiaccess relay channels. The technique offsets user transmissions by one block per user and achieves the same rate region as backward decoding but with a smaller delay.

Index Terms—Multiaccess communication, relaying, encoding, cooperative systems.

I. INTRODUCTION

The multiaccess relay channel (MARC) is a network where several users (source nodes) communicate with a single destination in the presence of a relay [1]. Several coding strategies for the relay channel [2], [3] extend readily to the MARC [4], [5]. For example, the strategy of [3, theorem 1], now often called *decode-and-forward* (DF), has a relay that decodes user messages before forwarding them to the destination [4], [5]. Similarly, the strategy in [3, theorem 6], now often called *compress-and-forward* (CF), has the relay quantize its output symbols and transmit the resulting quantized bits to the destination [5].

For the classic relay channel, several encoding and decoding techniques achieve the DF rate in [3, theorem 1] (see [4, Sec. I]): *irregular* block Markov encoding (different size codebooks at the source and relay) and *successive* decoding [3, theorem 1], *regular* block Markov encoding (same size codebooks at the source and relay) and *sliding-window* decoding [6], regular block Markov encoding and *backward decoding* [7]. These methods have all been generalized to multiple relay networks [4], [8]–[11]. For the MARC, however, the different DF encoding and decoding methods do not always yield the same rate region. For example, we show that backward decoding can give larger rates than sliding-window decoding. However, we prefer to use sliding-window decoding because it incurs much less delay. We hence develop an *offset encoding* technique for sliding-window decoding that recovers the backward decoding rate region. The delay of our method is

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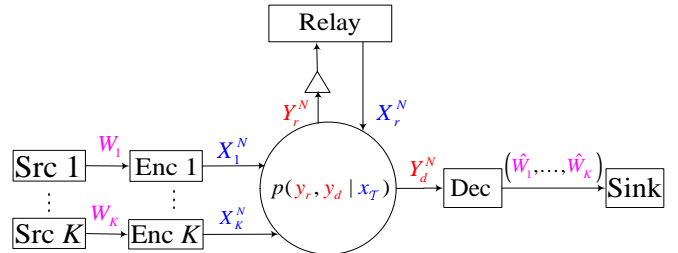


Fig. 1. A K -user multiaccess relay channel

only $K + 1$ transmit blocks per message rather than the $B + 1$ blocks required by backward decoding over $B + 1$ blocks. Note that the number of users, K , is usually much smaller than B .

This correspondence is organized as follows. In Section II we present the MARC model and summarize the DF random code construction of [4, Appendix A]. In Section III, we review the backward decoding rate region and compute the window decoding rate region. The latter region is in general smaller than the former. In Section IV, we describe offset encoding and compute its rate region when combined with sliding-window decoding. An error probability analysis is presented in the Appendix.

II. PRELIMINARIES

A. Model

The K -user MARC has K source nodes, one relay node, and one destination node (see Fig. 1). The messages W_k , $k = 1, 2, \dots, K$, are statistically independent and W_k takes on values uniformly in the set $\{1, 2, \dots, M_k\}$. The channel is used N times so that the rate of W_k is $R_{W_k} = B_{W_k}/N$ bits per channel use where $B_{W_k} = \log_2 M_k$ bits. The channel input $X_{k,i}$ from source k at time i , $i = 1, 2, \dots, N$, is a function of W_k , while the relay's channel input $X_{r,i}$ is a causal function of its received signals $Y_r^{i-1} = (Y_{r,1}, Y_{r,2}, \dots, Y_{r,i-1})$. The destination uses the N channel outputs Y_d^N to decode the K messages as $(\hat{W}_1, \hat{W}_2, \dots, \hat{W}_K)$. We write $\mathcal{K} = \{1, 2, \dots, K\}$, $\mathcal{T} = \mathcal{K} \cup \{r\}$, $X_{\mathcal{S}} = \{X_k : k \in \mathcal{S}\}$ for all $\mathcal{S} \subseteq \mathcal{K}$, and \mathcal{S}^c to denote the complement of \mathcal{S} in \mathcal{K} . The channel is time-invariant and memoryless with the conditional probability distribution

