1 Random Processes II

1.1 LINEAR SYSTEMS

$$y(t) = x(t) * h(t)$$

$$y(t) = \int h(t-\tau)x(\tau)d\tau$$

What can we say about $y(t), y(t_1), x(t_2)$ etc. when $x(t)$ is stationary.

Thm.

1. $$m_y = m_x \int_{-\infty}^{a} h(\tau)d\tau$$

2. $$R_{XY}(\tau) = R_X(\tau) * h(-\tau)$$

3. $$R_Y(\tau) = R_X(\tau) * h(\tau) * h(-\tau)$$

Def.

1. $$E(Y(t)) = \int_{-a}^{\infty} h(t-\tau)E(x(\tau))d\tau = m_x \int_{-\infty}^{\infty} h(\tau)d\tau$$

2.

$$\begin{aligned}
R_{XY}(t_1,t_2) &= E\left[\int_{-\infty}^{\infty} x(t_1)x(t_2-\sigma)h(\sigma)d\sigma\right] \\
&= \int_{-\infty}^{\infty} R_X(t_1-t_2+\sigma)h(\sigma) \\
&= \int_{-\infty}^{\infty} R_X(\tau+\sigma)h(\sigma)d\sigma \\
&= \int_{-\infty}^{\infty} R_X(\tau-\sigma)h(-\sigma)d\sigma \\
&= R_X(\tau) * h(-\tau)
\end{aligned}$$

3.

$$\begin{aligned}
R_Y(t_1,t_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[X(t_1-\sigma)X(t_2-\rho)\right] h(\sigma)h(\rho)d\rho d\sigma \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(t_1-\sigma-t_2+\rho)h(\sigma)h(\rho)d\rho d\sigma \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(\tau-\sigma+\rho)h(\sigma)h(\rho)d\rho d\sigma \\
&= R_X(\tau) * h(\tau) * h(-\tau)
\end{aligned}$$

Punchline → WSS in → WSS out

Also true that SSS in → SSS out, but we did not prove it

Power spectra

Def.

Define $x_T(t) = x(t)(u(t+T/2) - u(t-T/2))$

$$X_T(f) = F[x_T(t)]$$

$$S_X(f) = E \lim_{T\to\infty} \frac{|X_T(f)|^2}{T} = \lim_{T\to\infty} \frac{|X(f)|^2}{T}$$

Now need to relate $R_X(t_1, t_2)$ to $S_X(f)$

Thm. (Weiner Khinchin)

If $R_X(t+\tau, t)$ is integrable on any interval of length $|\tau|$ with $|\tau| < \infty$, i.e.

$$\left| \int_a^{a+\tau} R_X(t+\tau, t) dt \right| < \infty \qquad \forall a \in R$$

then

$$S_X(f) = F\left[< R_X(t+\tau, t) >\right]$$

where

$$< R_X(t+\tau, t) >= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} R_X(t+\tau, t) dt$$

Please look but don't touch

Corollaries

1. $X(t)$ stationary and $\tau R_X(\tau)$ finite for $|\tau| < \infty$

$$S_X(f) = F(R_X(\tau))$$

Pf.

W-K $\rightarrow$

$$\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_X(\tau)d\tau = R_X(\tau)$$

because $\tau R_X(\tau)$ finite, says

$$\int_a^{a+\tau} R_X(t+\tau,t)dt = \int_a^{a+\tau} R_X(t)dt = \tau R_X(\tau) < \infty$$

2. $X(f)$ cyclostationary w/ period

$$\left|\int_0^{T_0} R_X(t+\tau,t)dt\right| < \infty$$

then

$$S_X(f) = F(\bar{R_X}(\tau))$$

$$R_X(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} R_X(t+\tau,t)dt$$

pf. conditions satisfied for W-K if satisfied over a single period and

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_X(t+\tau,t)dt &= \lim_{n\to\infty} \frac{1}{nT_0} n\int_{-T_0/2}^{T_0/2} R_X(t+\tau,t)dt \\ &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} R_X(t+\tau,t)dt\end{aligned}$$

