Signals and Systems Review

1 Linear Systems Review

1.1 Signal types

- discrete/continuous
- real/complex
- odd/even
- periodic/aperiodic
- deterministic/random
- energy/power/other

1.2 Signal Menagerie

- sinusoids
- complex sinusoids
- step $u(t)$ (also $u_{-1}(t)$)
- ramp $u_{-2}(t)$
- $sinc(x) = sin\pi x/\pi x$
- $sign(x) = sgn(x)$
- $\delta(t)$

1.3 Properties of delta $\delta(t)$

$$\int_{-\infty}^{\infty} \delta(t-t_0)\phi(t)dt = \phi(t_0) \qquad \text{sifting}$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\delta(t) = 0 \qquad |t| > 0 \quad \delta(0) = \infty$$

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

$$\delta(t - t_0) * h(t) = h(t - t_0)$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \Leftrightarrow \delta(t) = \frac{d}{dt}u_{-1}(t)$$

- **Systems: (Continuous – but also applies to discrete)**

  Define transformations as

$$y(t) = T[x(t)]$$

- **Linearity**

  if

$$y_1(t) = T[x_1(t)]$$
$$y_2(t) = T[x_2(t)]$$

  and a and b are constant, then if

$$T[ax_1(t) + bx_2(t)] = ay_1(t) + by_2(t)$$

  the system is linear (superposition and scaling)

- **Time invariance**

$$T[x(t - t_0)] = y[t - t_0] \qquad \forall x(t), t_0$$

  LTI $\rightarrow$ linear, time invariant (has an impulse response)

Causality

$y(t_0) = T[x(t)]|_{t=t_0}$ depends only on $x(t), t \leq t_0$

1.4 LTI Systems: Time domain

$$h(t) = T[\delta(t)] \qquad \text{impulse response}$$

$$\text{Time Invariant} \qquad T[\delta(t-t_0)] = h(t-t_0)$$

$$x(t) = \int x(\sigma)\delta(t-\sigma)d\sigma$$

Integral is a sum of terms so is linear

$$T[\int] = \int T[\,]$$

$$\begin{aligned} T[x(t)] &= \int_{-\infty}^{\infty} T[x(\sigma)\delta(t-\sigma)]d\sigma \\ &= \int_{-\infty}^{\infty} x(\sigma)h(t-\sigma)d\sigma \\ &= \int_{-\infty}^{\infty} x(t-\sigma)h(\sigma)d\sigma \end{aligned}$$

This is the convolution integral and that's why impulse response is useful for LTI systems.

1.5 Orthogonal Signal Space

******IMPORTANT******

- **Vector Orthogonality:**

$$< \underline{x}, \underline{y} >= \sum_{i=1}^{N} x_i y_i^* = 0$$

if $\underline{x}$ and $\underline{y}$ orthogonal, $\underline{x} \perp \underline{y}$

Magnitude $||\underline{x}||^2 = \sum_{i=1}^{N} |x_i|^2 =< \underline{x}, \underline{x} >$ as defined above

$$\underline{a} \perp \underline{b} \Rightarrow ||a+b||^2 = ||a||^2 + ||b||^2 \qquad \text{(prove it yourself)}$$

triangle Inequality $||\underline{a} + \underline{b}|| \leq ||\underline{a}|| + ||\underline{b}||$

- **Linearity**

$$L[a\underline{x} + b\underline{y}] = aL[\underline{x}] + bL[\underline{y}]$$

- **Orthogonal expansion**

  Can always write

$$\underline{x} = \sum_{i=1}^{N} a_i \underline{e_i} \qquad \text{where} < \underline{e_i}, \underline{e_j} >= \begin{cases} 0 & i \neq j \\ \alpha_i & i = j \end{cases}$$

$\{\underline{e_i}\}$ is an orthogonal basis Therefore,

$$L[\underline{x}] = \sum_{i=1}^{N} a_i L[\underline{e_i}]$$

<u>**Eigenvector:**</u> $\underline{x}$ is an eigenvector of system $L[\,]$ if

$$L[\underline{x}] = \lambda \underline{x}$$

where λ is a constant.