$$p(y_r, y_d | x_{\mathcal{T}}). \quad (1)$$

| | Block 1 | Block 2 | Block B | Block B+1 |
|--------|-------------------------------|-------------------------------------|-----------------------------------------|-------------------------------------|
| User 1 | $\underline{x}_1(w_{1,1}, 1)$ | $\underline{x}_1(w_{1,2}, w_{1,1})$ | $\underline{x}_1(w_{1,B}, w_{1,B-1})$ | $\underline{x}_1(1, w_{1,B})$ |
| | $\underline{v}_1(1)$ | $\underline{v}_1(w_{1,1})$ | $\underline{v}_1(w_{1,B-1})$ | $\underline{v}_1(w_{1,B})$ |
| User 2 | $\underline{x}_2(w_{2,1}, 1)$ | $\underline{x}_2(w_{2,2}, w_{2,1})$ | $\underline{x}_2(w_{2,B}, w_{2,B-1})$ | $\underline{x}_2(1, w_{2,B})$ |
| | $\underline{v}_2(1)$ | $\underline{v}_2(w_{2,1})$ | $\underline{v}_2(w_{2,B-1})$ | $\underline{v}_2(w_{2,B})$ |
| Relay | $\underline{x}_r(1, 1)$ | $\underline{x}_r(w_{1,1}, w_{2,1})$ | $\underline{x}_r(w_{1,B-1}, w_{2,B-1})$ | $\underline{x}_r(w_{1,B}, w_{2,B})$ |

Fig. 2. Regular encoding for a two-user MARC.

The capacity region \mathcal{C}_{MARC} of a K -user MARC is the closure of the set of rate tuples $(R_{W_1}, R_{W_2}, \dots, R_{W_K})$ such that the destination can, for sufficiently large N , decode the K source messages with an arbitrarily small positive error probability [1]. We write $R_S = \sum_{k \in S} R_k$, $[m, n] = \{m, m+1, \dots, n\}$, $S \setminus \mathcal{A}$ for the set of elements in S but not in \mathcal{A} , and $|S|$ for the cardinality of S .

B. Random Code Construction

A DF code construction is presented in [4, Appendix A] and we review it below. We remark that this construction is common to all the decoding methods considered in the sequel.

Random Code Construction:

Consider the probability distribution

$$\left(\prod_{k=1}^K p(v_k) p(x_k | v_k) \right) \cdot p(x_r | v_K) \quad (2)$$

where the V_k , $k = 1, 2, \dots, K$, are statistically independent auxiliary random variables. Each V_k is used to generate a codebook with length- n codewords $\underline{v}_k(s_k)$, $s_k = 1, 2, \dots, 2^{nR_k}$ (see Fig. 2). Every codeword $\underline{v}_k(s_k)$ is next used to generate a codebook with length- n codewords $\underline{x}_k(w_k, s_k)$, $w_k = 1, 2, \dots, 2^{nR_k}$. Finally, one length- n relay codeword $\underline{x}_r(s_1, s_2, \dots, s_K)$ is generated for each choice of $\underline{v}_1(s_1)$, $\underline{v}_2(s_2)$, \dots , $\underline{v}_K(s_K)$.

The above procedure is repeated $B + 1$ times and the b^{th} codebook is used in block b , $b = 1, 2, \dots, B + 1$. The encoding procedure of [4, Appendix A] proceeds as follows. We change this procedure in Sec. IV.

Regular Block Markov Encoding:

Source k sends the messages $(w_{k,1}, w_{k,2}, \dots, w_{k,B})$, each having nR_k bits, over $B + 1$ consecutive blocks as shown in Fig. 2. The relay sends the codeword $\underline{x}_r(s_{1,b}, s_{2,b}, \dots, s_{K,b})$ in block b . The relay sets $s_{k,1} = 1$ while the sources set $w_{k,B+1} = 1$. We thus have $N = n(B + 1)$ and $BW_k = nR_k B$ such that the overall rate of user k is $R_{W_k} = R_k \cdot B / (B + 1)$ which approaches R_k for large B .