Useful Stuff

1. $$P_X = \int_{-\infty}^{\infty} S_X(f)df = < R_X(t+\tau,t) > \Big|_{\tau=0}$$

2. PSD's of each sample function are equal for an ergodic process

3. Turns out that $S_X(f)$ is nonnegative and even

LTI systems again

Remember

$$m_Y = m_X \int_{-\infty}^{\infty} h(\tau) d\tau$$

$$R_{XY}(\tau) = R_X(\tau) * h(-\tau)$$

$$R_Y(\tau) = R_X(\tau) * h(-\tau) * h(\tau)$$

So

$$m_Y = m_X H(0) = \left. \int_{-\infty}^{\infty} e^{-j2\pi f\tau} h(\tau) d\tau \right|_{f=0}$$

$$S_{XY}(f) = S_X(f) H^*(f)$$

$$S_{YX}(f) = S_X(f) H(f)$$

$$S_Y(f) = S_X(f) |H(f)|^2$$

NICE PICTURE

Sums

$$z(t) = x(t) + y(t)$$

$$m_Z = m_Y + m_X$$

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau) + R_{XY}(\tau) + R_{YX}(\tau)$$

$$S_Z(f) = S_X(f) + S_Y(f) + 2Re[S_{XY}(f)]$$

Uncorrelated?

$$R_{XY}(\tau) = m_X m_Y$$

and if m_Y or m_X is zero

$$S_Z(f) = S_X(f) + S_Y(f)$$

1.2 GAUSSIAN PROCESSES

Def.

$X(t)$ gaussian if $X(t_1), X(t_2), ...$ jointly gaussian RV's

Mean and covariance matrix completely specify joint distribution so

Thm.

A gaussian process is COMPLETELY characterized by $R_X(t_1, t_2)$ and $m_X(t)$ pf. Remember, $cov(X, Y) = E(XY) - E(X)E(Y)$ (x,y real)

By **Thm**

pf. LTI has

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \lim_{\Delta \to 0} \sum_{n=-\infty}^{\infty} x(n\Delta)h(t-n\Delta)\Delta \end{aligned}$$

ie.each $y(t)$ is a linear/superposition of joint gaussians. Therefore, any set of $y(t_i)$ are jointly gaussian (see pp.159 text)

Which is why everybody always beats you up w/ Gaussians

Thm

If gaussian WSS $\rightarrow$ SSS

pf. Gaussian process is completely characterized by its mean and autocov.

Thm

Useful Stuff. If

$$\int_{-\infty}^{\infty} |R_X(\tau)|\, d\tau < \infty$$

and $X(t)$ gaussian w/ $m_X(t) = 0$

Then $X(t)$ stationary. (read tje pf. LONG AGO but don't remember the time)

Joint Gauss Process

$X(t), Y(t)$ are jointly gauss if $X(t_1)...X(t_n)Y(t_1)...Y(t_n)$ are jointly gaussian.

Thm.

If $X(t)$ and $Y(t)$ jointly gaussian then uncorrelated $\Rightarrow$ indept.

pf. Covariance matrix has no diag. terms so determinent is product of diag terms and

$$e^{x^T Cx/2} = \Pi_{i=1}^N e^{x_i^2/2C_{ii}^2}$$

i.e. the joint distribution becomes a product dist.

WHITE Process

Def.

$S_X(f) = \text{const.} \Rightarrow X(t)$ is white (allcolor spectrum)

so $R_X(\tau) = \text{const } \delta(\tau)$

$\Rightarrow$ IMPORTANT $X(t_1)X(t_2)$ UNCORRELATED if $t_1 \neq t_2$

If gaussian process $\rightarrow X(t_1)$ indept.$X(t_2)$

Noise and LTI

$$S_{W_c}(f) = S_W(f)|H(f)|^2 = N_0|H(f)|^2$$

Noise equivalent BW

(Don't like this but it is a common def. and manufacturers like it)

1. Look at $|H(f)|^2$ (lets assume its a BP filter)
2. Find max $|H(f)|^2$ at $f = f^*$

$$B_{neq} = \int_{-\infty}^{\infty} |H(f)|^2 df/2|H(f^*)|^2$$

3. Power in output $N_0 B_{neq} |H(f^*)|^2$

   i.e. if they give you $B_{neq} + |H(f^*)|$ you don't have to do the integral (because they did it for you)

Thm.

