332:545 **Communications Theory** **Spring 1999**

QUIZ II Statement and Solution

You have the class period to answer the following three questions. I think they are easy, but you know I'm almost always mistaken. Read through the exam carefully before beginning work.

1. **Does Whitening Make it White?:** *(50 points)* A zero mean Gaussian stochastic process $X(t)$ is defined on $(0,T)$ Luckily for you, Rutgera Univera has already found K-L basis set of eigenfunctions $\{\phi_k(t)\}$ such that the set $\{X_i = < X(t), \phi_i(t) >\}$ is a set of mutually independent zero mean random variables with variances λ_i.

   We form the whitened process
$$X_w(t) = \sum_k \frac{X_k}{\sqrt{\lambda_k}} \phi_k(t)$$

   Show as carefully as you can whether $X_w(t)$ is a white zero mean Gaussian process. That is, does whitening actually create a white process?

   What does this say about the optimality (or lack thereof) about the whitening filter for signals in non-white noise?

   To make the problem well-behaved we first assume that all the λ_i are non-zero. If not, then the whitened process is ill-defined (division by zero). Another way to think about it is that for each $\lambda_k = 0$ we introduce an independent dummy random variable Z_k which has zero mean and variance 1 and then form
$$X_w(t) = \sum_{k \in \Lambda} \frac{X_k}{\sqrt{\lambda_k}} \phi_k(t) + \sum_{k \in \mathcal{Z}} Z_k \phi_k(t)$$
   where Λ is the set of k for which λ_k is nonzero and $\mathcal{Z}$ is its complement.

   In any event, the end result is that we have a COMPLETE basis set $\{\phi_k(t)\}$.

   Now let's establish that $X_w(t)$ is zero mean. $E[X_w(t)]$ is
$$E[X_w(t)] = \sum_k \frac{E[X_k]}{\sqrt{\lambda_k}} \phi_k(t)$$
   and we must have $E[X_k] = 0$ for all k since otherwise $E[X(t)] \neq 0$ for all t. That is, the basis waveforms $\phi_k(t)$ are orthogonal and no one of them can be expressed as a linear superposition of the others. Thus, $X_w(t)$ is zero mean.

   The process $X_w(t)$ is Gaussian and stationary because it's a linear superposition of stationary Gaussian processes. Now as for whiteness, we need to show that the autocorrelation function is an impulse of some height.

   From Mercer's theorem we know
$$K_w(t,\tau) = \sum_k \phi_k(t)\phi_k(\tau)$$
   and we wish to show that $K_w(t,\tau) = \delta(t-\tau)$. So we check by forming
$$\int K_w(t,\tau) g(\tau) d\tau = \sum_k \phi_k(t) \int \phi_k(\tau) g(\tau) d\tau = \sum_k g_k \phi_k(t)$$
   where the $\{g_k\}$ are the projections of any arbitrary function $g(\tau)$ onto the basis set $\phi_k(\tau)$. Since the set $\phi_k(t)$ is complete we know that $\sum_k g_k \phi_k(t) = g(t)$ and $K_s(t,\tau)$ satisfies the sifting property of the delta function. All that's left to show is that
$$\int_{\infty}^{\infty} K_w(t,\tau) d\tau = N_0$$

where N_0 is some constant. Let $g(t) = 1$ so that $g_k = \int \phi_k(t)dt$ and by the same completeness argument

$$\int_{\infty}^{\infty} K_w(t,\tau)d\tau = 1$$

and we're done.

Another approach is to show that projection of $X_w(t)$ onto ANY orhtonormal basis set results in a set of independent Gaussian random variables. Define a new arbitrary basis set $\{\psi_k(t)\}$. We obtain projections

$$X_{w_j} = \int \psi_j(t)X_w(t)dt = \int \psi_j(t)\sum_k \frac{X_k}{\sqrt{\lambda_k}}\phi_k(t)dt = \sum_k \frac{X_k}{\sqrt{\lambda_k}}a_{kj}$$

where $a_{kj} = \int \psi_j(t)\phi_k(t)dt$. Since the $\{\psi_j(t)\}$ are orthonormal we know that $\sum_k a_{kj}a_{ki} = \delta_{ij}$. Don't believe me? By definition $< \psi_j(t), \psi_i(t) >= 0$. Expanding

$$\delta_{ij} = \int \psi_j(t)\psi_i(t)dt = \int \sum_k a_{kj}\phi_k(t)\sum_n a_{ni}\phi_n(t)dt = \sum_k\sum_n a_{kj}a_{ni}\delta_{kn} = \sum_k a_{kj}a_{ki}$$