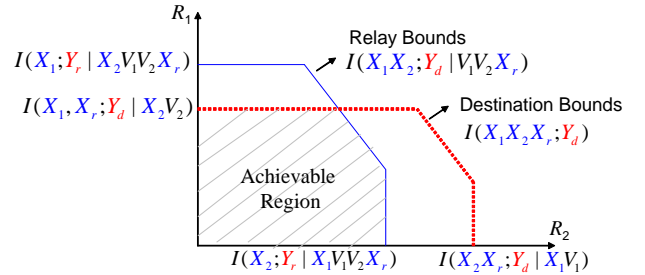


Fig. 3. Rate region achieved by DF for a two-user MARC.

III. DECODE-AND-FORWARD

A. Backward Decoding

Consider a 2-user MARC where the sources and the relay use the block Markov encoding method described above. The messages are decoded reliably by the relay if

$$R_1 \leq I(X_1; Y_r | X_2 V_1 V_2 X_r) \quad (3)$$

$$R_2 \leq I(X_2; Y_r | X_1 V_1 V_2 X_r) \quad (4)$$

$$R_1 + R_2 \leq I(X_1 X_2; Y_r | V_1 V_2 X_r). \quad (5)$$

The destination starts decoding after receiving all $B + 1$ blocks. The destination rate bounds are (see [4, Sec. IV-D])

$$R_1 \leq I(X_1 X_r; Y_d | X_2 V_2) \quad (6)$$

$$R_2 \leq I(X_2 X_r; Y_d | X_1 V_1) \quad (7)$$

$$R_1 + R_2 \leq I(X_1 X_2 X_r; Y_d) \quad (8)$$

An example of the MARC region defined by (3)-(8) is shown in Fig. 3. For a K -user MARC, the bounds (3)-(8) generalize as follows.

Proposition 1: The rate region for DF is the union of the set of rate tuples (R_1, R_2, \dots, R_K) that satisfy, for all $S \subseteq K$,

$$R_S \leq \min \left(\begin{array}{l} I(X_S; Y_r | X_{S^c} V_K X_r U), \\ I(X_S X_r; Y_d | X_{S^c} V_{S^c} U) \end{array} \right) \quad (9)$$

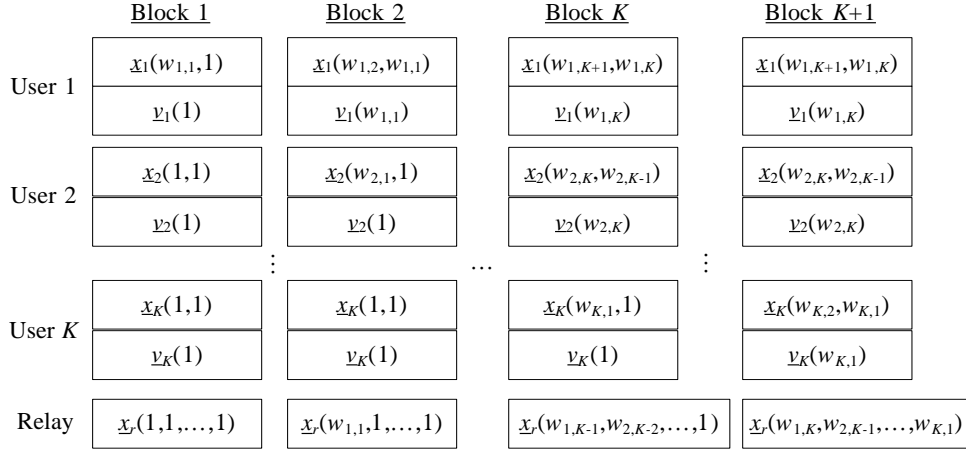


Fig. 4. Offset encoding for a K -user MARC.

where the union is over all distributions that factor as

$$p(u) \cdot \left(\prod_{k=1}^K p(v_k|u)p(x_k|v_k, u) \right) \cdot p(x_r|v_{\mathcal{K}}, u) \cdot p(y_r, y_d|x_{\mathcal{T}}). \quad (10)$$

Proof: See [4, Appendix A]. ■

Remark 1: The time-sharing random variable U convexifies the rate region of Proposition 1. For simplicity, we will develop the theory below only for a constant U .

