332:545 **Communications Theory** **Spring 1999**

QUIZ I Statement & Solutions

You have the class period to answer the following three questions. I think they are easy, but I'm almost always mistaken. Read through the exam carefully before beginning work. PLEASE put the problem in order in your booklet. You may DO the problems in any order you wish, but I need them to be IN ORDER in the booklet for ease of grading.

1. **Doing it RIGHT:** *(50 points)* You are given the following integral equation

$$\lambda\phi(t) = \int_0^t (1-(t-\tau))\phi(\tau)d\tau + \int_t^1 (1+(t-\tau))\phi(\tau)d\tau$$

The function $\phi(t)$ is assumed to exist only on $[0,1]$. After differentiation and application of Liebnitz' rule, you find (correctly) that

$$\lambda\frac{d^2\phi}{dt^2} + 2\phi = 0$$

Please carefully find in closed form a set of eigenfunctions and associated eigenvalues for this integral equation.

What is the maximum eigenvalue (if it exists)? What is the minimum eigenvalue (if it exists)? Show that $\phi(t)$ corresponding to different eigenvalues are indeed orthogonal. Does your set span the function space on $[0,1]$?

We know that the solution to the second order differential equation stated above is in general $Ae^{j\beta t} + Be^{-j\beta t}$ where A and B are constants and $\beta = \sqrt{2/\lambda}$.

First we substitute $e^{j\beta t}$

$$\int_0^t (1-(t-\tau))e^{j\beta\tau}d\tau + \int_t^1 (1+(t-\tau))e^{j\beta\tau}d\tau$$

This reduces to

$$\begin{array}{ll} & \frac{1-t}{j\beta}(e^{j\beta t}-1) + \frac{1+t}{j\beta}(e^{j\beta}-e^{j\beta t}) \\ + & \int_0^t \tau e^{j\beta\tau}d\tau - \int_t^1 \tau e^{j\beta\tau}d\tau \end{array}$$

Integration by parts yields $\int te^{j\beta t}dt = \frac{te^{j\beta t}}{j\beta} - \int \frac{e^{j\beta t}}{j\beta}dt$ which recduces to $\frac{t}{j\beta}e^{j\beta t} + \frac{1}{\beta^2}e^{j\beta t}$. We then have

$$\begin{array}{ll} & \frac{1-t}{j\beta}(e^{j\beta t}-1) + \frac{1+t}{j\beta}(e^{j\beta}-e^{j\beta t}) \\ + & \frac{t}{j\beta}e^{j\beta t} + \frac{1}{\beta^2}(e^{j\beta t}-1) \\ - & \frac{1}{j\beta}e^{j\beta} + \frac{t}{j\beta}e^{j\beta t} \\ - & \frac{1}{\beta^2}(e^{j\beta}-e^{j\beta t}) \end{array}$$

Simplifying we obtain

$$\frac{2}{\beta^2}e^{j\beta t} + \frac{j\beta - 1 - j\beta t(1+e^{j\beta}) - e^{j\beta}}{\beta^2}$$

and for $e^{-j\beta t}$

$$\frac{2}{\beta^2}e^{-j\beta t} + \frac{-j\beta - 1 + j\beta t(1+e^{-j\beta}) - e^{-j\beta}}{\beta^2}$$

So assume that $\phi(t) = Ae^{j\beta} + Be^{-j\beta}$. We then have

$$\begin{aligned}\lambda(Ae^{j\beta} + Be^{-j\beta}) &= (A+B)\frac{2}{\beta^2}(e^{j\beta t} + e^{-j\beta t}) \\ &+ A\frac{j\beta - 1 - j\beta t(1+e^{j\beta}) - e^{j\beta}}{\beta^2} \\ &+ B\frac{-j\beta - 1 + j\beta t(1+e^{-j\beta}) - e^{-j\beta}}{\beta^2}\end{aligned}$$

We then need

$$Aj\beta t(1+e^{j\beta}) - Bj\beta t(1+e^{-j\beta}) = 0$$

and

$$A(j\beta - 1 - e^{j\beta}) + B(-j\beta - 1 - e^{-j\beta}) = 0$$

We can see that $A = B$ and $\beta = (2n+1)\pi$ will do the trick. This implies that

$$\phi_n(t) = C_n \cos(2n+1)\pi t$$

and

$$\lambda = \frac{2}{(2n+1)^2\pi^2}$$

where C_n is a normalization constant ($C_n^2 = 1/ < \cos(2n+1)\pi t, \cos(2n+1)\pi t >$). These sinusoids are clearly orthogonal on $(0, 1)$.