We know $E[X_{w_j}] = 0$ since $E[X_k] = 0$. The X_{w_j} are also jointly Gaussian since they're linear superpositions of jointly Gaussian random variables X_k. So all that remains is to show that the correlation $E[X_{w_j}X_{w_i}] = \delta_{ij}$.

$$E[X_{w_j}X_{w_i}] = \sum_n\sum_k \frac{E[X_kX_n]}{\sqrt{\lambda_n\lambda_k}}a_{kj}a_{ki} = \sum_k \frac{\lambda_k}{\lambda_k}a_{kj}a_{ki} = \delta_{ij}$$

and we're done.

What this says about our detector is that it is indeed optimal. That is, we know what the optimal detector is for a signal embedded in a real white noise process. We now know that when we whiten a Gaussian noise process the result is a WHITE Gaussian noise process. Since our receiver (the actual detection process where the likelihood ratio test is evaluated) is based on the whitened process, the result is an OPTIMAL receiver, even though the noise is colored!

What is interesting is that by rescaling, we really have not changed anything. The decision region (as determined using a likelihood ratio) in the original "uncolored" signal space is just linear warping (by the scale factors) of that we obtain in the whitened space. It's simply that we can use our white results to claim optimality! Ain't that neato!?!?!?!?

2. **Incoherent Detection:** *(50 points)* A signaling scheme on an interval $(0,T)$ uses signals $s_0(t) = 0$ and $s_1(t) = \cos(\omega t + \phi)$ where ϕ is a uniformly distributed random variable on $(0,2\pi)$. The signals are equiprobable and used on an additive white Gaussian noise channel with noise spectral height N_0. You may assume that T is an integral multiple of periods of $\cos\omega t$.

   (a) *(10 points)* Find a simple 2-dimensional orthonormal signal representation for the signals $s_i(t)$.

   The $s_i(t)$ are sinusoids at frequency ω. So $s_i(t) = a\cos\omega t + b\sin\omega t$ for some a and b. Obviously we require $a^2 + b^2 = 1$ for $s_1(t)$ and $a = b = 0$ for $s_0(t)$. The normalized signals are $\sqrt{2}\cos\omega t$ and $\sqrt{2}\sin\omega t$ for $T = 2\pi/\omega$

   (b) *(10 points)* Assume the receiver takes the received signal $r(t) = s_i(t) + n(t)$ and projects it onto your basis functions to obtain r_1 and r_2. In the ABSENCE of noise, what is $f_{R_1|R_2}(r_1|r_2)$. Given R_2 when does the value of R_1 tell you anything more about which signal was sent?

   Projection yields

$$r_1 = \int_0^T \sqrt{2}a\cos^2\omega t dt = \frac{a}{\sqrt{2}}$$

and

$$r_2 = \int_0^T \sqrt{2} b \sin^2 \omega t dt = \frac{b}{\sqrt{2}}$$

and we know (for $s_1(t)$ sent) that $a^2+b^2=1$ so that $r_1^2+r_2^2=\frac{1}{2}$. For $s_0(t)$ sent we have $r_1=r_2=0$.

Thus, if we're given $R_2=r_2$ we have $R_1=\pm\sqrt{1/2-r_2^2}$. Thus if $s_1(t)$ is sent there are at most two possible values of R_1 given r_2 and these are equiproble since the phase random variable is uniform over $(0,2\pi)$. If $r_2=\pm 1/\sqrt{2}$ then R_1 is identically zero.

So knowing R_2 in the absence of noise is usually enough to determine whether $s_0(t)$ or $s_1(t)$ was sent, except when $R_2=0$ in which case we must examine R_1 to make the determination.

What is the functional form?

$$f_{R_1|R_2}(r_1|r_2) = p_0\delta(r_1) + (1-p_0)\left(\frac{1}{2}\delta(r_1-\sqrt{1/2-r_2^2}) + \frac{1}{2}\delta(r_1+\sqrt{1/2-r_2^2})\right)$$

where p_0 is the probability $s_0(t)$ was sent.

(c) *(10 points)* Derive a functional relationship between R_1 and R_2 in the absence of noise which captures all information about which signal was signal sent.