B. Sliding-Window Decoding

Suppose the destination uses sliding-window decoding, i.e., the destination decodes the message pair $(w_{1,b}, w_{2,b})$ transmitted in block b by using the signals $\underline{y}_{d,b}$ and $\underline{y}_{d,b+1}$ received in blocks b and $b+1$, respectively. It turns out that the bounds in (6)-(8) are replaced by

$$R_1 \leq I(X_1; Y_d | X_2 V_1 V_2 X_r) + I(V_1 X_r; Y_d | V_2) \quad (11)$$

$$R_2 \leq I(X_2; Y_d | X_1 V_1 V_2 X_r) + I(V_2 X_r; Y_d | V_1) \quad (12)$$

$$R_1 + R_2 \leq I(X_1 X_2 X_r; Y_d). \quad (13)$$

The same bounds result if the sliding window length is increased, unless the window includes block $B+1$, of course. The term $I(X_1; Y_d | X_2 V_1 V_2 X_r)$ in (11) results from $\underline{y}_{d,b}$ while the term $I(V_1 X_r; Y_d | V_2)$ is due to $\underline{y}_{d,b+1}$. The bounds (12) and (13) are obtained similarly. Observe that the message pair $(w_{1,b+1}, w_{2,b+1})$ in block $(b+1)$ is unknown while decoding $(w_{1,b}, w_{2,b})$. The error analysis used to obtain (11)-(13) is similar to that presented in the Appendix and is hence omitted.

We next compare (6)-(8) and (11)-(13). Obviously, the bounds (8) and (13) are the same. We can write (6) as

$$R_1 \leq I(X_1; Y_d | X_2 V_1 V_2 X_r) + I(V_1 X_r; Y_d | V_2 X_2). \quad (14)$$

Expanding the second term in (14), we have

$$I(V_1 X_r; Y_d | V_2 X_2) \geq H(V_1 X_r | V_2) - H(V_1 X_r | Y_d V_2) \quad (15)$$

$$= I(V_1 X_r; Y_d | V_2) \quad (16)$$

with equality if and only if

$$I(V_1 X_r; X_2 | V_2 Y_d) = 0. \quad (17)$$

The inequality (15) follows from the Markov relationships

$X_k - V_k - X_r$, $k = 1, 2$, and because conditioning cannot increase entropy. Backward decoding is thus at least as good as sliding-window decoding.

For example, consider a 2-user MARC with binary inputs X_1 , X_2 , and X_r . Suppose the relay output is $Y_r = 2X_1 + X_2$ where we use integer arithmetic so that $Y_r \in \{0, 1, 2, 3\}$. The destination output is

$$Y_d = \begin{cases} 0 & \text{if } (X_1 + X_2 + X_r) \leq 1 \\ 1 & \text{otherwise.} \end{cases} \quad (18)$$

Suppose V_1 and V_2 are binary. The bounds (5) and (8) are both maximized by choosing a uniform joint distribution $p(x_k, v_k)$, $k = 1, 2$, and uniform $p(x_r | v_1, v_2)$ for each choice of (v_1, v_2) . The resulting relay bounds in (3) and (5) are

$$\begin{aligned} R_1 &\leq 1 \\ R_1 + R_2 &\leq 2 \end{aligned} \quad (19)$$

while the destination bounds in (6) and (8) with backward decoding are

$$\begin{aligned} R_1 &\leq 0.8113 \\ R_1 + R_2 &\leq 1 \end{aligned} \quad (20)$$

However, the bound (11) on R_1 with sliding-window decoding is

$$R_1 \leq 0.6887. \quad (21)$$

By symmetry, the bounds on R_2 are the same as those on R_1 . Further, from the dependence of Y_d on X_2 in (18), we observe that (17) is not possible for all input distributions where both users transmit non-zero rates. Sliding-window decoding thus cannot achieve all backward decoding rate pairs.

For $K > 2$, the bounds (11)-(13) generalize to

$$R_S \leq I(X_S; Y_d | X_{S^c} V_S X_r) + I(V_S X_r; Y_d | V_{S^c}) \quad (22)$$

One can show that the bounds in (22) are in general more restrictive than those in (9) for all $S \subset \mathcal{K}$.