The maximum eigenvalue is $\lambda = 2/\pi^2$. There is no minimum eigenvalue (they approach 0 in n). As for completeness, the issue is whether we can represent ANY function on the interval $[0, 1]$. This is a tall order (even Fourier does not do this in general). So we'll settle for the class of functions which Fourier series can represent. As a test, let's see if we can represent $f(t) = 1$ on the interval. Well, the answer is obviously NO since $< 1, \cos(2n+1)\pi t >= 0$ on $[0, 1]$.

So the set is not complete. The problem is that the kernel $R_X(\tau)$ is not positive DEFINITE. In such a case, we'd need to "augment" the basis functions with the missing ones either directly (via Fourier in this case) or via Gram-Schmidt orthogonalization.

2. **Rutgera and the Shooting Gallery:** *(50 points)* Rutgera Univera, the world famous Rutgers University graduate student is down on her luck. A carnival side show owner has taken pity and offered her a job as a target for his shooting gallery. Rutgera would be attached to a rotating wheel wearing body armor and a BB-proof helmet (no, they're not using bullets for heaven's sake!). Having been subjected to far worse, Rutgera takes the job.

   The wheel moves at constant angular speed either clockwise or counterclockwise. When a BB strikes Rutgera, the wheel reverses direction of rotation with probability 1/2. BBs strike Rutgera at an average rate of 1 per second and the time between successive hits t is exponentially distributed and independent for any different pair of successive hits. That is,

$$f_T(t) = e^{-t}u(t)$$

   where $u(t)$ is the unit step function.

   Let $S(t)$ be Rutgera's angular speed and assume it takes on values $\pm\Omega$. Please find the autocorrelation function $R_S(t_1, t_2)$. Is the process wide sense stationary?

   Same idea as the example we did in class. If we have samples $S(t_1)$ and $S(t_2)$ in different epochs (one or more BB shots occurs between t_1 and t_2), then the values $S(t_1)$ and $S(t_2)$ are independent and $E[S(t_1)S(t_2)] = E[S(t_1)]E[S(t_2)] = 0$. So we only care about when there is NO BB shot between t_1 and t_2. In this case $E[S(t_1)S(t_2)] = \Omega^2$.

The probability of this event is the probability that starting from t_1 a BB shot occurs after t_2 (or the reverse if $t_2 < t_1$). Thus, we have $\int_{|t_1-t_2|}^{\infty} e^{-\tau} d\tau$ which simplifies to $e^{-|t_1-t_2|}$. So

$$R_S(t_1, t_2) = \Omega^2 e^{-|t_1-t_2|}$$

which may also be written as

$$R_S(\tau) = \Omega^2 e^{-|\tau|}$$

The process is wide sense stationary since $R_S()$ depends only on the time difference between samples.

3. **More Integral Equations:** *(50 points)* A wide sense stationary zero mean process $X(t)$ has autocorrelation $R_X(\tau) = e^{-|\tau|}$.

   Yup, this is the same one as in the previous problem :)

   (a) *(10 points)* Prove that $R_X(t_1, t_2) = R_X(t_1 - t_2)$ is positive definite. Recall that a positive definite correlation function satisfies

   $$\int\int g(t) R_X(t-\tau) g(\tau) dt d\tau > 0$$

   $\forall$ $g(t)$ with nonzero finite energy (assume $g(t)$ has unit energy for simplicity).
   Using our linear systems knowledge we see that regardless of whether we restrict the interval over which $g(t)$ is nonzero or let $g(t)$ be arbitrary we have

   $$\int\int g(t) R_X(t-\tau) g(\tau) dt d\tau = \mathcal{F}^{-1}\left[|G(f)|^2 S_X(f) \right]_{t=0} = \int_{-\infty}^{\infty} |G(f)|^2 S_X(f) df$$

   We find that $S_X(f) = \dfrac{1}{1 + (2\pi f)^2}$ so that the integrand is non-negative and non-zero at least over some range of the argument f if $g(t)$ has nonzero energy. Thus, the kernel $e^{|-\tau|}$ is positive definite.
   It's worthwhile to note that ANY autocorrelation function is at least positive SEMI-def. The simplest way to see this is to note that

   $$\int\int g(t) R_X(t-\tau) g(\tau) dt d\tau = E[\int\int g(t) X(t) X(\tau) g(\tau) dt d\tau] = E[(\int g(t) X(t) dt)^2] \geq 0$$

   NOTE: if we have complex signal we'd have to have complex conjugates in the definitions of $R_X(t, \tau)$ and in the positive definite definition ($g(\tau) \to g^*(\tau)$).