- **Gram Schmidt**

  Can we create personalized orthogonal sets? GET YOURS NOW! CALL 1-800-555,... :)

  Aside on independence: $\{\underline{x_i}\}$ Card$\{\underline{x_i}\} = J \leq N$ are mutually independent if

$$\underline{x_i} \neq \sum_{j=1, j\neq i}^{J} a_j \underline{x_j}$$

for all choices $\{a_j\}$

Given J mutually independent vectors $\underline{x_1}, \cdots, \underline{x_j}$ we can form mutually orthogonal set $\{\underline{\phi_i}\}$ using GRAM-SCHMIDT ORTHOGONALIZATION

- **Procedure**

  Start with $\underline{x}_1$

$$\underline{\phi_1} = \frac{\underline{x_1}}{||\underline{x_1}||}$$

  Then use $\underline{x}_2$ and subtract off projection onto $\underline{\phi}_1$

$$\underline{\phi_2} = \underline{x_2} - \frac{<\underline{\phi_1}, \underline{x_2}>}{<\underline{\phi_1}, \underline{\phi_1}>}\underline{\phi_1} = \underline{x_2} - \left\langle \frac{\underline{\phi_1}}{||\underline{\phi_1}||}, \underline{x_2} \right\rangle \frac{\underline{\phi_1}}{||\underline{\phi_1}||}$$

  etcetera and so on...

$$\underline{\phi_k} = \underline{x_k} - \sum_{j=1}^{k-1} \frac{<\underline{\phi_j}, \underline{x_k}>}{<\underline{\phi_j}, \underline{\phi_j}>}\underline{\phi_j} = \underline{x_k} - \sum_{j=1}^{k-1} \left\langle \frac{\underline{\phi_j}}{||\underline{\phi_j}||}, \underline{x_k} \right\rangle \frac{\underline{\phi_j}}{||\underline{\phi_j}||}$$

- **Orthogonality Check**

$$<\underline{\phi_1}, \underline{\phi_2}> = <\underline{x_1}, \underline{x_2}> - \frac{<\underline{x_1}, \underline{x_2}>}{<\underline{x_1}, \underline{x_1}>} <\underline{x_1}, \underline{x_1}> = 0$$

  can show recursively that $\underline{\phi_i} \perp \phi_j \qquad i \neq j$

  Then we make orthonormal:

$$\underline{\psi_i} = \frac{\underline{\phi_i}}{|\underline{\phi_i}|}$$

$$< \underline{\psi_i}, \underline{\psi_j} >= \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

- **What about Signals?**
  SIGNAL PICTURE

$$\underline{f} = (f_1, f_2, \cdots f_N)$$

$$\Delta < \underline{g}, \underline{f} >= (\sum_{i=1}^{N} g_i f_i^*)\Delta$$

Shrink $\Delta \to 0$

$$\frac{1}{T}\int_0^T g(t) f^*(t) dt \equiv < g(t), f(t) >$$

if $g(t) \perp f(t)$ on (a, b) then

$$\int_a^b g(t) f^*(t) dt = 0$$

- Example

$$\omega_0 = \frac{2\pi}{T}$$

$$\underbrace{< e^{j\omega_0 kt}, e^{j\omega_0 mt} >}_{\text{orthronormal set}} = \begin{cases} 0 & m \neq k \\ 1 & m = k \end{cases}$$

GRAM SCHMIDT WORKS TOO!