Nyquist sampling holds for random processes

i.e. the expected difference between $X(t)$ and $\hat{X}(t)$=reconstruction from samples

$$\begin{aligned}\hat{X}(t) &= \text{LPF}(\sum_n X(nT)\delta(t-nT)) \\ &= \sum_n X(nT) sinc(2W(T-NT))\end{aligned}$$

is zero if $T_s \leq \frac{1}{2W}$

HOWEVER can't say $\hat{X}^(t) = X(t)$ MS not pointwise

Real Interesting Theorem

If $S_X(f)$ flat over passband, then $X(nT)$ are uncorrelated

I think we should try to prove this same time (think about it).

$\Rightarrow$ Kind of tells you how long you need to want before you get new information.

1.3 BANDPASS PROCESSES

Must work about these for multiplying many signals using same channel (like radio)

Now need to worry about decomposition

Any X(t) can be decomposed into its sin portion and its cosine portion. What we want to do is represent $X(t)$ as a modulated signal

$$X(t) = X_c(t)\cos 2\pi f_0 t - X_s(t)\sin 2\pi f_0 t$$

$$\hat{X}(t) = X_c(t)\sin 2\pi f_0 t + X_s(t)\cos 2\pi f_0 t$$

The $X_c(t)$ and $X_s(t)$ are "baseband" processes

Hilbert tells us

$$\hat{X}^(t) = Hil(X(t))$$

$$X_c(t) = X(t)\cos 2\pi f_0 t + \hat{X}(t)\sin 2\pi f_0 t$$

$$X_s(t) = X(t)\cos 2\pi f_0 t - \hat{X}(t)\sin 2\pi f_0 t$$

FALLOUT

Thm.

$X(t)$ is a stationary BP process (BW=2W)

$\Rightarrow$ $X_s(t)$ and $X_c(t)$ are jointly stationary lowpass (baseband) processes each having the power of $X(t)$

$$R_{X_c}(\tau) = R_{X_s}(\tau) = R_X(\tau)\cos 2\pi f_0\tau + \hat{R}_X(\tau)\sin 2\pi f_0\tau$$

$$R_{X_cX_s}(\tau) = R_X(\tau)\sin 2\pi f_0\tau - \hat{R}_X(\tau)\cos 2\pi f_0\tau$$

$$\hat{R}_X(\tau) = Hil(R_X(\tau))$$

BP

$$S_{X_c}(f) = S_{X_s}(f) = \begin{cases} S_X(f-f_0) + S(f+F_0) & |f| < f_0 \\ 0 & \text{o.w.} \end{cases}$$

Since $S_X(f)$ limited to $|f-f_0| < W$ then $S_{X_{c/s}}(f)$ limited to $|f| < W$

$$S_{X_cX_s}(f) = \begin{cases} j(S_X(f+f_0) - S_X(f-f_0)) & |f| < f_0 \\ 0 & o.w. \end{cases}$$

same idea (notice $S_{X_sX_c}(0) =$ always)

Important $\Rightarrow$ if $X(t)$ is such that

$$S_X(f+f_0) = S_X(f-f_0) \qquad (|f| < f_0)$$

then cross corr is zero!

i.e. $R_{X_cX_s}(\tau) \equiv 0$ everywhere!

So, say $X(t)$ is also zero mean $\rightarrow X_c(t)$ and $X_s(t)$ also zero mean (easy to see)

Therefore, $X_c(t)$ and $X_s(t)$ uncorrelated

Say $X(t)$ is also gaussian $\rightarrow X_c$ and X_s jointly gaussian

Therefore, X_c and X_s INDEPT

FINALLY, $P_X = R_X(0) = R_{X_c}(0) = R_{X_s}(0)$

Powers in $X_c(t)$ nd $X_s(t)$ same as in $X(t)$

END