   (b) *(20 points)* Set up the integral equation for this kernel $R_x(\tau)$ and find the eigenfunctions and eigenvalues for this process. Assume that the eigenfunctions may exist for all t.
   The integral equation is

   $$\lambda \phi(t) = \int_{-\infty}^{\infty} R_X(t-\tau) \phi(\tau) d\tau$$

   where $\phi(t)$ is defined $\forall t$. This is a convolution integral. First we note that

   $$\lambda \Phi(f) = S_X(f) \Phi(f)$$

   and the solution to this equation is $\Phi(f) = \delta(f - f_0)$ with $\lambda = S_X(f_0)$. This implies $[e^{j2\pi f_0 t}, S_X(f_0)]$. Checking the integral equation we have

   $$\int_{-\infty}^{\infty} R_X(t-\tau) e^{j2\pi f_0 \tau} d\tau = \int_{-\infty}^{\infty} R_X(\tau) e^{j2\pi f_0 (t-\tau)} d\tau = e^{j2\pi f_0 t} \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f_0 \tau} d\tau = e^{j2\pi f_0 t} S_X(f_0)$$

and we're done.

(c) *(20 points)* Set up the integral equation for this kernel $R_x(\tau)$ and find the eigenfunctions and eigenvalues for this process when the eigenfunctions are only allowed to be nonzero on $[0, 1]$.

Same deal but now we have time limits. Let's try something quick. Write $\phi(t) = \psi(t)r(t)$ were $r(t)$ is a rectangular time gating function which is 1 on $[0, 1]$ and zero elsewhere. We then have

$$\lambda(R * \Psi)(f) = S_X(f)(R * \Psi)(f)$$

So we know we'll have the sinusoids again. In partcular, we'll have $\phi(t) = Ae^{j\beta(f)t} + Be^{-j\beta(f)t}$ with $\beta(f)$ chosen so that the resulting eigenvalue λ satisfies $\lambda - S_X(f) = 0$. It's just a question of which ones will actually satisfy the integral equation.

So only allowing $t \in [0, 1]$ we have

$$\int_0^1 e^{j\beta\tau}e^{-|t-\tau|}d\tau = \int_0^t e^{j\beta\tau}e^{-t+\tau}d\tau + \int_t^1 e^{j\beta\tau}e^{t-\tau}d\tau$$

Solving the integrals yields

$$\begin{aligned} & e^{-t}\frac{1}{j\beta+1}(e^{(j\beta+1)t} - 1) + e^t\frac{1}{j\beta-1}(e^{(j\beta-1)} - e^{(j\beta-1)t}) \\ = \; & \frac{1}{j\beta+1}e^{j\beta t} - e^{-t}\frac{1}{j\beta+1} - \frac{1}{j\beta-1}e^{j\beta t} + e^t\frac{1}{j\beta-1}e^{(j\beta-1)} \end{aligned}$$

Combining the solutions for β and $-\beta$ we have

$$\begin{aligned} & A\frac{1}{j\beta+1}e^{j\beta t} - B\frac{1}{j\beta-1}e^{-j\beta t} - Ae^{-t}\frac{1}{j\beta+1} + Be^{-t}\frac{1}{(j\beta-1)} \\ - \; & A\frac{1}{j\beta-1}e^{j\beta t} + B\frac{1}{j\beta+1}e^{-j\beta t} + Ae^t\frac{1}{j\beta-1}e^{(j\beta-1)} - Be^t\frac{1}{j\beta+1}e^{-(j\beta+1)} \end{aligned}$$

We need the terms in e^t and e^{-t} to go away which implies

$$A\frac{1}{j\beta+1} = B\frac{1}{j\beta-1}$$

and

$$A\frac{e^{j\beta-1}}{j\beta-1} = B\frac{e^{-(j\beta+1)}}{j\beta+1}$$

Combining yields

$$e^{2j\beta} = \frac{(1-j\beta)^2}{(1+j\beta)^2} = \frac{(1-j\beta)^4}{(1+\beta^2)^2}$$

or

$$e^{j\beta} = \pm 1\frac{(1-\beta^2-2j\beta)}{1+\beta^2}$$

which leads to

$$\cos\beta = \pm\frac{1-\beta^2}{1+\beta^2}$$

and

$$-\sin\beta = \pm\frac{2\beta}{1+\beta^2}$$

If you got anywhere near this far, you should have received full credit. Now, for the analysts,