$R^2=R_1^2+R_2^2$ tells us everything. If $R=0$ then $s_0(t)$ was sent and if $s_1(t)$ sent then $R^2=1/2$.

(d) *(20 points)* Use your functional relationship to derive an optimal receiver for this communications problem WITH NOISE.

Sketch a block diagram of your receiver.

The outputs of the correlation portion of the receiver are

$$r_1 = < s_i(t), \sqrt{2}\cos\omega t > + < n(t), \sqrt{2}\cos\omega t > = s_{i1}+n_1$$

and

$$r_1 = < s_i(t), \sqrt{2}\sin\omega t > + < n(t), \sqrt{2}\sin\omega t > = s_{i2}+n_2$$

where the n_i are independent zero mean Gaussian random variables with variance N_0. We form $r^2=r_1^2+r_2^2$ or

$$r^2 = \begin{cases} n_1^2+n_2^2 & \text{given } s_0(t) \\ (s_{i1}+n_1)^2+(s_{i2}+n_2)^2 & \text{given } s_1(t) \end{cases}$$

Notice that s_{ij} are random variables dependent on the distribution of ϕ the phase. Since ϕ is uniform on $(0,2\pi)$ we'll espect these two variables to have zero mean and identical marginal distributions independent of the noise random variables n_1 and n_2. BUT, please note that the s_{ij} are not mutually independent.

So our decision statistic is r^2 and all we have to do is calculate the distibutions for the two cases, form the likelihood ratio test and we're done. You were not expected to derive closed forms for these distributions, although it's reasonably clear that there will be Bessel functions floating about.

First off, one might recognize $n_1^2+n_2^2$ (where n_1 and n_2 are zero mean Gaussians) as the square of a Rayleigh random variable (we'll derive it).

For now, however, let's assume ignorance and derive everything from from scratch. To find the distribution on r^2 we first find the CDF – prob($R\le r$). Using the usual hypothesis notation we have

$$F_R(r|H_0) = \int_0^{n_1^2+n_2^2=r} \frac{1}{2\pi N_0} e^{-\frac{n_1^2+n_2^2}{2N_0}} dn_1 dn_2$$

This is a natural integration in polar coordinates so

$$F_R(r|H_0) = \int_0^{2\pi}\int_0^r \frac{1}{2\pi N_0} e^{-\frac{R^2}{2N_0}} R dR d\theta = \int_0^r \frac{1}{N_0} e^{-\frac{R^2}{2N_0}} R dR d\theta$$

so that

$$f_R(r|H_0) = \frac{r}{N_0} e^{-\frac{r^2}{2N_0}}$$

which is a Rayleigh distribution.

The next part is little harder than I thought so I gave full credit for even the most basic answers. We have to examine the random variable $R = \sqrt{(s_{i1}+n_1)^2 + (s_{i2}+n_2)^2}$ and remember that $s_{i1} = \frac{1}{\sqrt{2}}\cos\phi$ and $s_{i2} = \frac{1}{\sqrt{2}}\sin\phi$ where ϕ is the phase random variable. Thus we have

$$R = \sqrt{(\frac{1}{\sqrt{2}}\cos\phi + n_1)^2 + (\frac{1}{\sqrt{2}}\sin\phi + n_2)^2}$$

Now let's get devious and start first with the conditional distribution $F_{R|\phi,H_1}()$. Then we can generate the distribution for $F_{R|H_1}()$ by multiplying by the distribution (uniform) for ϕ and integrating. So,

$$F_{R|\phi,H_1}(r|\phi, H_1) = \int_{R<r} \frac{1}{2\pi N_0} e^{-\frac{(x_1-\frac{1}{\sqrt{2}}\cos\phi)^2+(x_2-\frac{1}{\sqrt{2}}\sin\phi)^2}{2N_0}} dx_1 dx_2$$

where $x_1 = \frac{1}{\sqrt{2}}\cos\phi + n_1$, $x_2 = \frac{1}{\sqrt{2}}\cos\phi + n_2$, so that $R^2 = x_1^2 + x_2^2$. Once again, it's easier to work in polar with $x_1 = R\cos\theta$ and $x_2 = R\sin\theta$,

$$F_{R|\phi,H_1}(r|\phi, H_1) = \int_0^{2\pi}\int_0^r \frac{1}{2\pi N_0} e^{-\frac{R^2+\frac{1}{2}-R\sqrt{2}(\cos\theta\cos\phi+\sin\theta\sin\phi)}{2N_0}} R dR d\theta$$