- **Completeness**

  Complete set means any signal in the signal space could be expanded as a series, i.e.

  $$f(t) = \sum_{i=1}^{\infty} a_i \phi_i(t)$$

  where $< \phi_i(t), \phi_j(t) >= \delta_{i,j}$

  $\{\phi_i(t)\}$ forms a basis for the signal set if any $f(t)$ can be represented this way

  Complex exponentials are a complete orthonormal which is why Fourier series works

  DECOMPOSITION

  $$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$$

  PROJECTION

  $$a_k = \frac{1}{T}\int_0^T f(t)e^{-j\omega_0 kt}dt$$

  which form the **Fourier series pair** when $f(t) = f(t+T)$, $\omega_0 = 2\pi/T$ and if $f(t)$ satisfies Dirichlet (see book)

1.6 LTI Systems and Periodic Excitation

define $H(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt$

$$y(t) = T[x(t)] = T[\sum_k x_k e^{j\omega_0 kt}] = \sum_k x_k T[e^{j\omega_0 kt}]$$

$$T[e^{j\omega_0 kt}] = \int_{-\infty}^{\infty} h(\tau)e^{j\omega_0 k(t-\tau)}d\tau = e^{j\omega_0 kt}H(\frac{k}{T})$$

Therefore,

$$y(t) = \sum_k x_k H(\frac{k}{T}) e^{j\omega_0 kt}$$

Which is why the frequency response of a system H(f) is so useful for sinusoidal signal inputs

- **Parseval**

$$\int_0^T |x(t)|^2 dt = \sum_k |x_k|^2$$

  i.e. energy in a period $\Rightarrow$ add up energy in spectral components

- **Fourier Xform**

  Just spread period out:

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

  Look! no $1/2\pi$ and integrals have f not $\omega = 2\pi f$. This is the FT pair - $f(t)$ must satisfy Dirichlet

- **Useful stuff**

  1. $F[\delta(t)] = 1$
     therefore

$$\delta(t) = \int_{-\infty}^{\infty} e^{j2\pi ft} df$$

  2. $x(t)$ real $\Rightarrow X(f) = X(^*(-f))$ (Hermitian)

3. $x(t)$ odd $\quad X(f)$ imag
4. $x(t)$ even $\quad X(f)$ real
5. **Linearity**

$$F[ax(t) + by(t)] = aX(f) + bX(f)$$

6. **Duality**
   If $X(f) = F[x(t)]$, then
   $F[X(-t)] = x(f)$
   and $F[X(t)] = x(-f)$
   notice the absence of 2π factors! Isn't it lovely!?!?
7. <u>Time shift</u>

$$F[x(t - t_0)] = e^{-j2\pi f t_0} X(f)$$

8. **Scaling**

$$F[x(at)] = \frac{1}{|a|} X(\frac{f}{a}) \qquad a \neq 0 \quad \text{and real}$$

9. **Convolution**

$$F[x(t) * y(t)] = X(f)Y(f)$$

10. **Modulation**

$$F[x(t)e^{j2\pi f_0 t}] = X(f - f_0)$$

11. **Parseval**

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$$

which is the general form. This implies

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

which is the more usual (and useful) form.

12. **Autocorrelation**

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt$$

(There are others comparable definition) then

$$\underbrace{F}_{w.r.t.\tau}[R_x(\tau] = |X(f)|^2$$

13. **Differentiation**

$$x(t) \to X(f)$$

$$\frac{dx}{dt} \to (j2\pi f)X(f)$$

$$\frac{d^n x}{dt^n} \to (j2\pi f)^n X(f)$$

Likewise,

$$\frac{d^n X(f)}{df^n} \Rightarrow (-j2\pi t)^n x(t)$$

14. **Integration**

$$\int_{-\infty}^{\tau} x(\tau)d\tau = \frac{1}{j2\pi f}X(f) + \underbrace{\frac{1}{2}X(0)\delta(f)}$$

