Derivation of Pascal Distribution

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Hi Folks,

Manasi asked me a question and though I made noises in the right direction, I pretty much screwed up in what I wrote down in an email. This will serve to fix my errors and be an indirect apology to Manasi.

Suppose you have a set of identically distributed random variables K_i where each is geometric with parameter p. That is

$$p_{K_i}(k) = p(1-p)^{k-1}$$

for $k = 1, 2, ...$ We can think of K_i as the number of trials up to and including a first "success."

Now, suppose we run the same "success" experiment independently r times. That is, we're looking for the number of trials up to and including the r^{th} success. Then we have a new random variable $K = K_1 + K_2 + \cdots + K_r$. Since the K_i are independent, the distribution of the sum is the convolution of the distributions (A VERY IMPORTANT FACT! HINT HINT!!!).

However, unless you're a masochist and like doing convolution as opposed to multiplication, the speediest way to figure out the distribution on K is to go to "frequency domain" and use moment generating functions. So, the moment generating function for K_i is

$$\phi_{K_i}(s) = E[e^{sk}] = \sum_{k=1}^{\infty} p(1-p)^{k-1}e^{sk} = \frac{pe^s}{1-(1-p)e^s}$$

which immediately means that

$$\phi_K(s) = \left(\frac{pe^s}{1-(1-p)e^s}\right)^r = p^r e^{sr} \left(\frac{1}{1-(1-p)e^s}\right)^r$$

And I dare you to try to take the inverse transform directly to get back to $p_K(k)$. (I have a personal dislike of doing contour integrations, so inverse Z transforms and inverse Laplace transforms often bedevil me. Luckily, I have lots of company in my laziness.)

When I'm confronted with something I don't know (or don't want to do using brute force), I nose around for an easy way out. Looking at the fraction raised to a power makes me immediately think of

$$\frac{d}{d\theta}\frac{1}{1-\theta} = \left(\frac{1}{1-\theta}\right)^2$$

Therefore

$$\frac{d^\ell}{d\theta^\ell}\frac{1}{1-\theta} = \ell!\left(\frac{1}{1-\theta}\right)^{\ell+1}$$

Using this lovely fact we have (setting $\theta = (1-p)e^s$)

$$\phi_K(s) = p^r e^{sr}\frac{1}{(r-1)!}\frac{d^{r-1}}{d\theta^{r-1}}\frac{1}{1-\theta}$$

Suddenly, I'm much happier because I immediately remember that

$$\sum_{m=0}^{\infty} q^m = \frac{1}{1-q}$$

which leads to

$$\phi_K(s) = p^r e^{sr}\frac{1}{(r-1)!}\frac{d^{r-1}}{d\theta^{r-1}}\sum_{k=1}^{\infty}\theta^{k-1} = p^r e^{sr}\frac{1}{(r-1)!}\sum_{k=r}^{\infty}\frac{(k-1)!}{(k-1-(r-1))!}\theta^{k-1-(r-1)}$$

which we simplify as

$$\phi_K(s) = p^r e^{sr}\frac{1}{(r-1)!}\sum_{k=r}^{\infty}\frac{(k-1)!}{(k-r)!}\theta^{k-r}$$

We then substitute for θ to obtain

$$\phi_K(s) = \sum_{k=r}^{\infty} p^r e^{sr}\frac{1}{(r-1)!}\frac{(k-1)!}{(k-r)!}(1-p)^{k-r}e^{s(k-r)}$$

and rearrange a little to get

$$\phi_K(s) = \sum_{k=r}^{\infty} e^{sk} \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

Hmmmmm! This looks just like $E[e^{sk}]$ if

$$p_K(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

$k = r, r+1, \cdots$. AND WE'RE DONE since that's the Pascal distribution!

Laziness has its rewards!