so that

$$f_{R|\phi,H_1}(r|\phi, H_1) = \int_0^{2\pi} \frac{r}{2\pi N_0} e^{-\frac{r^2+\frac{1}{2}-r\sqrt{2}(\cos\theta\cos\phi+\sin\theta\sin\phi)}{2N_0}} d\theta = \frac{r}{2\pi N_0} e^{-\frac{r^2+1/2}{N_0}} \int_0^{2\pi} e^{r\sqrt{2}\frac{\cos(\theta-\phi)}{2N_0}} d\theta$$

so that

$$f_{R|H_1}(r|H_1) = \frac{r}{(2\pi)^2 N_0} e^{-\frac{r^2+1/2}{N_0}} \int_0^{2\pi}\int_0^{2\pi} e^{r\sqrt{2}\frac{\cos(\theta-\phi)}{2N_0}} d\theta d\phi$$

We can substitute $q = \theta - \phi$ to obtain

$$f_{R|H_1}(r|H_1) = \frac{r}{(2\pi)^2 N_0} e^{-\frac{r^2+1/2}{N_0}} \int_0^{2\pi}\int_{-\phi}^{2\pi-\phi} e^{r\sqrt{2}\frac{\cos q}{2N_0}} dq d\phi$$

which can also be written as

$$f_{R|H_1}(r|H_1) = \frac{r}{(2\pi)^2 N_0} e^{-\frac{r^2+1/2}{N_0}} \int_0^{2\pi}\int_0^{\pi} 2e^{r\sqrt{2}\frac{\cos q}{2N_0}} dq d\phi = \frac{r}{N_0} e^{-\frac{r^2+1/2}{N_0}} J_0(-jr/\sqrt{2}N_0)$$

where $J_0(z) = \frac{1}{\pi}\int_0^{\pi} e^{jz\cos x} dx$ – a Bessel function of the first kind and zero$^{\text{th}}$ order.

Amazingly enough, the probability of error (which was not part of the problem) works out to a simple exponential of the ratio of the signal energy to N_0 (not even a Q function!). This would actually be a really good homework problem because you can short circuit the evaluation of the Bessel function by being clever in how you take your integrals.

In any case, this problem was more of a brain beater than I thought. It took me 1.5 hours to type out the solutions (which means about 30 minutes for me by hand, which in turn means about 1 hour for you). So don't feel too bad if you had to punt on this one.

3. **Signal to Noise and Probability of Error :** *(50 points)*

(a) *(20 points)* Derive a union (upper) bound on the probability of error for M equiprobable mutually orthogonal signals, each with energy E based on the Q function: $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$. Assume additive Gaussian zero mean white noise with spectral height N_0 and a maximum likelihood detection rule.

This is a nice symmetric problem. The probability of mistaking one signal for another is $Q(\sqrt{(d/2)^2/N_0})$ where d is the distance between signals – $d^2 = 2E$ since each signal has energy E and thus a single nonzero coordinate of $\sqrt{E}$. This probability is the same for any pair of signals. So assuming equally likely signals, the probability of error is upper bounded by

$$P_e \le (M-1)Q(\sqrt{E/2N_0}) \le MQ(\sqrt{E/2N_0})$$

(b) *(10 points)* Use the approximation for the Q function $Q(x) \le e^{-x^2/2}/2$ to provide a looser upper bound.

$$P_e \le MQ(\sqrt{E/2N_0}) \le \frac{M}{2} e^{-E/4N_0}$$

(c) *(20 points)* Let $M = 2^k$. Thus, k bits are sent during each transmission. Define $E_b = E/k$, the energy used per bit. Rewrite your bound in terms of E_b and k and N_0. Then find the minimum ratio E_b/N_0 (if it exists) such that your upper bound approaches zero as $k \to \infty$.

You'll want to remember the ratio E_b/N_0 since it crops up often in communications theory.

Rewriting we have

$$P_e \le \frac{2^k}{2} e^{-kE_b/4N_0}$$

Noting that $2^k = e^{k \log 2}$ we have

$$P_e \le \frac{1}{2} e^{-k(E_b/4N_0 - \log 2)}$$

which goes to zero in k as long as $(E_b/4N_0 - \log 2) > 0$. Thus, we require $E_b/N_0 > 4 \log 2$.