IV. OFFSET ENCODING

To improve window decoding, we offset the message transmissions from the K sources by one block per source. Let π denote a permutation (order) of the source indices. We let

source $\pi(i)$ start transmitting in block i . Thus, $w_{\pi(i),b} = 1$ for all $b < i$ and $b \in [B + i, B + K]$. The resulting message-to-codeword mapping for a K -user MARC with offset order $\pi = (1, 2, \dots, K)$ is shown in Fig. 4.

The relay decodes at the end of each block as before. We thus require

$$R_{\mathcal{S}} \leq I(X_{\mathcal{S}}; Y_r | X_{\mathcal{S}^c} V_{\mathcal{K}} X_r) \quad (23)$$

for all $\mathcal{S} \subseteq \mathcal{K}$ as in (9). In block b , the relay sends the codeword $\underline{x}_r(s_{\mathcal{K},b})$ as shown in Fig. 4. The destination uses a sliding window of length $K + 1$ to jointly decode the messages $w_{k,b}$, $k = 1, 2, \dots, K$, from the K sources. We summarize the resulting rate bounds below. A detailed analysis of the error probability is presented in the Appendix.

A. Two Users

Consider $K = 2$ and suppose the offset order is $\pi = (1, 2)$. The message pair $(w_{1,b}, w_{2,b})$ is decoded reliably using blocks $b, b + 1$, and $b + 2$ if

$$\begin{aligned} R_1 &\leq I(X_1; Y_d | X_2 V_1 V_2 X_r) + I(V_1 X_r; Y_d | X_2 V_2) \\ &= I(X_1 X_r; Y_d | X_2 V_2) \end{aligned} \quad (24)$$

$$R_2 \leq I(X_2; Y_d | X_1 V_1 V_2 X_r) + I(V_2; Y_d) \quad (25)$$

$$\begin{aligned} R_1 + R_2 &\leq I(V_2; Y_d) + I(X_2 V_1 X_r; Y_d | V_2) \\ &\quad + I(X_1; Y_d | X_2 V_{\mathcal{K}} X_r) \\ &= I(X_1 X_2 X_r; Y_d). \end{aligned} \quad (26)$$

Comparing (24)-(26) with (6)-(8) and using the Markov chain $X_k - V_k - X_r$, we can verify that bound in (25) is at most as large as that achieved by backward decoding in (7). However, we show that the bound on R_2 in (25) suffices to obtain a corner point where R_1 achieves its maximum single-user rate. Observe that the bound (26) can also be written as

$$R_1 + R_2 \leq I(V_2 X_2; Y_d) + I(X_1 X_r; Y_d | X_2 V_2). \quad (27)$$

Consider the corner point labeled “ $\pi = (1, 2)$ ” in Fig. 5. This corner point is achievable provided that $I(V_2 X_2; Y_d)$ is less than the right hand side in (25). But we have

$$\begin{aligned} I(V_2 X_2; Y_d) &= I(V_2; Y_d) + I(X_2; Y_d | V_2) \end{aligned} \quad (28)$$

$$\leq I(V_2; Y_d) + H(X_2 | V_2) - H(X_2 | Y_d X_1 V_1 V_2) \quad (29)$$

$$= I(V_2; Y_d) + I(X_2; Y_d | X_1 V_1 V_2 X_r) \quad (30)$$

where (30) follows from the Markov relationship

$$X_2 - V_2 - [X_r, V_1, X_1]. \quad (31)$$

Thus, we obtain the corner point under consideration. For the offset order $\pi = (2, 1)$, we similarly obtain the corner point labeled “ $\pi = (2, 1)$ ” in Fig. 5. These corner points are the same as those obtained via backward decoding. All other non-corner points can be achieved by time-sharing. Thus, we obtain the same rate region as backward decoding.