Note that if x(t) has no DC component then this disappears

15. **Moments**

(will be useful for probability functions)

$$\int_{-\infty}^{\infty} t^n x(t) dt = \left(\frac{j}{2\pi}\right)^n \frac{d^n X(f)}{df^n}\bigg|_{f=0}$$

from differentiation theorem

- **Useful Transforms**

(might want to memorize them)

$$\begin{bmatrix} \delta(t) \rightarrow & 1 \\ 1 & \rightarrow \delta(f) \end{bmatrix}$$

$$\begin{bmatrix} \delta(t-t_0) \rightarrow & e^{-j2\pi f t_0} \\ e^{j2\pi f t} & \rightarrow \delta(f-f_0) \end{bmatrix}$$

$$\cos(2\pi f_0 t) \rightarrow \frac{\delta(f-f_0)+\delta(f+f_0)}{2}$$

$$\sin(2\pi f_0 t) \rightarrow \left(\frac{-\delta(f+f_0)+\delta(f-f_0)}{2j}\right)$$

$$rect(t) \rightarrow sinc(f) = \frac{\sin \pi f}{\pi f}$$

$$sinc(t) \rightarrow rect(f)$$

$$e^{-\alpha t} u(t) \rightarrow \frac{1}{\alpha + j2\pi f} \quad \alpha > 0$$

$$sgn(t) \rightarrow \frac{1}{j\pi f}$$

$$u(t) \rightarrow \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

(from integration theorem)

- **Periodic signals**

  Qualitative $\Rightarrow$ energy concentrated at harmonics
  So $x'(t) = x(t+T)$

$$X(f) = X\left(\frac{k}{T}\right) = \sum_{k=-\infty}^{\infty} x_k \delta(f - \frac{k}{T})$$

$$x_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-j\frac{2\pi}{T}kt} dt$$

  i.e. Fourier series coeff.

- **FT and LTI systems**

$$H(f) = F[h(t)]$$

$$Y(f) = F[\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau] = H(f)Y(f)$$

  as we used before
  If $x(t) = e^{j2\pi f_0 t}$

$$Y(f) = \delta(f - f_0)H(f) \Rightarrow y(t) = \underbrace{H(f_0)}_{\text{const}}\underbrace{e^{j2\pi f_0 t}}_{\text{x(t)}}$$

  $e^{j2\pi f_0 t}$ is an **EIGENFUNCTION** of LTI systems
  $\rightarrow$ that's why we decompose signals onto $e^{j\omega t}$

- **Signal energy**

  $voltage^2 \Rightarrow \propto$ units of watts=joules/sec

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \text{joules}$$

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \text{watts}$$

  energy signal $E_x < \infty$
  power signal $0 < P_x < \infty$
  or OTHER signals (don't really care for this designation, but it's analytically useful sometimes)

- **E-type**

$$\begin{aligned} R_x(\tau) &\equiv x(\tau) * x^*(-\tau) \\ &= \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt \\ &= \int_{-\infty}^{\infty} x(t+\tau)x^*(t)dt \\ &\overset{F[]}{\Rightarrow} |X(f)|^2 \end{aligned}$$

  Clearly,

$$E_x = R_x(0) \qquad \text{and can use Parseval as well}$$

$$y(t) = x(t) * h(t) \Rightarrow R_x(\tau) * R_h(\tau) = R_y(\tau)$$

  since

$$R_x(\tau) \to |X(f)|^2 \Rightarrow \text{notation} G_x(f) = \text{ESD of x(t)}$$

$$|X(f)|^2 \equiv \text{energy spectral density}$$

—————————————————-

$$G_y(f) = |G_x(f)||H(f)|^2$$

—————————————————-

rough units: F[joules(t)]⇒ joules.sec=joules/(1/sec)=joules/Hz

- **Power signals**

$$R_x(\tau) \equiv \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x^*(t-\tau)dt$$

$$P_x = R_x(0) \qquad \text{(parseval too)}$$

define

$$\begin{aligned} S_X(f) &\equiv PSD(\qquad \text{power spectral density}) \\ &= F[R_x(\tau)] \end{aligned}$$

as with energy signals

$$y(t) = x(t) * h(t) \Rightarrow R_y(\tau) = R_x(\tau) * \underbrace{h(\tau) * h^*(-\tau)}_{\text{ESD of system}}$$

so

$$S_y(f) = S_x(f)|H(f)|^2$$

- *aside*

  Periodic signals are power signals $x(t)$ periodic, $y(t) = x(t) * h(t)$

$$P_y = \sum_n |x_n|^2 |H\left(\frac{n}{T}\right)|^2 \qquad \text{obtained via Parseval}$$