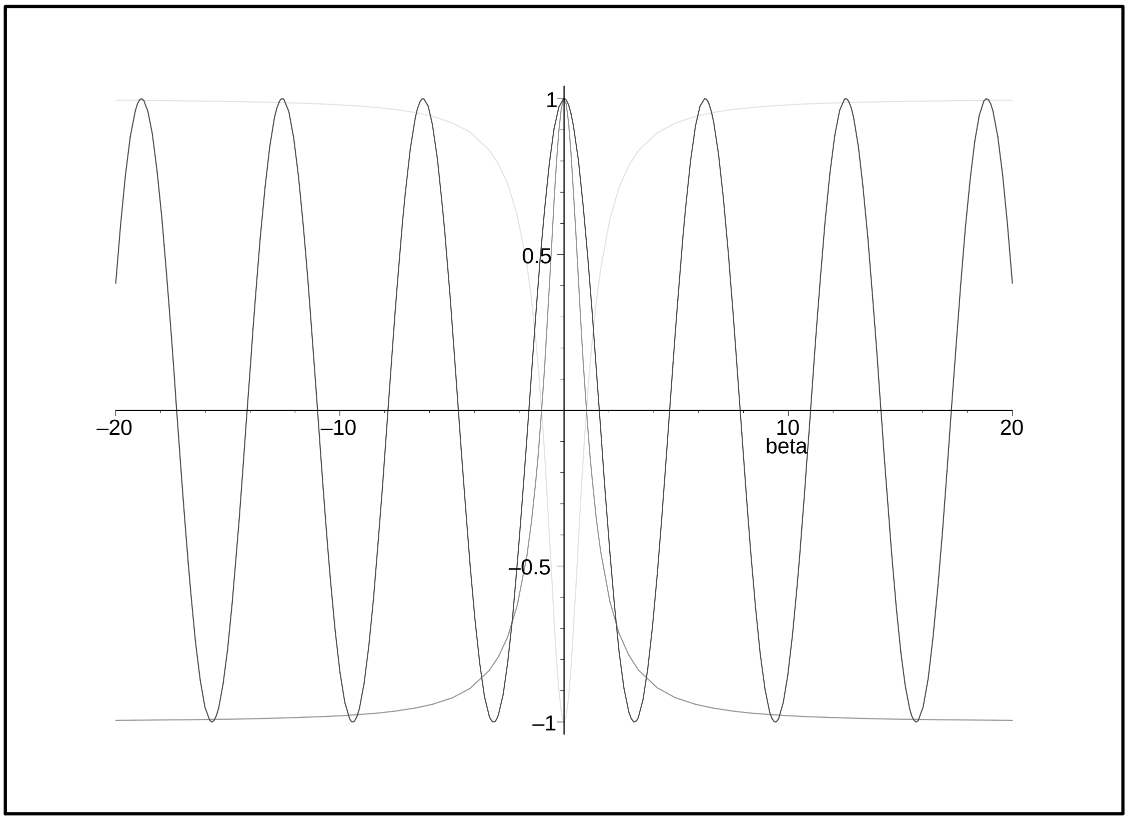

Figure 1: Intercepts of $\dfrac{1-\beta^2}{1+\beta^2}$ and $\cos\beta$.

just because we have expressions for β does not mean that such β exist. So the first thing is to see whether the two equations are consistent. We square both sides of each and add them. The result should be 1.

$$\cos^2\beta + \sin^2\beta = \frac{(1-\beta^2)^2}{(1+\beta^2)^2} + \frac{4\beta^2}{(1+\beta^2)^2} = \frac{(1+\beta^2)^2}{(1+\beta^2)^2} = 1$$

so if β satisfies one of the equations it will satisfy the other (within a sign which we change). Now as for existence, we just plot $\cos\beta$ and $\pm\dfrac{1-\beta^2}{1+\beta^2}$ and note where they cross over (see FIGURE 3c).
Regrouping the terms in $e^{\pm j\beta t}$ we have

$$\frac{A}{\beta^2+1}(e^{j\beta t} + \frac{j\beta-1}{j\beta+1}e^{-j\beta t})$$

We note that as β gets large we find the solutions go toward $n\pi$ and the eigenfunctions approach $\cos n\pi t$.