B. K Users

We extend the the above analysis to $K > 2$. Without loss of generality, suppose the offset order is $\pi = (1, 2, \dots, K)$. The

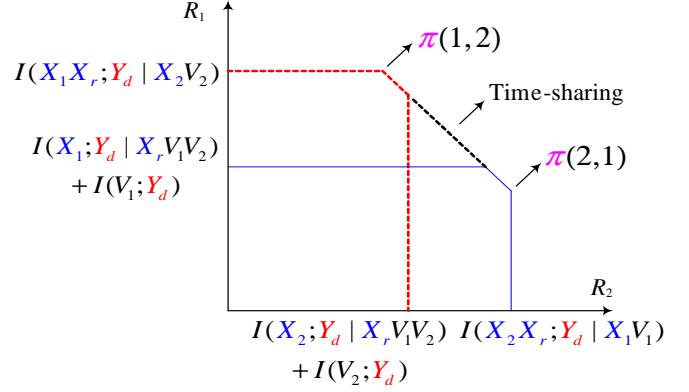


Fig. 5. Rate region with sliding-window decoding and offset encoding.

destination decodes $w_{\mathcal{K},b}$ by using the $K + 1$ output blocks $\underline{y}_{d,b}, \underline{y}_{d,b+1}, \dots, \underline{y}_{d,b+K}$.

Upon inspection of Fig. 4, we find that block $b + 1$ must be treated specially. We define

$$\tilde{I}_2 = \sum_{k \geq 3} I(X_{k\mathcal{S}} V_{(k-1)\mathcal{S}}; Y_d | X_{[k+1,K]} V_{[k,K]} X_{k\mathcal{S}^c} V_{(k-1)\mathcal{S}^c}) \quad (32)$$

where

$$k_{\mathcal{A}} = \begin{cases} \{k\}, & k \in \mathcal{A} \\ \emptyset, & \text{else.} \end{cases} \quad (33)$$

We have the following rate bounds for all $\mathcal{S} \subseteq \mathcal{K}$:

$$R_{\mathcal{S}} \leq \begin{cases} \tilde{I}_2 & \text{if } 1 \notin \mathcal{S}, 2 \notin \mathcal{S} \\ \tilde{I}_2 + I(X_2; Y_d | X_r X_{[3,K]} V_{\mathcal{K}}) & \text{if } 1 \notin \mathcal{S}, 2 \in \mathcal{S} \\ \tilde{I}_2 + I(X_1 X_r; Y_d | X_{[2,K]} V_{[2,K]}) & \text{if } 1 \in \mathcal{S}, 2 \notin \mathcal{S} \\ \tilde{I}_2 + I(X_1 X_2 X_r; Y_d | X_{[3,K]} V_{[2,K]}) & \text{if } 1 \in \mathcal{S}, 2 \in \mathcal{S} \end{cases} \quad (34)$$

For singleton sets, the bounds in (34) simplify as

$$R_1 \leq I(X_1 X_r; Y_d | X_{[2,K]} V_{[2,K]}) \quad (35)$$

$$R_k \leq \begin{cases} \left(\begin{array}{l} I(X_2; Y_d | X_r X_{[3,K]} V_{\mathcal{K}}) \\ + I(V_2; Y_d | X_{[3,K]} V_{[3,K]}) \end{array} \right) & k = 2 \\ \left(\begin{array}{l} I(X_k; Y_d | X_{[k+1,K]} V_{[k-1,K]}) \\ + I(V_k; Y_d | X_{[k+1,K]} V_{[k+1,K]}) \end{array} \right) & k \geq 3 \end{cases} \quad (36)$$

Let $I_{k,\max}$ denote the mutual information expression bounding R_k in (35) and (36) for all k . We show that the bound (34) for any \mathcal{S} of the form $[1, k]$, $k \geq 2$, results in a corner point such that source 1 achieves $I_{1,\max}$. We refer to any such corner point as that *associated* with source 1. For all sets \mathcal{S} of the form $[1, m]$, $m \in [1, K]$, the bounds in (34) simplify to the backward decoding bound

$$R_{\mathcal{S}} \leq I(X_{\mathcal{S}} X_r; Y_d | X_{\mathcal{S}^c} V_{\mathcal{S}^c}) \quad (37)$$

$$\begin{aligned} &= I(X_1 X_r; Y_d | X_{[2,K]} V_{[2,K]}) \\ &\quad + \sum_{k=2}^m I(X_k V_k; Y_d | X_{[k+1,K]} V_{[k+1,K]}). \end{aligned} \quad (38)$$

