THE STATE UNIVERSITY OF NEW JERSEY

RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:559 **Spectrum Management and Wireless Systems** **Fall 2003**

Take Home Examination

Each problem PART is worth the same number of points and the whole exam is half of the grade – the other half will be determined from your conference paper proposals.

1. **Eigenfunctions and Signal Space** Suppose a waveform $\phi_k(t)$ is applied to linear time-invariant filter with impulse response $h(t)$ and output $r(t)$.

   (a) If $\phi_k(t)$ is zero outside $t \in [0,T]$ and we only consider the output on the same interval, find an expression which can be solved to find the set of all $\phi_k(t)$ such that

   $$r(t) = (\phi_k * h)(t) = \lambda_k \phi_k(t)$$

   where $\lambda_k \in \mathcal{C}$.

   Such $\phi_k(t)$ are called eigenfunctions of the system.

   (b) For $h(t) = e^{-t}$ provide numerical examples of a few $\phi_k(t)$ and the associated λ_k for $T = 1$. Plot the two with the largest eigenvalues and the two with the smallest eigenvalues.

   You may numerically calculate any integrals.

   (c) Suppose $\phi_k(t)$ and $r(t)$ are now defined $\forall t$. Show (or disprove) that e^{st} is an eigenfunction of any $h(t)$ where $s \in \mathcal{C}$.

   (d) Now suppose we require the $\phi_k(t)$ to be orthonormal. Then we cannot guarantee that $\phi_k(t)$ is an eigenfunction of the system $h(t)$. However, if we define $\psi_k(t)$ as the output associated with application of $\phi_k(t)$ to $h(t)$, then we can require

   $$\langle \phi_k(t), \phi_\ell(t) \rangle = \delta_{k\ell}$$

   and

   $$\langle \psi_k(t), \psi_\ell(t) \rangle = \lambda_k \delta_{k\ell}$$

   Derive an expression from which the $\phi_k(t)$ and thence the $\psi_k(t)$ can be found. Assume the $\phi_k()$ and $\psi_k()$ are defined only on $(0,T)$.

   (e) Let $h(t) = e^{-t}u(t)$. Derive an expression for the $\phi_k(t)$, find a few of them numerically. Plot the two with the largest eigenvalues and the two with the smallest eigenvalues. Assume $T = 1$.

   (f) Let $h(t) = \frac{\sin 2\pi t}{\pi t}$ and repeat the previous part for $T = 1$ and then $T = 100$.

   (g) Let $\phi_k(t)$ and $\psi_k(t)$ now be defined $\forall t$. Analytically find appropriate $\phi_k(t)$ for arbitrary $h(t)$.

2. **Waterfilling Math** You are given a power budget P which you can split over N Gaussian channels. Let the noise level in each channel be σ_i and the power used on that channel p_i. The capacity of each channel is

$$C_i = \frac{1}{2}\log\left(1+\frac{p_i}{\sigma_i}\right)$$

and the total capacity using all channels is

$$C = \sum_{i=1}^{N} C_i = \frac{1}{2}\sum_{i=1}^{N}\log\left(1+\frac{p_i}{\sigma_i}\right)$$

(a) Suppose you are only allowed to place power into a single dimension, but you can choose which one. Show that any choice of dimension k must satisfy $\sigma_k \leq \sigma_\ell$, $\ell = 1,2,\cdots,N$ to attain maximum capacity C_k.

(b) Prove using Lagrange Multipliers that

$$p_i = [c - \sigma_i]^+$$

where c is a constant chosen to satisfy $\sum_i p_i = P$.

Derive a condition on P so that no p_i is zero.

You can find writeups on Lagrange Multipliers in most optimization texts and some applied math texts. My favorite is Advanced Calculus for Applications (Hildebrand).

(c) Let $d_i = p_i + \sigma_i$. Show using Jensen's inequality that to maximize capacity the d_i should be a constant c, or if not, σ_i. What is the constant c? Is this the same result as the previous part?