1.7 Deterministic Sampling and Reconstruction

$x(t)$ bandlimited to $\pm W Hz$
Sample at at least $1/2W$ second intervals and $x(t)$ can be obtained from samples.WHY?

no aliasing

$$\frac{1}{T} \geq 2W \rightarrow T \leq \frac{1}{2W} \qquad \text{NYQUIST RATE}$$

<u>*aside*</u>

$$x(t)s(t) \Rightarrow X(f) * S(f)$$

$$s(t) = \sum_n \delta(t - nT) \text{is periodic so t has FS}$$

Therefore,

$$S(f) = \sum_k s_k \delta(f - kf_0)$$

$$s_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-j\frac{2\pi}{T}kt} dt = \frac{1}{T}$$

$$S(f) = \frac{1}{T}\sum_k \delta(f - \frac{k}{T})$$

Thererfore

$$X(f) * S(f) = \frac{1}{T}\sum_k X(f - \frac{k}{T})$$

<u>**Recovery**</u> $\Rightarrow$ *******FIG.7*********

1.8 Bandpass and Hilbert

$$x(t) = \cos 2\pi f_0 t$$

$$z(t) = e^{j2\pi f_0 t}$$

*******FIG.8************

- **In general:**

  $Z(f) = 2U(f)X(f)$ and $x(t)$ is assumed real usually. So,

  $$z(t) = 2(\frac{1}{2}\delta(t) + \frac{j}{2\pi t}) * x(t) = x(t) + j\hat{x}(t)$$

  where

  $$\underbrace{\hat{x}(t) = x(t) * \frac{1}{\pi t}}_{\text{Hilbert Xform}}$$

  $$\hat{X}(f) = -jsgn(-f)X(f) = -jsgn(f)X(f)$$

  In examples above

  $$x(t) = \cos 2\pi f_0 t$$

  $$\hat{x}(t) = \sin 2\pi f_0 t$$

Hilbert Xform gives a 90^o phase shift.

$$e^{j2\pi f_0 t} = z(t) = x(t) + j\hat{x}(t) = \underbrace{\cos 2\pi f_0 t}_{\text{in phase}} + j \underbrace{\sin 2\pi f_0 t}_{\text{quadrature}}$$

- **Hilbert neatos**

1. $x(t)$ even, $\hat{x}(t)$ odd
   $x(t)$ odd, $\hat{x}(t)$ even

2. $\hat{\hat{x}}(t) = -x(t)$
3. $E_x = E_{\hat{x}}$
4. $x(t) \perp \hat{x}(t)$ (if $x(t)$ real)

   This is worth a proof.

$$\begin{aligned} < x(t), \hat{x}(t) > &= \int_{-\infty}^{\infty} x(t) \left(\int_{-\infty}^{\infty} x^*(t-\tau) \frac{1}{\pi\tau} d\tau \right) dt \\ &= \int_{-\infty}^{\infty} \frac{d\tau}{\pi\tau} \underbrace{\left(\int_{-\infty}^{\infty} x(t) x^*(t-\tau) dt \right)}_{R_x(\tau)} \\ &= \int_{-\infty}^{\infty} \frac{R_x(\tau)}{\pi\tau} d\tau \\ &= \underbrace{\int_{-\infty}^{\infty} \frac{Re[R_x(\tau)] d\tau}{\pi\tau} d\tau}_{=0} + \int_0^{\infty} \frac{jIm[R_x(\tau)]}{\pi\tau} d\tau \\ &+ \int_{-\infty}^{0} \frac{jIm[R_x(\tau)]}{\pi\tau} d\tau \end{aligned}$$

and if $x(t)$ real (or $X(f) = X^*(f)$, i.e. hermitian),
$Im[R_x(\tau)] = 0$ QED