The expression (38) suggests that the sources in \mathcal{S} can be successively decoded to obtain a corner point associated with

source 1. We define

$$I_k = I(X_k V_k; Y_d | X_{[k+1, K]} V_{[k+1, K]}) \quad (39)$$

for all $k \geq 2$. Expanding I_k , we have

$$\begin{aligned} I_k &= I(V_k; Y_d | X_{[k+1, K]} V_{[k+1, K]}) \\ &\quad + H(X_k | V_k) - H(X_k | Y_d, X_{[k+1, K]} V_{[k, K]}) \quad (40) \\ &\leq I_{k, \max} \quad (41) \end{aligned}$$

where (41) follows from using the Markov chains $X_k - V_k - X_r$ for all k , and because the sources are independent. Thus, we obtain a corner point associated with source 1. Observe that this corner point is the same as that achieved via backward decoding since the bound on R_S in (37) is the same as the R_S bound at the destination in (9) for a constant U . The remaining corner points associated with source 1 are then obtained by fixing source 1 as the first source in π while choosing all possible permutations of the remaining $K - 1$ sources.

For example, consider a 3-user MARC. The destination rate region in Proposition 1 has two corner points associated with each source. For both of these corner points, we decode source 1 after all other sources. From (37)-(41), we see that the offset order $\pi = (1, 2, 3)$ gives the corner point where source 3 is decoded first. One can similarly verify that the offset order $\pi = (1, 3, 2)$ gives the corner point where source 2 is decoded first.

For a K -user MARC, all $(K - 1)!$ corner points associated with the k^{th} source, $k \in [1, K]$, are obtained by using those offset orders where source k transmits first. Thus, we obtain all the $K!$ corner points in the K -dimensional rate polytope achieved at the destination in (9). The non-corner points are achieved by time-sharing.

V. CONCLUSIONS

We presented an offset encoding technique for DF that improves the rate region of sliding-window decoding. The technique achieves the same rate region as backward decoding but avoids the excessive delay associated with backward decoding. Offset encoding will clearly generalize to other multi-terminal problems [12]–[14].

APPENDIX ERROR PROBABILITY ANALYSIS

We derive the rate bounds for the DF strategy employing offset encoding and window decoding. Without loss of generality, we develop the bounds at the destination for a 2-user MARC with the offset order $\pi = (1, 2)$. The random code construction is described in Section II-B.

Encoding: Consider block b .

- 1) The source k transmits $\underline{x}_k(w_{k,b}, w_{k,b-1})$.
- 2) The relay transmits $\underline{x}_r(s_{1,b}, s_{2,b}, \dots, s_{K,b})$ where $s_{K,b}$ is the set of messages decoded at the relay in block $(b - 1)$.

Decoding:

- 1) *At the relay:* The relay decodes the messages $w_{S,k}$ in block k , $k = 1, 2, \dots, B + K$, by using $\underline{y}_{r,k}$ and by assuming that its message estimates in the previous blocks are correct. This decoding problem is equivalent to that

of a multiaccess channel (MAC) with side information $V_{\mathcal{K}}$ and X_r (see [15, pg. 403]) where reliable decoding is achieved when, for all $S \subseteq \mathcal{K}$, we have

$$R_S < I(X_S; Y_r | X_{S^c} X_r V_{\mathcal{K}}). \quad (42)$$

- 2) *At the destination:* Assuming that no errors were made up to block b , the destination decodes the message set $w_{\mathcal{K},b} = (w_{1,b}, w_{2,b}, \dots, w_{K,b})$ using $\underline{y}_{d,b}, \underline{y}_{d,b+1}, \dots, \underline{y}_{d,b+K}$ (see [10, Sec. IV]). For $K = 2$, three kinds of errors occur in decoding $(w_{1,b}, w_{2,b})$. The first type of error has the decoder estimating $\hat{w}_{2,b} = w_{2,b}$ but $\hat{w}_{1,b} \neq w_{1,b}$. Invoking the usual typicality arguments, we require