HINT: This is a little of a mind-bender. For those of you who've done a formal optimization course, this is an example of optimization through a "slack variable" (I think). Once you get the basic idea, it might help when finishing up to recognize that $\log 1 = 0$ and $\log d_i \geq 0$ by the definition of d_i and capacity.

3. **Single Base Interference Avoidance** In class and in the assigned Interference Avoidance paper (see web page) we learned the following facts:

$$\mathbf{S} = \begin{bmatrix} | & | & & | \\ \mathbf{s}_1 & \mathbf{s}_2 & \cdots & \mathbf{s}_M \\ | & | & & | \end{bmatrix}$$

where the $\{\mathbf{s}_k\}$ are codewords of dimension N. We usually assume $M \geq N$ but that's written in stone. We also have

$$\mathbf{S}\mathbf{S}^\top = \mathbf{S}\mathbf{S}^\top$$

$$\mathbf{R} = \mathbf{S}\mathbf{S}^\top + \mathbf{W}$$

$$C_s = \frac{1}{2}\log|\mathbf{R}| + \frac{1}{2}\log|\mathbf{W}|$$

Greedy interference avoidance applied to codeword k replaces $\mathbf{s}_k$ with $\mathbf{x}$ where $\mathbf{x}$ is a minimum eigenvalue eigenvector of $\mathbf{R} - \mathbf{s}_k\mathbf{s}_k^\top$ and we derived a bunch of conditions for fixed points and optimality in class and in the paper, especially in relation to a quantity called the TSC (or GSC), defined as

$$\text{TSC} = \text{Trace}\left[\left(\mathbf{S}\mathbf{S}^\top\right)^2\right]$$

$$\text{GSC} = \text{Trace}\left[\mathbf{R}^2\right]$$

(a) We know that in white noise, greedy interference avoidance does not increase TSC. Show that greedy interference avoidance does not increase GSC in colored noise ($\mathbf{W} \neq \alpha\mathbf{I}$).

(b) Show that greedy interference avoidance does not decrease sum capacity.

(c) Write an interference avoidance algorithm (in Matlab or C). You may assume white unit variance noise $\mathbf{W} = \mathbf{I}$. Starting from randomly chosen unit norm codewords, sketch (plot) the codeword set attained by interference avoidance for $M = 3$ users in $N = 2$ dimensions. Do the same for $M = 8$ users in $N = 2$ dimensions. Can you find examples of initial codeword sets (for $M = 3$) which do not attain minimum TSC even after repeated application of your algorithm?

Plot the codeword covariance spectra in each case.

4. **Multi-Base Interference Avoidance** Now, suppose we have a TWO base system and two user groups with codeword matrices $\mathbf{S}_1$ and $\mathbf{S}_2$. The users are arranged so that the covariance at base 1 is

$$\mathbf{R}_1 = \mathbf{S}_1\mathbf{S}_1^\top + g\mathbf{S}_2\mathbf{S}_2^\top + \mathbf{I}$$

and

$$\mathbf{R}_2 = \mathbf{S}_2\mathbf{S}_2^\top + g\mathbf{S}_1\mathbf{S}_1^\top + \mathbf{I}$$

at base 2 where g is a non-negative gain factor representing the interference level seen at base i from user set j (a symmetric interference system). Assume $N = 3$ and $M = 3$ for each set of users (for a total of six users over the two bases).

Write code which applies greedy interference avoidance at each base. For $g = 0.1$ and $g = 2$ first try sequential application (one base then the other). Then try interleaved application and answer the following questions.

(a) One could define two TSC's for this problem:

$$\text{TSC}_1 = \text{Trace}\left[\left(\mathbf{S}_1\mathbf{S}_1^\top\right)^2\right]$$

and

$$\text{TSC}_2 = \text{Trace}\left[\left(\mathbf{S}_2\mathbf{S}_2^\top\right)^2\right]$$

Plot each TSC on the same graph as a function of algorithm step. When (if ever) do either or both of the TSC's converge? Is there a pattern of convergence as a function of the interference gain factor g?

(b) If the codewords converge, plot the converged spectra for both user sets. Do the codeword spectra overlap in signal space? If so, when and how much?

5. **Potential Research Paper Topics** COMING!