- $(\underline{v}_1(w_{1,b-1}), \underline{v}_2(w_{2,b-2}), \underline{x}_1(\hat{w}_{1,b}, w_{1,b-1}), \underline{x}_2(w_{2,b-1}, w_{2,b-2}), \underline{x}_r(w_{1,b-1}, w_{2,b-2}), \underline{y}_{d,b}) \in T_\epsilon(V_{\mathcal{K}}, X_{\mathcal{K}}, X_r, Y_d)$ and
- $(\underline{v}_1(\hat{w}_{1,b}), \underline{x}_r(\hat{w}_{1,b}, w_{2,b-1}), \underline{v}_2(w_{2,b-1}), \underline{x}_2(w_{2,b}, w_{2,b-1}), \underline{y}_{d,b+1}) \in T_\epsilon(V_{\mathcal{K}}, X_2, X_r, Y_d)$.

Since the codebooks in consecutive blocks are independent, the above two errors are independent. Using [4, Lemma 1], the error probability of the intersection of these events is upper bounded by

$$2^{n(R_1 - I(X_1; Y_d | X_2 V_{\mathcal{K}} X_r) - I(X_1 X_r; Y_d | X_2 V_2) + 12\epsilon)}. \quad (43)$$

Thus, $w_{1,b}$ can be decoded reliably if

$$R_1 < I(X_1, X_r; Y_d | X_2 V_2). \quad (44)$$

For the case where $\hat{w}_{1,b} = w_{1,b}$ but $\hat{w}_{2,b} \neq w_{2,b}$ we require

- $(\underline{x}_2(\hat{w}_{2,b}, w_{2,b-1}), \underline{v}_1(w_{1,b}), \underline{v}_2(w_{2,b-1}), \underline{x}_r(w_{1,b}, w_{2,b-1}), \underline{y}_{d,b+1}) \in T_\epsilon(V_{\mathcal{K}}, X_2, X_r, Y_d)$
- $(\underline{v}_2(\hat{w}_{2,b}), \underline{y}_{d,b+2}) \in T_\epsilon(V_2, Y_d)$

The probability of intersection of these events is upper bounded by

$$2^{n(R_2 - I(X_2; Y_d | V_{\mathcal{K}} X_r) - I(V_2; Y_d) + 9\epsilon)}. \quad (45)$$

We thus require

$$R_2 < I(X_2; Y_d | X_1 V_1 V_2 X_r) + I(V_2; Y_d). \quad (46)$$

Finally, we have $\hat{w}_{1,b} \neq w_{1,b}$ and $\hat{w}_{1,b} \neq w_{1,b}$ if

- $(\underline{v}_1(w_{1,b-1}), \underline{v}_2(w_{2,b-2}), \underline{x}_1(\hat{w}_{1,b}, w_{1,b-1}), \underline{x}_2(w_{2,b-1}, w_{2,b-2}), \underline{x}_r(w_{1,b-1}, w_{2,b-2}), \underline{y}_{d,b}) \in T_\epsilon(V_{\mathcal{K}}, X_{\mathcal{K}}, X_r, Y_d)$
- $(\underline{v}_1(\hat{w}_{1,b}), \underline{v}_2(w_{2,b-2}), \underline{x}_2(\hat{w}_{2,b}, w_{2,b-2}), \underline{x}_r(\hat{w}_{1,b}, w_{2,b-2}), \underline{y}_{d,b+1}) \in T_\epsilon(V_{\mathcal{K}}, X_2, X_r, Y_d)$
- $(\underline{v}_2(\hat{w}_{2,b}), \underline{y}_{d,b+2}) \in T_\epsilon(V_2, Y_d)$

The resulting error probability is upper bounded as

$$2^{n(R_{\mathcal{K}} - I(X_{\mathcal{K}} X_r; Y_d) + 15\epsilon)}. \quad (47)$$

For reliable decoding, we thus require

$$R_{\mathcal{K}} < I(X_{\mathcal{K}} X_r; Y_d). \quad (48)$$

Combining (44), (46), and (48), we have the bounds (24)-(27). One can extend this analysis for $K > 2$. The random variable U in (9) and (10) is the usual time-sharing random variable [15, p. 397].

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